

# Limit sets of unfolding paths in Outer space

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## Abstract

We construct an unfolding path in Outer space which does not converge in the boundary, and instead it accumulates on the entire 1-simplex of projectivized length measures on a non-geometric arational  $\mathbb{R}$ -tree  $T$ . We also show that  $T$  admits exactly two dual ergodic projective currents.

## 1 Introduction

For the once-punctured torus, the Thurston compactification of the Teichmüller space by projective measured laminations coincides with the visual compactification of the hyperbolic plane. In this case, every geodesic ray has a unique limit point, and the dynamical behavior of the ray in moduli space is governed by the continued fraction of its limit point. For hyperbolic surfaces of higher complexity, Teichmüller space with the Teichmüller metric is no longer negatively curved [Mas75, MW95] (or even Riemannian), and the Thurston boundary is no longer its visual boundary [Ker80]. More surprisingly, geodesic rays do not always converge [Len08, LLR18].

For hyperbolic surfaces of higher complexity, another interesting phenomenon is the existence of nontrivial simplices in the Thurston boundary which correspond to measures on non-uniquely ergodic laminations. Particularly interesting is the case when the underlying lamination is minimal and filling, also called *arational*. Constructions of non-uniquely ergodic arational laminations have a long history, and typically used flat structures on surfaces [Vee69], [Sat75], [KN76], [Kea77]. A topological construction was introduced in [Gab09]. In [LLR18], Leininger, Lenzhen and Rafi combined a topological approach with some arithmetic parameters akin to continued fractions. This allowed them to show that it is possible for the full simplex of measures on a non-uniquely ergodic arational lamination to be realized as the limit set of a Teichmüller geodesic ray.

In this paper, we take the above construction into Culler-Vogtmann's Outer space [CV86]. A Thurston-type boundary for Outer space is given by the set of projective classes of minimal, very small  $\mathbb{F}_n$ -trees [CM87], [BF94], [CL95], [Hor17] and the action of  $\text{Out}(\mathbb{F}_n)$  extends continuously to the compactified space. The analogue of arational laminations are *arational trees*; for example, trees dual to arational laminations on a once-punctured surface fall into this category. There are other examples, such as trees dual to minimal laminations on finite 2-complexes that are not surfaces, called *Levitt type*; and yet others, called *non-geometric*, that do not come from the latter two constructions. The non-uniquely ergodic phenomenon for laminations has two natural analogues for  $\mathbb{F}_n$ -trees: one in terms of length measures on trees, giving rise to *non-uniquely ergometric trees* [Gui00], and the other in terms of currents, giving *non-uniquely ergodic trees*, see [CHL07]. It is an open problem to determine whether these two notions coincide. An example of a Levitt type non-uniquely ergometric arational tree, modeled on Keane's construction, was given in [Mar97].

In Outer space, the analogue of Teichmüller metric is the Lipschitz metric and that of Teichmüller geodesics are folding paths. However, unlike Teichmüller geodesics, a folding path in Outer space has a forward direction, reflecting the asymmetry of the Lipschitz metric. Even though the boundary of Outer space is not a visual boundary, a folding path always converges

along its forward direction. Our main result is that this nice behavior does not persist in the backward direction; in fact, in the backward direction, folding paths can behave as badly as Teichmüller geodesics. Define an *unfolding path* in Outer space to be a folding path with the backward direction. Our main result, as follows, is a direct analogue of the results of [LLR18].

**Theorem 1.1.** *There exists an unfolding path in Outer space of free group of rank 7 which does not converge to a point in the boundary of Outer space. In fact, the limit set is a 1-simplex consisting of the full set of length measures on a non-geometric and arational tree  $T$ . Moreover, the set of projective currents dual to  $T$  is also a 1-dimensional simplex. In particular,  $T$  is neither uniquely ergometric nor uniquely ergodic.*

We use the framework of folding and unfolding sequences. Every such sequence tracks the combinatorics of an appropriate folding path, resp. unfolding path, in Outer space. An infinite folding sequence has a naturally associated limiting tree in the boundary of Outer space and an unfolding sequence has a naturally associated algebraic lamination, called the *legal lamination*. The graphs in the folding sequence can be given compatible metrics, which are then used to parametrize the different length measures supported on the limiting tree. Compatible edge thicknesses on the graphs of the unfolding sequence parametrize the different currents with support contained in the legal lamination. The latter can then be used to study the currents dual to the trees in the limit set of the unfolding sequence. See [NPR14] or our [Section 3](#) for definitions and more precise statements.

Modeling the construction of [LLR18] on a 5-holed sphere, the folding and unfolding sequences we consider come from explicit sequences of automorphisms of the free group of rank 7. More explicitly, fix a non-geometric fully irreducible automorphism on three letters and extend it to an automorphism  $\phi$  of  $\mathbb{F}_7$  by identity on the other four basis elements. Also let  $\rho$  be a finite order automorphism of  $\mathbb{F}_7$  that rotates the support of  $\phi$  off itself. For an integer  $r$ , set  $\phi_r = \rho\phi^r$ . Given a sequence  $(r_i)_{i \geq 1}$  of positive integers, define a sequence of automorphisms by

$$\Phi_i = \phi_{r_1} \circ \cdots \circ \phi_{r_i}.$$

From  $(\Phi_i)_i$  we get an unfolding sequence using the train track map induced by  $\phi_{r_i}$ , and from  $(\Phi_i^{-1})_i$  we get a companion folding sequence. The parameters  $(r_i)_i$  play the role of the continued fraction expansion for the limiting tree of the folding sequence, and adjusting them produces different types of trees and behaviors of the unfolding sequence. In particular, we show that if the sequence  $(r_i)_i$  satisfies certain arithmetic conditions and grows sufficiently fast, then the limiting tree is arational, non-geometric, non-uniquely ergodic, and non-uniquely ergometric. Moreover, the limit set of the unfolding sequence is the full simplex of length measures on the tree. We refer to [Section 10](#) for the full technical statement.

To see how the parameters  $(r_i)_i$  come into play, it is informative to look at the sequence of free factors  $A_i = \Phi_i(A)$ , where  $A$  is the support of  $\phi$ . The  $A_i$ 's are the projection of the folding sequence to the free factor complex  $\mathcal{FF}_7$ . By our construction,  $A_i$  and  $A_{i+1}$  are disjoint (meaning  $\mathbb{F}_7 = A_i * A_{i+1} * B_i$  for some  $B_i$ ), but  $A_i, A_{i+2}$  are not, and  $r_i$  measures the distance between the projections of  $A_{i-2}$  and  $A_{i+2}$  to the free factor complex of  $A_i$ . Morally, if  $r_i$ 's are sufficiently large, then  $(A_i)_i$  forms a quasi-geodesic in  $\mathcal{FF}_7$ . Hence, by [BR15, Ham16], the limiting tree of the folding sequence is arational. In addition, we show that the tree is non-geometric. To get two currents on the tree, we take loops in the  $A_i$ 's, which correspond to currents on  $\mathbb{F}_n$  and take projective limits of the odd and even subsequences. Non-unique ergometricity of the tree follows a similar principle.

Although our construction is general in spirit, the case of rank 7 is already fairly involved, and some computations used computer assistance. One issue is that there is no known algorithm to tell if a collection of free factors has a common complement. This issue appears in the proof of arationality of the limiting tree that led to the peculiar looking arithmetic conditions on the parameters; see [Section 5](#).

## Outline

- In [Section 2](#), we review some background material, including train track maps, Outer space, currents, length measures, and arational trees.
- In [Section 3](#) we discuss folding and unfolding sequences. We relate length measures on a folding sequence with the length measures on the limiting tree when it is arational. We also define the legal lamination for an unfolding sequence and state a result from [\[NPR14\]](#) relating the currents supported on the legal lamination with those of the unfolding sequence.
- In [Section 4](#), we discuss our main construction to generate from a sequence  $(r_i)_i$  of positive integers a sequence of automorphisms of  $\mathbb{F}_7$ . The associated transition matrices for these automorphisms have block shapes which we use to analyze their asymptotic behavior. From each sequence of automorphisms and their inverses, we get a folding and unfolding sequence of graphs of rank 7 induced by their train track maps.
- In [Section 5](#), we show that under the right conditions on  $(r_i)_i$ , the folding sequence converges to a non-geometric and arational tree  $T$  in boundary of Outer space of rank 7. To show arationality, we project the folding sequence to the free factor complex and show it is a quasi-geodesic.
- In [Section 6](#), we study the behavior of the unfolding sequence. The main result is that if the sequence  $(r_i)_i$  grows sufficiently fast, then the legal lamination of the unfolding sequence supports a 1-simplex of projective currents.
- In [Section 7](#), we show that if the sequence  $(r_i)_i$  grows sufficiently fast, then the limiting tree of the folding sequence supports a 1-simplex of projective length measures. In particular, the limiting tree is not uniquely ergodic.
- In [Section 8](#), we relate the legal lamination of the unfolding sequence to the dual lamination of the limiting tree of the folding sequence. This shows the limiting tree is not uniquely ergodic.
- In [Section 9](#), we show that the unfolding sequence limits onto the full simplex of length measures on the limiting tree of the folding sequence, and thus does not have a unique limit in the boundary of Outer space.
- In [Section 10](#), we collect the results to prove the main theorem.
- In [Section 11](#), we prove a technical lemma about convergence of products of matrices.

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## 2 Background

Let  $\mathbb{F}_n$  be the free group of rank  $n$ . We review some background on train track maps, Outer space, laminations, currents, and the free factor complex.

### 2.1 Train track maps

We recall some basic definitions from [\[BH92\]](#). Identify  $\mathbb{F}_n$  with  $\pi_1(\mathbb{R}_n, *)$  where  $\mathbb{R}_n$  is a rose with  $n$  petals. A *marked graph*  $G$  is a graph of rank  $n$ , all of whose vertices have valence at least three, equipped with a homotopy equivalence  $m: \mathbb{R}_n \rightarrow G$  called a *marking*.

A *length vector* on  $G$  is a vector  $\lambda \in \mathbb{R}^{|EG|}$  that assigns a positive number, i.e. a length, to every edge of  $G$ . The volume of  $G$  with respect to  $\lambda$  is the total length of all the edges of  $G$ . This induces a path metric on  $G$  where the length of an edge  $e$  is  $\lambda(e)$ .

A direction  $d$  based at a vertex  $v \in G$  is an oriented edge of  $G$  with initial vertex  $v$ . A *turn* is an unordered pair of distinct directions based at the same vertex. A *train track structure* on

$G$  is an equivalence relation on the set of directions at each vertex  $v \in G$ . The classes of this relation are called *gates*. A turn  $(d, d')$  is *legal* if  $d$  and  $d'$  do not belong to the same gate, it is called *illegal* otherwise. A path is legal if it only crosses legal turns.

A map  $f: G \rightarrow G'$  between two graphs is called a *morphism* if it is locally injective on open edges and sends vertices to vertices. If  $G$  and  $G'$  are metric graphs, then we can homotope  $f$  relative to vertices such that it is linear on edges. Similarly, for an  $\mathbb{R}$ -tree  $T$ , a map  $\tilde{G} \rightarrow T$  from the universal cover of  $G$  is a morphism if it is injective on edges. To a morphism  $f: G \rightarrow G'$  we associate the *transition matrix* as follows. Enumerate the (unoriented) edges  $e_1, e_2, \dots, e_m$  of  $G$  and  $e'_1, e'_2, \dots, e'_n$  of  $G'$ . Then the transition matrix  $M$  has size  $n \times m$  and the  $ij$ -entry is the number of times  $f(e_j)$  crosses  $e'_i$ , i.e. it is the cardinality of the set  $f^{-1}(x) \cap e_j$  for a point  $x$  in the interior of  $e'_i$ . If  $f$  is in addition a homotopy equivalence, then  $f$  is a *change-of-marking*.

A homotopy equivalence  $f: G \rightarrow G$  induces an outer automorphism of  $\pi_1(G)$  and hence an element  $\phi$  of  $\text{Out}(\mathbb{F}_n)$ . If  $f$  is a morphism then we say that  $f$  is a *topological representative* of  $\phi$ . A topological representative  $f: G \rightarrow G$  induces a *train track structure* on  $G$  as follows: The map  $f$  determines a map  $Df$  on the directions in  $G$  by defining  $Df(e)$  to be the first (oriented) edge in the edge path  $f(e)$ . We then declare  $e_1 \sim e_2$  if  $(Df)^k(e_1) = (Df)^k(e_2)$  for some  $k \geq 1$ .

A topological representative  $f: G \rightarrow G$  is called a *train track map* if every vertex has at least two gates, and  $f$  maps legal turns to legal turns and legal paths (equivalently, edges) to legal paths. Equivalently, every positive power  $f^k$  is a topological representative. If  $f$  is a train track map with transition matrix  $M$ , then the transition matrix of  $f^k$  is  $M^k$  for every  $k \geq 1$ . If  $M$  is *primitive*, i.e.  $M^k$  has positive entries for some  $k \geq 1$ , then Perron-Frobenius theory implies that there is an assignment of positive lengths to all the edges of  $G$  so that  $f$  uniformly expands lengths of legal paths by some factor  $\lambda > 1$ , called the *stretch factor* of  $f$ .

If  $\sigma$  is a path (or a circuit) in  $G$ , we denote by  $[\sigma]$  the reduced path homotopic to  $\sigma$  (rel endpoints if  $\sigma$  is a path). A path or circuit  $\sigma$  in  $G$  is called a *periodic Nielsen path* if  $[f^k(\sigma)] = \sigma$  for some  $k \geq 1$ . If  $k = 1$ , then  $\sigma$  is a *Nielsen path*. A Nielsen path is *indivisible*, denoted INP, if it cannot be written as a concatenation of nontrivial Nielsen paths.

The following lemma is an important property of train track maps. For a very rudimentary form, see [BH92, Lemma 3.4] showing that INPs have exactly one illegal turn, and for a more involved version see [BFH97] (some details can also be found in [KL14, Proposition 3.27, 3.28]). We will need it for the proof of Lemma 4.8 and include a proof here.

**Lemma 2.1.** *Let  $h: G \rightarrow G$  be a train track map with a primitive transition matrix. There exists a constant  $R > 0$  such that for any edge path  $\gamma$ , either,*

1. *the number of illegal turns in  $[h^R(\gamma)]$  is less than that of  $\gamma$ , or,*
2.  *$\gamma = u_1 v_1 u_2 v_2 \dots u_n$  where each  $u_i$  is a legal subpath, possibly degenerate, and each  $[h^R(v_i)]$  is a periodic INP.*

*Proof.* Let  $\lambda > 1$  be the stretch factor of  $h$ , and equip  $G$  with the metric so that  $h$  uniformly expands the length of every legal path by  $\lambda$ . It goes back to the work of Thurston (see [Coo87]) that there is a constant  $BCC(h)$ , called the bounded cancellation constant for  $h$ , such that if  $\alpha\beta$  is a reduced edge path then  $[h(\alpha)][h(\beta)]$  have cancellation bounded by  $BCC(h)$ . The existence of this constant is really a consequence of the Morse lemma and the fact that  $h$  is a quasi-isometry. Define  $C = BCC(h)/(\lambda - 1)$ .

Here is the significance of  $C$ . To fix ideas let us assume that  $\gamma$  has only one illegal turn, so  $\gamma = \alpha\beta$  with both  $\alpha, \beta$  legal. Say  $\alpha$  has length  $|\alpha| = C + \epsilon > C$ . Then  $h(\alpha)$  has length  $\lambda|\alpha|$  and after cancellation with  $h(\beta)$  the length is  $\geq \lambda|\alpha| - BCC(h) = |\alpha| + \lambda\epsilon$ . Thus assuming  $[h^i(\gamma)]$  still has an illegal turn, the length of the initial subpath to the illegal turn has length growing exponentially in  $i$ , assuming it is long enough.

We now prove the lemma for paths  $\gamma = \alpha\beta$  with one illegal turn, with  $\alpha, \beta$  legal. Consider the finite collection of paths consisting of those with length at most  $C$  with both endpoints at vertices, or with length exactly  $C$  with only one endpoint at a vertex. Let  $R$  be a number larger than the square of the size of this collection. If  $[h^i(\gamma)] = \alpha_i\beta_i$  has one illegal turn (with  $\alpha_i, \beta_i$  legal) for  $i = 1, 2, \dots, R$ , then by the pigeon-hole principle there will be  $i < j$  in this range so

that the  $C$ -neighborhoods of the illegal turns of  $[h^i(\gamma)]$  and  $[h^j(\gamma)]$  are the same (if  $\alpha_i$  or  $\beta_i$  has length  $< C$  this means  $\alpha_i = \alpha_j$  or  $\beta_i = \beta_j$ ). We can lift  $h^{j-i}$  and  $\gamma$  to the universal cover of the graph and arrange that (the lift of)  $\gamma$  and  $[h^{j-i}(\gamma)]$  have the same illegal turn. Thus  $h^{j-i}$  maps the terminal  $C$ -segment of  $\alpha_i$  (or  $\alpha_i$  itself) over itself (by the above calculation) and therefore fixes a point in  $\alpha_i$ , and similarly for  $\beta_i$ . The subpath of  $[h^i(\gamma)]$  between these fixed points is a periodic INP, proving the lemma in the case  $\gamma$  has one illegal turn.

The general case is similar. Write  $\gamma = \gamma_1\gamma_2 \cdots \gamma_s$  with all  $\gamma_k$  legal and with the turn between  $\gamma_k$  and  $\gamma_{k+1}$  illegal. Also assume that  $[h^i(\gamma)]$  has the same number of illegal turns for  $i = 1, \dots, R$ . We can write  $[h^i(\gamma)] = \gamma_1^i\gamma_2^i \cdots \gamma_s^i$  with all  $\gamma_k^i$  legal and the turns between them illegal. For each illegal turn corresponding to the pair  $(k, k+1)$  there will be  $i < j$  in this range so that the  $C$ -neighborhoods of the illegal turn in  $[h^i(\gamma)]$  and in  $[h^j(\gamma)]$  are the same. This gives fixed points of  $h^{j-i}$  in  $\gamma_k^i$  and  $\gamma_{k+1}^i$  and these fixed points split  $\gamma$  into periodic INPs and legal segments, as claimed.  $\square$

We will use the lemma in the situation that  $h$  has no periodic INPs, in which case the conclusion is that whenever  $\gamma$  is not legal, then  $[h^R(\gamma)]$  has fewer illegal turns than  $\gamma$ .

## 2.2 Outer space and its boundary

An  $\mathbb{F}_n$ -tree is an  $\mathbb{R}$ -tree with an isometric action of  $\mathbb{F}_n$ . An  $\mathbb{F}_n$ -tree  $T$  has dense orbits if some (every) orbit is dense in  $T$ . An  $\mathbb{F}_n$ -tree is called *very small* if the action is minimal, arc stabilizers are either trivial or maximal cyclic, and tripod stabilizers are trivial. We review the definition of Outer space first introduced in [CV86].

*Unprojectivized Outer space*, denoted by  $cv_n$ , is the set of free, minimal, and simplicial  $\mathbb{F}_n$ -trees. By considering the quotient graphs,  $cv_n$  is also equivalently the set of marked metric graphs, i.e. the set of triples  $(G, m, \lambda)$ , where  $G$  is a graph of rank  $n$  with all valences at least 3,  $m: \mathbb{R}_n \rightarrow G$  is a marking, and  $\lambda$  is a positive length vector on  $G$ . By [CM87], the map of  $cv_n \rightarrow \mathbb{R}^{\mathbb{F}_n}$  given by  $T \mapsto (\|g\|_T)_{g \in \mathbb{F}_n}$ , where  $\|g\|_T$  is the translation length of  $g$  in  $T$ , is an inclusion. This endows  $cv_n$  with a topology. The closure  $\overline{cv}_n$  in  $\mathbb{R}^{\mathbb{F}_n}$  is the space of very small  $\mathbb{F}_n$ -trees [BF94, CL95]. The boundary  $\partial cv_n = \overline{cv}_n - cv_n$  consists of very small trees that are either not free or not simplicial.

Culler Vogtmann's *Outer space*,  $CV_n$ , is the image of  $cv_n$  in the projective space  $\mathbb{P}\mathbb{R}^{\mathbb{F}_n}$ . Elements in  $CV_n$  can also be described as free, minimal, simplicial  $\mathbb{F}_n$ -trees with unit covolume. Topologically,  $CV_n$  is a complex made up of simplices with missing faces, where there is an open simplex for each marked graph  $(G, m)$  spanned by positive length vectors on  $G$  of unit volume. The closure  $\overline{CV}_n$  of  $CV_n$  in  $\mathbb{P}\mathbb{R}^{\mathbb{F}_n}$  is compact and the boundary  $\partial CV_n = \overline{CV}_n - CV_n$  is the projectivization of  $\partial \overline{cv}_n$ .

The spaces  $cv_n$  and  $CV_n$  and their closures are equipped with a natural (right) action by  $\text{Out}(\mathbb{F}_n)$ . That is, for  $\Phi \in \text{Out}(\mathbb{F}_n)$  and  $T \in \overline{cv}_n$  the translation length function of  $T\Phi$  on  $\mathbb{F}_n$  is  $\|g\|_{T\Phi} = \|\phi(g)\|_T$ , where  $\phi$  is any lift of  $\Phi$  to  $\text{Aut}(\mathbb{F}_n)$ .

## 2.3 Laminations, currents and non-uniquely ergodic trees

In [BFH00], Bestvina, Feighn and Handel defined a dynamic invariant called the attracting lamination associated to a train track map. In this article, we will consider the more modern definition of a lamination as given in [CHL08a].

Let  $\partial\mathbb{F}_n$  denote the Gromov boundary of  $\mathbb{F}_n$  and let  $\Delta$  be the diagonal in  $\partial\mathbb{F}_n \times \partial\mathbb{F}_n$ . The *double boundary* of  $\mathbb{F}_n$  is  $\partial^2\mathbb{F}_n = (\partial\mathbb{F}_n \times \partial\mathbb{F}_n - \Delta)/\mathbb{Z}_2$ , which parametrizes the space of unoriented bi-infinite geodesics in a Cayley graph of  $\mathbb{F}_n$ . By an *(algebraic) lamination*, we mean a non-empty, closed and  $\mathbb{F}_n$ -invariant subset of  $\partial^2\mathbb{F}_n$ .

Associated to  $T \in \overline{cv}_n$  is a *dual lamination*  $L(T)$ , defined as follows in [CHL08b]. For  $\epsilon > 0$ , let

$$L_\epsilon(T) = \overline{\{(g^{-\infty}, g^{\infty}) \mid \|g\|_T < \epsilon, g \in \mathbb{F}_n\}},$$

so  $L_\epsilon(T)$  is a lamination and set  $L(T) = \bigcap_{\epsilon>0} L_\epsilon(T)$ . Elements of  $L(T)$  are called *leaves*. For trees in  $cv_n$ ,  $L(T)$  is empty.

A *current* is an additive, non-negative,  $\mathbb{F}_n$ -invariant function on the set of compact open sets in  $\partial^2\mathbb{F}_n$ . Equivalently, it is an  $\mathbb{F}_n$ -invariant Radon measure on the  $\sigma$ -algebra of Borel sets of  $\partial^2\mathbb{F}_n$ . Let  $\text{Curr}_n$  denote the space of currents, equipped with the weak\* topology. The quotient space of  $\mathbb{P}\text{Curr}_n$  of projectivized currents (i.e. homothety classes of nonzero currents) is compact.

For  $\mu \in \text{Curr}_n$ , let  $\text{supp}(\mu) \subset \partial^2\mathbb{F}_n$  denote the support of  $\mu$ , which is in fact a lamination. For  $T \in \overline{cv}_n$  and  $\mu \in \text{Curr}_n$ , if  $\text{supp}(\mu) \subseteq L(T)$ , then we say  $\mu$  is *dual* to  $T$ . Denote by  $\mathcal{C}(T)$  the convex cone of currents dual to  $T$  and by  $\mathbb{P}\mathcal{C}(T)$  the set of projective currents dual to  $T$ .  $\mathbb{P}\mathcal{C}(T)$  is a compact, convex space and its extremal points are called the *ergodic currents* dual to  $T$ . We say  $T$  is *uniquely ergodic* if there is only one projective class of currents dual to  $T$ , and *non-uniquely ergodic* otherwise. In [CH16], the authors show that if  $T \in \partial cv_n$  has dense orbits, then  $\mathbb{P}\mathcal{C}(T)$  is the convex hull of at most  $3n - 5$  projective classes of ergodic currents dual to  $T$ .

In [KL09], Kapovich and Lustig established a length pairing,  $\langle \cdot, \cdot \rangle$ , between  $\overline{cv}_n$  and the space of measured currents  $\text{Curr}_n$ . They also showed in [KL10, Theorem 1.1] that for  $T \in \overline{cv}_n$  and  $\mu \in \text{Curr}_n$ ,  $\langle T, \mu \rangle = 0$  if and only if  $\mu$  is dual to  $T$ .

Given two trees  $T$  and  $T'$ , we say a map  $h: T \rightarrow T'$  is *alignment-preserving* if whenever  $b \in T$  is contained in an arc  $[a, c] \subset T$ , then  $h(b)$  is contained in the arc  $[h(a), h(c)]$ .

**Theorem 2.2** ([CHL07]). *Let  $T, T' \in \partial CV_n$  be two trees with dense orbits. The following are equivalent:*

- $L(T) = L(T')$ .
- There exists an  $\mathbb{F}_n$ -equivariant alignment-preserving bijection between  $T$  and  $T'$ .

## 2.4 Length measures and non-uniquely ergometric trees

Since  $\mathbb{R}$ -trees need not be locally compact, classical measure theory is not well-suited for them. In [Pau95], a *length measure* was introduced for  $\mathbb{R}$ -trees. See [Gui00] for details.

A *length measure* on an  $\mathbb{F}_n$ -tree  $T$  is a collection of finite Borel measures  $\lambda_I$  for every compact interval  $I$  in  $T$  such that if  $J \subset I$ , then  $\lambda_J = (\lambda_I)|_J$ . We require the length measure to be invariant under the  $\mathbb{F}_n$  action. The collection of the Lebesgue measures of the intervals of  $T$  is  $\mathbb{F}_n$ -invariant, and this will be called the Lebesgue measure of  $T$ . A length measure  $\lambda$  is *non-atomic* or *positive* if every  $\lambda_I$  is non-atomic or positive. If every orbit is dense in some segment of  $T$ , then  $T$  cannot have an invariant measure with atoms. Further, if  $T$  is *indecomposable*, that is, if for any pair of nondegenerate arcs  $I$  and  $J$  in  $T$ , there exist  $g_1, \dots, g_m \in \mathbb{F}_n$ , such that  $I \subset \bigcup g_i J$  and  $g_i J \cap g_{i+1} J$  is nondegenerate, then every nonzero length measure is positive (in fact, the condition of mixing [Gui00] suffices).

Let  $\mathcal{D}(T)$  be the cone of  $\mathbb{F}_n$ -invariant length measures on  $T$ , with projectivization  $\mathbb{P}\mathcal{D}(T)$ , i.e. the homothety classes of  $\mathbb{F}_n$ -invariant length measures on  $T$ .  $\mathbb{P}\mathcal{D}(T)$  is a compact convex set and we will call its extremal points the *ergodic length measures* on  $T$ . When  $T$  has dense orbits there are at most  $3n - 4$  such measures for any  $T$ , see [Gui00, Corollary 5.2, Lemma 5.3] and  $\mathcal{D}(T)$  is naturally a subset of  $\partial cv_n$ . In fact,

**Lemma 2.3.** [Gui00] *If  $T \in cv_n$  is indecomposable then  $\mathcal{D}(T)$  is in one-to-one correspondence with the set of isometry classes of  $\mathbb{F}_n$ -invariant metrics on  $T$ , denoted  $X_T \subset cv_n$ .*

*Proof.* Let  $\lambda \in \mathcal{D}(T)$  be a length measure on  $T$ . Consider the pseudo-metric  $d_\lambda$  on  $T$  where  $d_\lambda(x, y) = \lambda([x, y])$  for  $x, y \in T$ . In fact, since  $T$  is indecomposable,  $d_\lambda$  is a metric on  $T$ . For the converse, let  $T' \in X_T$ . Then the pull back of Lebesgue measure on  $T'$  under identity map  $\text{id}: T \rightarrow T'$  gives a positive length measure on  $T$ .  $\square$

We say  $T$  is *uniquely ergometric* if there is only one projective class of length measures on  $T$ , which necessarily is the homothety class of the Lebesgue measure on  $T$ . It is called *non-uniquely ergometric* otherwise.

## 2.5 Arational trees and the free factor complex

For a tree  $T \in \overline{\text{cv}}_n$  and a free factor  $H$  of  $\mathbb{F}_n$ , let  $T_H$  denote the minimal  $H$ -invariant subtree of  $T$  (this tree is unique unless  $H$  fixes an arc). A tree  $T \in \partial\text{cv}_n$  is *arational* if every proper free factor  $H$  of  $\mathbb{F}_n$  has a free and simplicial action on  $T_H$ . By [Rey12] every arational tree is free and indecomposable or it is the dual tree to an arational measured lamination on a surface with one puncture. The arational trees of the first kind are either Levitt type or non-geometric.

Let  $\mathcal{AT} \subset \partial\text{CV}_n$  denote the set of arational trees with the subspace topology. Using Lemma 2.3, define an equivalence relation  $\sim$  on  $\mathcal{AT}$  by “forgetting the metric”, that is,  $T \sim T'$  if  $T' \in \mathbb{PD}(T)$ , and endow  $\mathcal{AT}/\sim$  with the quotient topology. The following lemma is implicit in [Gui00] and we include a proof for completeness.

**Lemma 2.4.** *Let  $T, T'$  be arational trees. Then  $T \sim T'$  if and only if  $L(T) = L(T')$ .*

*Proof.* If  $T \sim T'$  then the identity map is an alignment preserving bijection  $\text{id}: T \rightarrow T'$ . Therefore, by Theorem 2.2,  $L(T) = L(T')$ .

If  $L(T) = L(T')$ , then by Theorem 2.2 there is an alignment preserving bijection  $f: T \rightarrow T'$ . Pulling back the Lebesgue measure on  $T'$  induces a length measure on  $T$ , and the corresponding metric  $d_\mu$  on  $T$  is isometric to  $T'$ , so  $T' \sim T$ .  $\square$

The *free factor complex*  $\mathcal{FF}_n$  is a simplicial complex whose vertices are given by conjugacy classes of proper free factors of  $\mathbb{F}_n$  and a  $k$ -simplex is given by a nested chain  $[A_0] \subset [A_1] \subset \dots \subset [A_k]$ . When the rank  $n = 2$  the definition is modified and an edge connects two conjugacy classes of rank 1 factors if they have complementary representatives. The free factor complex can be given a metric as follows: identify each simplex with a standard simplex and endow the resulting space with path metric. By result of [BF14a], the metric space  $\mathcal{FF}_n$  is Gromov hyperbolic. The Gromov boundary of  $\mathcal{FF}_n$  was identified with  $\mathcal{AT}/\sim$  in [BR15] and [Ham16].

There is a projection map  $\pi: \text{CV}_n \rightarrow \mathcal{FF}_n$  defined as follows [BF14a, Section 3]: for  $G \in \text{CV}_n$ ,  $\pi(G)$  is the collection of free factors given by the fundamental group of proper subgraphs of  $G$  which are not forests. This map is coarsely well defined, that is,  $\text{diam}_{\mathcal{FF}_n}(\pi(G)) \leq K$  for some universal  $K$ . Note that if  $G, G'$  belong to the same open simplex of  $\text{CV}_n$ , then  $\pi(G) = \pi(G')$ , so the projection of a simplex of  $\text{CV}_n$  has uniformly bounded diameter.

## 3 Folding and unfolding sequences

In this section we introduce (un)folding sequences and review some work of Namazi-Pettet-Reynolds [NPR14].

A *folding/unfolding sequence* is a sequence

$$G_a \longrightarrow \dots \longrightarrow G_{-1} \longrightarrow G_0 \longrightarrow G_1 \longrightarrow \dots \longrightarrow G_b$$

of graphs, together with maps  $f_i: G_i \rightarrow G_{i+1}$  such that for any  $j \leq i$ ,  $f_{i-1} \circ f_{i-2} \circ \dots \circ f_j: G_j \rightarrow G_i$  is a change-of-marking morphism. Equivalently, a sequence as above is called a folding/unfolding sequence, if there exists a train track structure on each  $G_i$  and  $f_{i-1} \circ f_{i-2} \circ \dots \circ f_j$  maps legal paths to legal paths. We allow the sequence to be infinite in one or both directions. We assume that a marking on  $G_0$  has been specified, so a folding/unfolding sequence determines a sequence of open simplices in Outer space.

Let  $Q_i$  be the transition matrix of  $f_i$ . A *length measure* for a folding/unfolding sequence  $(G_i)_{a \leq i \leq b}$  is a sequence  $(\lambda_i)_{a \leq i \leq b}$ , where  $\lambda_i \in \mathbb{R}^{|EG_i|}$  is a length vector on  $G_i$ , and for  $a \leq i < b$ ,

$$\lambda_i = Q_i^T \lambda_{i+1}.$$

In this way,  $f_i$  restricts to an local isometry on every edge of  $G_i$ . When  $b < \infty$ , a length vector on  $G_b$  determines a length measure on the sequence. When the sequence is infinite in the forward direction we denote by  $\mathcal{D}((G_i)_i)$  the space of length measures on  $(G_i)_i$ , and  $\mathbb{PD}((G_i)_i)$  its projectivization. Observe that the dimension of  $\mathcal{D}((G_i)_i)$  is bounded by  $\liminf_{i \rightarrow \infty} |EG_i|$ .

A *current* for a folding/unfolding sequence  $(G_i)_{a \leq i \leq b}$  is a sequence  $(\mu_i)_{a \leq i \leq b}$ , where  $\mu_i \in \mathbb{R}^{|EG_i|}$  is a length vector on  $G_i$  (but thought of as a vector of thicknesses of edges), and for  $a \leq i < b$ , we require

$$\mu_{i+1} = Q_i \mu_i.$$

Likewise, when the sequence is infinite in the backward direction, we denote by  $\mathcal{C}((G_i)_i)$  the space of currents on  $(G_i)_i$ , and  $\mathbb{P}\mathcal{C}((G_i)_i)$  its projectivization. The dimension of  $\mathcal{C}((G_i)_i)$  is bounded by  $\liminf_{i \rightarrow -\infty} |EG_i|$ .

### 3.1 Isomorphism between length measures

In this section, we identify the space of length measures on a folding sequence with that of the limiting tree, when it is an arational tree.

Consider a folding sequence of marked graphs of rank  $n$

$$G_0 \xrightarrow{f_1} G_1 \longrightarrow \cdots \xrightarrow{f_i} G_i \xrightarrow{f_{i+1}} \cdots$$

Let  $\tilde{G}_i$  be the universal cover of  $G_i$ , and let  $\tilde{f}_i$  be a lift of  $f_i$ . For any positive length measure  $(\lambda_i)_i \in \mathcal{D}((G_i)_i)$ , we can realize  $(\tilde{G}_i, \tilde{\lambda}_i)_i$  as a sequence in  $\text{cv}_n$ , which can be “filled in” by a folding path in  $\text{cv}_n$  (see [BF14a] for details on folding paths). In particular,  $(\tilde{G}_i, \tilde{\lambda}_i)_i$  always converges to a point  $T \in \partial\text{cv}_n$ . Furthermore, we have morphisms  $h_i: \tilde{G}_i \rightarrow T$  such that  $h_i = h_{i+1} \tilde{f}_{i+1}$ . With respect to the length measure  $\tilde{\lambda}_i$ ,  $\tilde{f}_i$  and  $h_i$  restrict to isometries on edges [BR15, Lemma 7.6].

Let  $(U_i)_i$  be the sequence of open simplices  $\text{CV}_n$  associated to the sequence  $(G_i)_i$ . Recall the projection map  $\pi: \text{CV}_n \rightarrow \mathcal{FF}_n$  is coarsely well-defined on simplices of  $\text{CV}_n$ . We will say the folding sequence  $(G_i)_i$  converges to an arational tree  $T$  if  $\pi(U_i)$  converges to  $[T] \in \partial\mathcal{FF}_n$ .

**Proposition 3.1.** *Suppose a folding sequence  $(G_i)_i$  converges to an arational tree  $T$ . Then there is a linear isomorphism between  $\mathcal{D}((G_i)_i)$  and  $\mathcal{D}(T)$ .*

*Proof.* Fix a positive length measure  $(\lambda_i)_i \in \mathcal{D}((G_i)_i)$  and let  $T \in \partial\text{cv}_n$  be the limiting tree of  $(\tilde{G}_i, \tilde{\lambda}_i)$  with corresponding morphism  $h_i: \tilde{G}_i \rightarrow T$ . Recall from Section 2.5 that if  $T$  is arational, then we can identify  $\mathcal{D}(T)$  with the subspace of  $\mathbb{F}_n$ -metrics on  $T$  in  $\partial\text{cv}_n$ . We will let  $\lambda \in \mathcal{D}(T)$  be a length measure, and  $T_\lambda$  its image in  $\partial\text{cv}_n$ .

By [BR15], if  $\pi(U_i)$  converges to  $[T''] \in \partial\mathcal{FF}_n$ , then for any positive  $(\lambda'_i)_i \in \mathcal{D}((G_i)_i)$ ,  $(\tilde{G}_i, \tilde{\lambda}'_i)$  also converges to an arational tree  $T' \in \partial\text{cv}_n$ , such that  $[T''] = [T'] = [T]$ ; in other words,  $T' = T_{\lambda'}$  for some  $\lambda' \in \mathcal{D}(T)$ . This gives a linear map  $\mathcal{D}((G_i)_i) \rightarrow \mathcal{D}(T)$ .

Conversely, for any positive length measure  $\lambda' \in \mathcal{D}(T)$ , we can use the morphism  $h_i$  to pull back  $\lambda'$  from  $T$  to a length measure  $\lambda'_i$  on  $\tilde{G}_i$ . The fact that  $h_i = h_{i+1} \tilde{f}_{i+1}$  implies  $(\lambda'_i)_i \in \mathcal{D}((G_i)_i)$ . Moreover, the sequence  $(\tilde{G}_i, \tilde{\lambda}'_i)_i$  converges to  $T_{\lambda'} \in \text{cv}_n$ . This gives a linear map  $\mathcal{D}(T) \rightarrow \mathcal{D}((G_i)_i)$  which is the inverse of  $\mathcal{D}((G_i)_i) \rightarrow \mathcal{D}(T)$  defined above. This shows  $\mathcal{D}((G_i)_i) \rightarrow \mathcal{D}(T)$  is an isomorphism.  $\square$

*Remark 3.2.* A more general statement of Proposition 3.1 which doesn't involve the assumption that  $T$  is arational can be found in [NPR14, Proposition 5.4], but we will not need such a general statement here.

### 3.2 Isomorphism between currents

In this section, we state an analogous result identifying the space of currents on an unfolding sequence with the space of currents of a legal lamination associated to a unfolding sequence. We record some definitions from [NPR14] first.

Consider an unfolding sequence of marked graphs of rank  $n$

$$\cdots \xrightarrow{f_{i+1}} G_i \xrightarrow{f_i} \cdots \xrightarrow{f_2} G_1 \xrightarrow{f_1} G_0.$$



Denote the composition  $F_i = f_1 \circ \dots \circ f_i$ . Let  $\Omega_\infty^L(G_i)$  denote the set of bi-infinite legal paths in  $G_i$ . Define the *legal lamination* of the unfolding sequence  $(G_i)_i$  to be

$$\Lambda = \bigcap_i F_i(\Omega_\infty^L(G_i)) \subseteq \Omega_\infty^L(G_0).$$

Use the marking on  $G_0$  to identify  $\partial^2\pi_1(G_0)$  with  $\partial^2\mathbb{F}_n$ . The preimage, in  $\partial^2\mathbb{F}_n$ , of the lift of  $\Lambda$  to  $\partial^2\pi_1(G_0)$  is a lamination  $\tilde{\Lambda}$ . We denote by  $\mathcal{C}(\Lambda)$  the convex cone of currents supported on  $\tilde{\Lambda}$ , with projectivization  $\mathbb{P}\mathcal{C}(\Lambda)$ .

An *invariant sequence of subgraphs* is a sequence of nondegenerate (i.e. not forests) proper subgraphs  $H_i \subset G_i$  such that  $f_i$  restricts to a morphism  $H_i \rightarrow H_{i-1}$ . We will need the following theorem from [NPR14].

**Theorem 3.3** (Theorem 4.4 [NPR14]). *Given an unfolding sequence  $(G_i)_{i \geq 0}$  without an invariant sequence of subgraphs and with legal lamination  $\Lambda$ , then there is a natural linear isomorphism between  $\mathcal{C}((G_i)_i)$  and  $\mathcal{C}(\Lambda)$ .*

## 4 Main setup

In this section, we will construct an unfolding sequence  $(\tau_i)_i$  and a folding sequence  $(\tau'_i)_i$  in  $\text{CV}_7$  that intersect the same infinite set of simplices, which we will eventually use to show the existence of a non-uniquely ergodic and ergometric tree. The construction is done via a family of outer automorphisms. We will describe these automorphisms and then analyze the asymptotic behavior of their train track maps.

### 4.1 The automorphisms

Let  $\mathbb{F}_7 = \langle a, b, c, d, e, f, g \rangle$ . Denote by  $\bar{x}$  the inverse of  $x \in \mathbb{F}_7$ . First consider the map induced on the 3-petaled rose by the automorphism

$$\theta: a \mapsto b, b \mapsto c, c \mapsto ca \in \text{Aut}(\mathbb{F}_3)$$

and the map induced by the inverse automorphism

$$\vartheta: a \mapsto \bar{b}c, b \mapsto a, c \mapsto b.$$

Using  $\theta$  and  $\vartheta$  to also denote the corresponding graph maps, and using the convention that  $a$  also denotes the initial direction of the oriented edge  $a$ , while  $\bar{a}$  denotes the terminal direction, we have

$$D\theta^3: \begin{array}{c} a \longrightarrow c \longleftarrow b \\ \quad \quad \quad \curvearrowright \end{array} \quad \bar{a} \curvearrowright \quad \bar{b} \curvearrowright \quad \bar{c} \curvearrowright$$

$$D\vartheta^3: a \longrightarrow \bar{c} \curvearrowright \quad b \longrightarrow \bar{a} \curvearrowright \quad c \longrightarrow \bar{b} \curvearrowright$$

*Observation 4.1.* From the structure of the above maps, for  $n \equiv 0 \pmod{3}$ ,  $D\theta^n = D\theta^3$  and  $D\vartheta^n = D\vartheta^3$ .

**Lemma 4.2.** *The map on the 3-petaled rose labeled  $a, b, c$  induced by  $\vartheta$  is a train track map with respect to the train track structure with gates  $\{a, \bar{c}\}, \{b, \bar{a}\}, \{c, \bar{b}\}$ . Moreover, this train track map does not have any periodic INPs.*

*The map on the 3-petaled rose labeled  $a, b, c$  induced by  $\theta$  is also a train track map with respect to the train track structure with gates  $\{a, b, c\}, \{\bar{a}\}, \{\bar{b}\}, \{\bar{c}\}$  and it has one periodic Nielsen path (see [BF94, Example 3.4]).*

*Proof.* The train track structure on the rose induces a metric on the graph coming from Perron-Frobenius theory. Every INP has length at most twice the volume of the graph, one illegal turn, and the endpoints are fixed. Since there are only finitely many fixed points in  $G$ , it is easy to enumerate all such paths and check if they are Nielsen. For periodic INPs one knows that the period is bounded by a function of the rank of  $\mathbb{F}_n$  [FH18], so one can take a suitable power and check for INPs (though there are more efficient ways, see [Kap19]). Coulbois' train track package [Cou] for the mathematics software system Sage [Sag] computes periodic INPs of train track maps.  $\square$

Now let  $\phi \in \text{Aut}(\mathbb{F}_7)$  be the automorphism:

$$a \mapsto b, b \mapsto c, c \mapsto ca, d \mapsto d, e \mapsto e, f \mapsto f, g \mapsto g$$

and  $\rho \in \text{Aut}(\mathbb{F}_7)$  be the rotation by 4 clicks:

$$a \mapsto e, b \mapsto f, c \mapsto g, d \mapsto a, e \mapsto b, f \mapsto c, g \mapsto d$$

Thus  $\phi$  is the extension of  $\theta$  by identity, and  $\rho$  rotates the support of  $\phi$  off itself.

**Lemma 4.3.** *For any  $r \geq 3$ , the map on the 7-petaled rose induced by  $\phi_r = \rho\phi^r$  is a train track map with respect to the train track structure with gates*

$$\{a, b, c\}, \{d, e, f\}$$

and 8 more gates consisting of single half edges. The transition matrix  $M_r$  has block form

$$\begin{pmatrix} 0 & I \\ B^r & 0 \end{pmatrix}$$

where  $I$  is the  $4 \times 4$  identity matrix, and  $B$  is the transition matrix of  $\theta$ :

$$B = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

*Proof.* By **Observation 4.1**, we only have to check the lemma for  $\phi_3, \phi_4, \phi_5$ , which can be done by hand or using the train track package for Sage.  $\square$

**Lemma 4.4.** *For any  $r \geq 3$  and  $r \equiv 0 \pmod{3}$ , the map on the 7-petaled rose induced by  $\psi_r = (\rho\phi^r)^{-1}$  is a train track map with respect to the train track structure with gates*

$$\{a, e, \bar{g}\}, \{b, \bar{d}\}, \{c, \bar{b}\}, \{d, \bar{c}\}, \{f, \bar{e}\}, \{g, \bar{f}\}, \{\bar{a}\}$$

The transition matrix  $N_r$  has block form

$$\begin{pmatrix} 0 & C^r \\ I & 0 \end{pmatrix}$$

where  $I$  is the  $4 \times 4$  identity matrix, and  $C$  is the transition matrix of  $\vartheta$ :

$$C = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

*Proof.* By **Observation 4.1**, we only have to check the lemma for  $\psi_3$ , which can be done by hand or using the train track package for Sage.  $\square$

## 4.2 Asymptotics of transition matrices

Let  $\theta, \vartheta, \phi_r, \psi_r$  be the maps defined in the last section. We now analyze the behavior of the transition matrices  $M_r$  and  $N_r$  for  $\phi_r$  and  $\psi_r$  respectively.

**Lemma 4.5.** *Let  $B$  be the transition matrix for  $\theta$ , with Perron-Frobenius eigenvalue  $\lambda_B$ . There exists a constant  $\kappa_B > 0$  such that if  $r, s - r \rightarrow \infty$ , then*

$$\frac{1}{\kappa_B \lambda_B^s} M_r M_s \rightarrow Y$$

where  $Y$  is an idempotent matrix of the form

$$Y = \begin{pmatrix} u & pu & qu & 0 & 0 & 0 & 0 \end{pmatrix} \text{ with } u = (0, u_1, u_2, u_3, 0, 0, 0)^T \text{ and } p, q > 0,$$

and  $(u_1, u_2, u_3)^T$  is a Perron-Frobenius eigenvector of  $B$ .

*Proof.* There exists a Perron-Frobenius eigenvector  $x = (x_1, x_2, x_3)^T$  for  $B$  and constants  $p, q > 0$  such that

$$P = \lim_{s \rightarrow \infty} \frac{B^s}{\lambda_B^s} = \begin{pmatrix} x & px & qx \end{pmatrix}.$$

We have

$$M_r M_s = \left( \begin{array}{ccc|ccc|c} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline & B^s & & 0 & & & 0 \\ & & & & 0 & & 0 \\ & & & & & & 0 \\ \hline 0 & & & B^r & & & 0 \\ & & & & 0 & & 0 \\ & & & & & & 0 \end{array} \right) \implies \frac{1}{\lambda_B^s} M_r M_s \rightarrow \left( \begin{array}{ccc|ccc|c} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline & P & & 0 & & & 0 \\ & & & & 0 & & 0 \\ & & & & & & 0 \\ \hline 0 & & & 0 & & & 0 \\ & & & & 0 & & 0 \\ & & & & & & 0 \end{array} \right)$$

The square of the limiting matrix above has a nonzero block where  $P$  is, of the form

$$(px_1 + qx_2)P,$$

and zero elsewhere, so we set

$$\kappa_B = px_1 + qx_2 \quad \text{and} \quad (u_1, u_2, u_3)^T = \frac{1}{\kappa_B} (x_1, x_2, x_3)^T. \quad \square$$

We have a similar statement for the matrices  $N_r$ .

**Lemma 4.6.** *Let  $C$  be the transition matrix for  $\vartheta = \theta^{-1}$ , with Perron-Frobenius eigenvalue  $\lambda_C$ . There exists a constant  $\kappa_C > 0$  such that if  $r, s - r \rightarrow \infty$ , then*

$$\frac{1}{\kappa_C \lambda_C^s} N_s N_r \rightarrow Z,$$

where  $Z$  is an idempotent matrix of the form

$$Z = \begin{pmatrix} 0 & v & pv & qv & 0 & 0 & 0 \end{pmatrix} \text{ with } v = (v_1, v_2, v_3, 0, 0, 0, 0)^T \text{ and } p, q > 0,$$

and  $(v_1, v_2, v_3)^T$  is a Perron-Frobenius eigenvector of  $C$ .

*Proof.* We observe that the matrix  $N_s N_r$  has shape that is the transpose of the matrix in [Lemma 4.5](#), with powers of the PF matrix  $C$  forming the nonzero blocks:

$$N_s N_r = \left( \begin{array}{c|cc|ccc} 0 & & & & & & \\ 0 & C^s & & & & & 0 \\ 0 & & & & & & \\ \hline 0 & & & 0 & & & C^r \\ 0 & & & & & & \\ 0 & & & & & & \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right) \quad \square$$

For future reference, we also record the following. Let  $P = \lim_{r \rightarrow \infty} B^r / \lambda_B^r$  and  $Q = \lim_{r \rightarrow \infty} C^r / \lambda_C^r$ . Set

$$M_\infty = \lim_{r \rightarrow \infty} \frac{M_r}{\lambda_B^r} = \left( \begin{array}{c|c} 0 & 0 \\ \hline P & 0 \end{array} \right) \quad \text{and} \quad N_\infty = \lim_{r \rightarrow \infty} \frac{N_r}{\lambda_C^r} = \left( \begin{array}{c|c} 0 & Q \\ \hline 0 & 0 \end{array} \right)$$

**Lemma 4.7.** *There are  $p, q, r, s > 0$  such that*

$$M_\infty Y = (y \quad py \quad qy \quad 0 \quad 0 \quad 0 \quad 0) \quad \text{with } y = (0, 0, 0, 0, y_1, y_2, y_3)^T,$$

$(y_1, y_2, y_3)^T$  is a Perron-Frobenius eigenvector of  $B$ ; and

$$ZN_\infty = (0 \quad 0 \quad 0 \quad 0 \quad z \quad rz \quad sz) \quad \text{with } z = (z_1, z_2, z_3, 0, 0, 0, 0)^T,$$

and  $(z_1, z_2, z_3)^T$  is a Perron-Frobenius eigenvector of  $C$ .

### 4.3 Folding and unfolding sequence

Consider a sequence of positive integers  $(r_i)_{i \geq 1}$  and the sequence of automorphisms  $\phi_{r_i}$ , with transition matrix  $M_{r_i}$  and  $\phi_{r_i}^{-1} = \psi_{r_i}$  with transition matrix  $N_{r_i}$ . Let  $\tau_i \rightarrow \tau_{i-1}$  (resp.  $\tau'_{i-1} \rightarrow \tau'_i$ ) be the train track map induced on the rose by  $\phi_{r_i}$  (resp.  $\psi_{r_i}$ ) as given by [Lemma 4.3](#) (resp. [Lemma 4.4](#)). Thus we have an unfolding sequence

$$\cdots \longrightarrow \tau_{i+1} \xrightarrow{\phi_{r_{i+1}}} \tau_i \xrightarrow{\phi_{r_i}} \tau_{i-1} \longrightarrow \cdots \xrightarrow{\phi_{r_3}} \tau_2 \xrightarrow{\phi_{r_2}} \tau_1 \xrightarrow{\phi_{r_1}} \tau_0,$$

and a folding sequence

$$\cdots \longleftarrow \tau'_{i+1} \xleftarrow{\psi_{r_{i+1}}} \tau'_i \xleftarrow{\psi_{r_i}} \tau'_{i-1} \longleftarrow \cdots \xleftarrow{\psi_{r_3}} \tau'_2 \xleftarrow{\psi_{r_2}} \tau'_1 \xleftarrow{\psi_{r_1}} \tau'_0.$$

Let  $\Phi_i = \phi_{r_1} \circ \cdots \circ \phi_{r_i}$  and  $\Phi_i^{-1} = \Psi_i = \psi_{r_i} \circ \cdots \circ \psi_{r_1}$ . Here,  $\tau_0$  is a rose with petals labeled by elements in  $\{a, b, c, d, e, f, g\}$  and hence for  $i \geq 1$ ,  $\tau_i$  is a rose labeled by  $\{\Phi_i(a), \dots, \Phi_i(g)\}$ . Also,  $\tau'_0$  is a rose labeled by  $\{a, b, c, d, e, f, g\}$ , so  $\tau'_i$  is also a rose labeled by  $\{\Phi_i(a), \dots, \Phi_i(g)\}$ . Thus, for every  $i \geq 0$ ,  $\tau_i$  and  $\tau'_i$  have the same marking but different train track structures. In other words, they belong to the same simplex in  $\text{CV}_7$ .

The next lemma studies the behavior of illegal turns in a path along the folding sequence. This will be used in the proof of [Proposition 5.10](#) to show that the limit tree of the folding sequence is non-geometric.

**Lemma 4.8.** *Let  $(r_i)_{i \geq 1}$  be strictly increasing such that  $r_i \equiv 0 \pmod{3}$  and  $r_1 > R$ , where  $R$  is the constant from [Lemma 2.1](#). Let  $(\tau'_i)_i$  be the corresponding folding sequence. Then for any edge path  $\beta$  in  $\tau'_j$  with at least one illegal turn, the number of illegal turns in  $[\psi_{r_{j+3}} \psi_{r_{j+2}} \psi_{r_{j+1}}(\beta)]$  is less than the number of illegal turns in  $\beta$ .*

*Proof.* By [Lemma 4.4](#), the illegal turns in  $\tau'_j$  are

$$\{a, e\}, \{a, \bar{g}\}, \{e, \bar{g}\}, \{b, \bar{d}\}, \{c, \bar{b}\}, \{d, \bar{c}\}, \{f, \bar{e}\}, \{g, \bar{f}\}$$

and we have

$$\begin{aligned} \{a, e\} &\xrightarrow{\psi_{r_{j+1}}} \{d, \bar{c}\} \xrightarrow{\psi_{r_{j+2}}} \{g, \bar{f}\} \\ \{a, \bar{g}\} &\xrightarrow{\psi_{r_{j+1}}} \{d, \bar{c}\} \xrightarrow{\psi_{r_{j+2}}} \{g, \bar{f}\} \\ \{b, \bar{d}\} &\xrightarrow{\psi_{r_{j+1}}} \{e, \bar{g}\} \\ \{c, \bar{b}\} &\xrightarrow{\psi_{r_{j+1}}} \{f, \bar{e}\} \\ \{d, \bar{c}\} &\xrightarrow{\psi_{r_{j+1}}} \{g, \bar{f}\} \end{aligned}$$

Thus, for any illegal edge path  $\beta \subset \tau'_j$ , one of  $\beta, \psi_{r_{j+1}}(\beta), \psi_{r_{j+2}}\psi_{r_{j+1}}(\beta)$  has a illegal turn  $\{x, y\}$  where  $x, y \in \{e, f, g, \bar{e}, \bar{f}, \bar{g}\}$ .

Consider the automorphism  $\vartheta$  and corresponding train track map  $h: \mathbb{R}_3 \rightarrow \mathbb{R}_3$  as in [Lemma 4.2](#). Then  $h$  does not have any periodic INPs. Since  $R$  is the constant from [Lemma 2.1](#), we get that one of  $[\psi_{r_{j+1}}(\beta)], [\psi_{r_{j+2}}\psi_{r_{j+1}}(\beta)], [\psi_{r_{j+3}}\psi_{r_{j+2}}\psi_{r_{j+1}}(\beta)]$  has fewer illegal turns than  $\beta$ .  $\square$

## 5 Limiting tree of folding sequence

In this section, we will show that for appropriate choices of  $(r_i)_i$ , the projection of the folding sequences  $(\tau'_i)_i$  to the free factor complex  $\mathcal{FF}_7$  is a quasi-geodesic and hence converges to the equivalence class of an arational tree. We will also show that this tree is non-geometric.

### 5.1 Sequence of free factors

Given a sequence  $(r_i)_{i \geq 1}$ , recall that  $\Phi_i = \phi_{r_1}\phi_{r_1} \cdots \phi_{r_i}$ , where  $\phi_r = \rho\phi^r$ . For convenience, also set  $\Phi_0 = \text{id}$ . We have the folding sequence

$$\cdots \longleftarrow \tau'_{i+1} \xleftarrow{\psi_{r_{i+1}}} \tau'_i \xleftarrow{\psi_{r_i}} \tau'_{i-1} \longleftarrow \cdots \xleftarrow{\psi_{r_3}} \tau'_2 \xleftarrow{\psi_{r_2}} \tau'_1 \xleftarrow{\psi_{r_1}} \tau'_0.$$

where  $\tau'_i$  is a rose labeled by  $\{\Phi_i(a), \dots, \Phi_i(g)\}$ , and  $\psi_r = \phi_r^{-1}$ . From the markings, we can associate  $\tau'_i$  to an open simplex  $U_i$  in  $\text{CV}_7$ . Consider a sequence of free factors  $A_i \in \pi(U_i)$ , where  $\pi: \text{CV}_7 \rightarrow \mathcal{FF}_7$ . For an appropriate sequence of  $(r_i)_i$ , we will see that  $(A_i)_i$  is a quasi-geodesic (with infinite diameter). The key will be [Lemma 5.3](#) which is the main goal of this section.

We now consider the following explicit sequence of free factors. Let  $A_0 = \langle d, e, f \rangle$  be the free factor in  $\mathbb{F}_7$ , and define

$$A_i := \Phi_i A_0 = \langle \Phi_i(d), \Phi_i(e), \Phi_i(f) \rangle.$$

Note that for any  $r, s, t > 0$ , the following holds:

$$\begin{aligned} A_0 &= \langle d, e, f \rangle \\ A_1 &= \phi_r(A_0) = \langle a, b, c \rangle \\ A_2 &= \phi_s\phi_r(A_0) = \langle e, f, g \rangle \\ A_3 &= \phi_t\phi_s\phi_r(A_0) = \langle b, c, d \rangle \end{aligned} \tag{1}$$

Thus, for any sequence  $(r_i)_i$ ,

$$A_i = \Phi_i(A_0) = \Phi_{i-1}(A_1) = \Phi_{i-2}(A_2) = \Phi_{i-3}(A_3). \tag{2}$$

We say two free factors  $A$  and  $A'$  are *disjoint* if (possibly after conjugating)  $\mathbb{F}_n = A * A' * B$  for a (possibly trivial) free factor  $B$ , and  $A'$  is *compatible* with  $A$  if it either contains  $A$  (up to conjugation) or is disjoint from  $A$ .

**Lemma 5.1.** *For any sequence  $(r_i)_{i \geq 1}$ , if  $|i - j| = 1$ , then  $A_i, A_j$  are disjoint; and if  $|i - j| = 2$  or  $3$ , then they are distinct and not disjoint.*

*Proof.* We see from [Equation \(1\)](#) that the statement of the lemma holds for  $A_0, A_1, A_2$  and  $A_3$ . Now for  $i \geq 1$  and  $k \in \{1, 2, 3\}$ , by [Equation \(2\)](#), the pair  $A_i, A_{i+k}$  differs from  $A_0, A_k$  by the automorphism  $\Phi_i$ , whence the lemma.  $\square$

Recall the transition matrix  $M_r$  for  $\phi_r$ , and the  $3 \times 3$  matrix  $B$  whose power  $B^r$  forms a block of  $M_r$ . For each  $i \geq 1$ , let  $\bar{M}_i = M_i \text{ mod } 2$ . By a simple computation, we see that  $B^7 = I \text{ mod } 2$ . Thus, when  $i = j \text{ mod } 7$ ,  $\bar{M}_i = \bar{M}_j$ . We have the following lemma.

**Lemma 5.2.** *Let  $V_0$  be the 3-dimensional vector space of  $(\mathbb{Z}/2\mathbb{Z})^7$  spanned by the vectors  $(0, 0, 0, 1, 0, 0, 0)^T$ ,  $(0, 0, 0, 0, 1, 0, 0)^T$ ,  $(0, 0, 0, 0, 0, 1, 0)^T$ . Then for all  $i \geq 0$ ,*

$$V_0 \cup \left( \bigcup_{j=0}^{107} \overline{M}_i \overline{M}_{i+1} \dots \overline{M}_{i+j} V_0 \right) = (\mathbb{Z}/2\mathbb{Z})^7.$$

*Proof.* Since  $\overline{M}_i = \overline{M}_j$  whenever  $i = j \pmod{7}$ , it is enough to verify the statement for  $i \in \{0, \dots, 6\}$ . In these cases, we can check the validity of the statement using Sage with the following code:

```

1 B = matrix(GF(2), [
2     [0,0,1],
3     [1,0,0],
4     [0,1,1]
5 ])
6
7 def M(i):
8     return block_matrix([
9         [ matrix(4,3,0) , identity_matrix(4) ],
10        [ B^i           , matrix(3,4,0)       ]
11    ])
12
13 V0 = set(
14     (0,0,0,i,j,k,0)
15     for i in (0,1)
16     for j in (0,1)
17     for k in (0,1)
18 )
19
20 for i in range(0,7):
21     W = set(V0)
22     P = identity_matrix(7)
23     for j in range(i,200):
24         P = P*M(j)
25         for v in V0:
26             w = tuple(P*vector(v))
27             W.add(w)
28             if len(W) >= 2^7:
29                 break
30     print(i,j)
31
32 # Output:
33 # 0 107
34 # 1 107
35 # 2 107
36 # 3 107
37 # 4 107
38 # 5 107
39 # 6 107

```

□

**Lemma 5.3.** *For any sequence  $(r_i)_i$ , if  $r_i \equiv i \pmod{7}$ , then 109 consecutive  $A_i$ 's cannot be contained in the same free factor or be disjoint from a common factor.*

*Proof.* For any  $i \geq 1$  and  $k \geq 0$ , let

$$B_{i+k} = \phi_i \phi_{i+1} \dots \phi_{i+k} A_0.$$

Abelianizing and reducing mod 2, we have  $A_0 \equiv V_0$ , and  $B_{i+k} \equiv \overline{M}_i \cdots \overline{M}_{i+k} V_0$ . Thus, by [Lemma 5.2](#), the sequence  $\{A_0, B_i, \dots, B_{i+107}\}$  cannot be contained in the same free factor or be disjoint from a common factor.

Now consider any sequence  $(r_i)_i$  with  $r_i \equiv i \pmod{7}$ , so that  $\overline{M}_{r_i} = \overline{M}_i$  for all  $i$ . Let  $A_i = \Phi_i A_0 = \phi_{r_1} \cdots \phi_{r_i} A_0$ . Set  $\Phi_0 = \text{id}$ . For any  $i \geq 1$ , by applying the automorphism  $\Phi_{i-1}^{-1}$ , the sequence of free factors  $\{A_{i-1}, \dots, A_{i+107}\}$  is isomorphic to the sequence

$$\{A_0, \phi_{r_1} A_0, \dots, \phi_{r_i} \phi_{r_{i+1}} \cdots \phi_{r_{i+107}} A_0\}.$$

The latter sequence after abelianization and reducing mod 2 is equivalent to the sequence  $\{A_0, B_i, \dots, B_{i+107}\}$ . Thus  $\{A_{i-1}, \dots, A_{i+107}\}$  cannot be contained in the same factor or be disjoint from a common factor.  $\square$

## 5.2 Subfactor projection

We will now use subfactor projection theory originally introduced in [\[BF14b\]](#) and further developed in [\[Tay14\]](#) to show that  $(A_i)_i$  is a quasi-geodesic for appropriate choices of sequence  $(r_i)_i$ .

We first define subfactor projection and recall the main results about them. For  $G \in \text{CV}_n$  and a rank  $\geq 2$  free factor  $A$ , let  $A|G$  denote the core subgraph of the cover of  $G$  corresponding to the conjugacy class of  $A$ . Pulling back the metric on  $G$ , we obtain  $A|G \in \text{CV}(A)$ . Denote by  $\pi_A(G) := \pi(A|G) \subset \mathcal{F}(A)$  the projection of  $A|G$  to  $\mathcal{F}(A)$ . Here  $\text{CV}(A)$  is the Outer space of the free group  $A$  and  $\mathcal{F}(A)$  is the corresponding free factor complex.

Recall two free factors  $A$  and  $B$  are *disjoint* if they are distinct vertex stabilizers of a free splitting of  $\mathbb{F}_n$ . If  $B$  is not compatible with  $A$ , then we say  $B$  *meets*  $A$ , that is,  $B$  and  $A$  are not disjoint and  $A$  is not contained in  $B$ , up to conjugation. In this case, define the projection of  $B$  to  $\mathcal{F}(A)$  as follows:

$$\pi_A(B) := \{\pi_A(G)|G \in \text{CV}_n \text{ and } B|G \subset G \text{ is embedded}\}$$

If  $B$  is compatible with  $A$ , then define  $\pi_A(B)$  to be empty. If  $A$  meets  $B$  and  $B$  meets  $A$ , then we say  $A$  and  $B$  *overlap*.

**Theorem 5.4** ([\[Tay14\]](#)). *Let  $A, B, C$  be free factors of  $\mathbb{F}_n$ . There is a constant  $D$  depending only on  $n$  such that the following statements hold.*

1. *If  $\text{rank}(A) \geq 2$ , then either  $A \subseteq B$  (up to conjugation),  $A$  and  $B$  are disjoint, or  $\pi_A(B) \subset \mathcal{F}(A)$  is defined and has diameter  $\leq D$ .*
2. *If  $\text{rank}(A) \geq 2$ ,  $B$  and  $C$  meet  $A$ , and  $B$  is compatible with  $C$ , then*

$$d_A(B, C) = \text{diam}_{\mathcal{F}(A)}(\pi_A(B) \cup \pi_A(C)) \leq D.$$

3. *If  $A$  and  $B$  overlap, have rank at least 2, and  $C$  meets both, then*

$$\min\{d_A(B, C), d_B(A, C)\} \leq D.$$

**Theorem 5.5** (Bounded geodesic image theorem [\[Tay14\]](#)). *For  $n \geq 3$ , there exists  $D' \geq 0$  such that if  $A$  is a free factor with  $\text{rank}(A) \geq 2$  and  $\gamma$  is a geodesic of  $\mathcal{FF}_n$  with each vertex of  $\gamma$  having a well-defined projection to  $\mathcal{F}(A)$ , then  $\text{diam}(\pi_A(\gamma)) \leq D'$ .*

We now prove the following lemma.

**Lemma 5.6.** *For any  $K > 0$ , there exists a constant  $r = r(K)$  such that for any sequence  $(r_i)_{i \geq 1}$ , if  $r_i \geq r$  for all  $i$ , then the following statements hold:*

1. *For any  $j \geq 2$ , the projections of  $A_{j-2}$  and  $A_{j+2}$  to the free factor complex  $\mathcal{F}(A_j)$  are defined and the distance between them is at least  $K$ .*

2. Let  $D$  be the constant of [Theorem 5.4](#). If  $K > 3D$ , then for any  $i < j < k$ , if  $j - i \geq 2$  and  $k - j \geq 2$ , the projections of  $A_i$  and  $A_k$  to  $\mathcal{F}(A_j)$  are defined and have distance at least  $K - 2D$ .

*Proof.* Recall for any  $r$ ,  $\phi_r = \rho\phi^r$ , where  $\phi$  restricts to a fully irreducible outer automorphism of  $\langle a, b, c \rangle$ . In particular,  $\phi$  acts as a loxodromic isometry of the free factor complex  $\mathcal{F}(\langle a, b, c \rangle)$ . Thus, for any  $K$ , there exists  $r = r(K)$  such that for all  $s \geq r$ , the distance between  $\phi^s(\langle b, c \rangle)$  is at least  $K + 2D$  away from  $\langle a, b \rangle$  in  $\mathcal{F}(\langle a, b, c \rangle)$ .

Now consider any sequence  $(r_i)_i$  with  $r_i \geq r$  for all  $i$ . By [Lemma 5.1](#) and [Theorem 5.4](#), the projections of  $A_{j-2}$  and  $A_{j+2}$  to  $\mathcal{F}(A_j)$  are defined. Moreover, by [Equation \(2\)](#), we see that, by applying an automorphism, the distance between projections of  $A_{j-2}$  and  $A_{j+2}$  in  $\mathcal{F}(A_j)$  is the same as the distance between the projections of  $A_0 = \langle d, e, f \rangle$  and  $\phi_{r_{j-1}}(A_3) = \langle \phi_{r_{j-1}}(b), \phi_{r_{j-1}}(c), a \rangle$  to  $\mathcal{F}(A_2) = \mathcal{F}(\langle e, f, g \rangle)$ . Note that the rotation  $\rho$  sends the free factor  $\langle a, b, c \rangle$  to  $A_2$ , thus inducing an isometry from  $\mathcal{F}(\langle a, b, c \rangle)$  to  $\mathcal{F}(A_2)$ . The projection of  $A_0$  to  $\mathcal{F}(A_2)$  is  $D$ -close to the factor  $\langle e, f \rangle = \rho(\langle a, b \rangle)$ , and the projection of  $\phi_{r_{j-1}}(A_3)$  to  $\mathcal{F}(A_2)$  is  $D$ -close to the factor  $\rho\phi^{r_{j-1}}(\langle b, c \rangle)$ . Thus, the distance in  $\mathcal{F}(A_2)$  of the two projections is at least  $K$ . This shows the first statement of the Lemma.

Now fix  $K > 3D$  and let  $(r_i)_i$  be any sequence with  $r_i \geq r(K)$  for all  $i$ . We will prove the second statement by inducting on  $l = k - i$  with the previous statement giving the base case  $l = 4$ . Suppose we are given  $A_i, A_j, A_k$  with  $l = k - i > 4$ ,  $j - i, k - j \geq 2$ . We first claim that projections of  $A_{j+2}, A_{j+3}, \dots, A_k$  to  $\mathcal{F}(A_j)$  are defined, i.e. none of them are equal to or disjoint from  $A_j$ . For suppose  $A_s$  is the first on the list that is equal to or disjoint from  $A_j$ . By [Lemma 5.1](#) we have  $4 \leq s - j < k - i$ . By induction, the projections of both  $A_j$  and  $A_s$  to  $\mathcal{F}(A_{j+2})$  are defined and the distance between their projections is  $\geq K - 2D > D$ . Using statement 2 of [Theorem 5.4](#), this implies that  $A_s$  and  $A_j$  cannot coincide or be disjoint, proving the claim. By the same argument, we also have that the projections of  $A_i, A_{i+1}, \dots, A_{j-2}$  to  $\mathcal{F}(A_j)$  are all defined.

By the first statement of the lemma, we have  $d_{A_j}(A_{j-2}, A_{j+2}) \geq K$ . We now claim that  $d_{A_j}(A_{j+2}, A_k) \leq D$ . If  $k = j + 3$ , then  $A_{j+2}$  and  $A_k$  are disjoint, and the claim holds by statement 2 of [Theorem 5.4](#). If  $k \geq j + 4$ , then applying induction again to  $j, j + 2$  and  $k$ , we see that  $A_j$  and  $A_k$  have well-defined projections to  $\mathcal{F}(A_{j+2})$  and  $d_{A_{j+2}}(A_j, A_k) \geq K - 2D > D$ . Now, the claim follows by the third statement of [Theorem 5.4](#). By the same argument, we also see that  $d_{A_j}(A_i, A_{j-2}) \leq D$ . We now conclude  $d_{A_j}(A_i, A_k) \geq K - 2D$  by the triangle inequality.  $\square$

We are now ready to prove the main results of this section.

**Proposition 5.7.** *There exists  $R > 0$  such for any sequence  $(r_i)_{i \geq 1}$ , if  $r_i \geq R$ , and  $r_i \equiv i \pmod{7}$ , then the sequence  $(A_i)_{i \geq 0}$  is a quasi-geodesic in  $\mathcal{FF}_7$ .*

*Proof.* Let  $D$  be the constant of [Theorem 5.4](#) and let  $D'$  be the constant of [Theorem 5.5](#). Fix  $K = 4D + D'$ . Let  $R = r(K)$  be the constant of [Lemma 5.6](#). Let  $(r_i)_{i \geq 1}$  be any sequence with  $r_i \geq R$  and  $r_i \equiv i \pmod{7}$  for all  $i$ . We will show that the sequence  $(A_i)_i$  goes to infinity with linear speed. More precisely, we will show that for any  $d > 0$ , if  $k - i \geq 110d + 4$ , then  $d_{\mathcal{FF}_7}(A_i, A_k) \geq d$ . Suppose not. Let  $\gamma$  be a geodesic between  $A_i$  and  $A_k$  of length  $< d$ .

For every  $j \in \{i + 2, \dots, k - 2\}$ , there exists a free factor in  $\gamma$  that is compatible with  $A_j$ . Indeed, if every free factor in  $\gamma$  meets  $A_j$ , then by [Theorem 5.5](#), projection of  $\gamma$  to  $A_j$  will be well-defined and has diameter bounded by  $D'$ . However, by [Lemma 5.6](#), the projections of  $A_i$  and  $A_k$  to  $\mathcal{F}(A_j)$  has distance at least  $K - 2D > D'$ .

By the pigeonhole principle, there exists a vertex  $B$  of  $\gamma$  compatible with at least 110 free factors among  $\{A_{i+2}, \dots, A_{k-2}\}$ . By [Lemma 5.3](#), it is not possible for  $B$  to be compatible with 109 consecutive  $A_j$ 's. Therefore, it must be possible to find  $i + 2 \leq i' < j' < k' \leq k - 2$  with  $j' - i' \geq 2$  and  $k' - j' \geq 2$ , such that  $B$  is compatible with  $A_{i'}$  and  $A_{k'}$ , but  $B$  meets  $A_{j'}$ . In particular,  $\pi_{A_{j'}}(B)$  is defined. By [Lemma 5.6](#),  $A_{i'}$ ,  $A_{k'}$  also have well-defined projections to  $\mathcal{F}(A_{j'})$  with  $d_{A_{j'}}(A_{i'}, A_{k'}) \geq K - 2D > 2D$ . On the other hand, since  $B$  is compatible with both  $A_{i'}$  and  $A_{k'}$ , we have  $d_{A_{j'}}(A_{i'}, B) \leq D$  and  $d_{A_{j'}}(A_{k'}, B) \leq D$  by [Theorem 5.4](#). This is a contradiction, finishing the proof that  $d_{\mathcal{FF}_7}(A_i, A_k) \geq d$  for all  $k - i \geq 110d + 4$ .  $\square$



Recall that  $\mathcal{FF}_n$  is Gromov hyperbolic and that its Gromov boundary is the space of equivalence class of arational trees. Also recall we say a folding sequence  $(G_i)_i$  converges to an arational tree  $T$ , if  $\pi(U_i)$  converges to  $[T] \in \partial\mathcal{FF}_n$ , where  $U_i$  is the open simplex in  $\mathcal{CV}_n$  associated to  $G_i$ . We have the following corollary.

**Corollary 5.8.** *Given any strictly increasing sequence  $(r_i)_{i \geq 1}$  satisfying  $r_i \equiv i \pmod{7}$ , the folding sequence  $(\tau'_i)_i$  converges to an arational tree  $T$ .*

### 5.3 Non-geometric tree

We will now show that the arational tree obtained in the previous section as the limit of the free factors  $(A_i)_i$  is non-geometric. This section will use the terminology of band complexes and resolutions, for the details see [BF95].

*Definition 5.9* (Geometric tree). [BF94][LP97] Let  $X$  be a band complex and  $T$  a  $G = \pi_1(X)$ -tree. A resolution  $f : \tilde{X} \rightarrow T$  is *exact* if for every  $G$ -tree  $T'$  and equivariant factorization

$$\tilde{X} \xrightarrow{f'} T' \xrightarrow{h} T$$

of  $f$  with  $f'$  a surjective resolution it follows that  $h$  is an isometry onto its image. We say  $T$  is *geometric* if every resolution is exact.

The proof of the following proposition is based on [BF94, Proposition 3.6].

**Proposition 5.10.** *For any strictly increasing sequence  $(r_i)_{i \geq 1}$ , if the corresponding folding sequence  $(\tau'_i)_i$  converges to an arational tree  $T$ , then  $T$  is not geometric.*

*Proof.* Let  $\tilde{\psi}_i : \tilde{\tau}'_{i-1} \rightarrow \tilde{\tau}'_i$  be a lift of the train track map to the universal covers fixing a base vertex. Pick a length measure on  $(\tau'_i)_i$  so we get a folding sequence  $\tilde{\tau}'_0 \xrightarrow{\tilde{\psi}_1} \tilde{\tau}'_1 \xrightarrow{\tilde{\psi}_2} \dots$  in  $\text{cv}_7$  that converges to  $T$ . Recall that there are morphisms  $h_i : \tilde{\tau}'_i \rightarrow T$  such that  $h_i = h_{i+1} \tilde{\psi}_{i+1}$ . Since  $T$  is arational,  $h_i$ 's are not isometries though they restrict to isometries on edges. Let  $\tilde{X}$  be a finite band complex with resolution  $f : \tilde{X} \rightarrow T$ . We will show that the resolution factors through  $\tilde{\tau}'_i$  for sufficiently large  $i$ . This will imply  $T$  is not geometric.

Let  $\tilde{\Gamma}$  be the underlying real graph of  $\tilde{X}$  (disjoint union of metric arcs) with preimage  $\tilde{\Gamma}$  in  $\tilde{X}$ . We may assume  $f$  embeds the components of  $\tilde{\Gamma}$ . A vertex  $v$  of  $\tilde{X}$  is either a vertex of  $\tilde{\Gamma}$  or a corner of a band or a 0-cell of  $\tilde{X}$ . For every such vertex  $v$  choose a point  $f_0(v) \in \tilde{\tau}'_0$  so that  $f_0$  is equivariant and  $f = h_0 f_0$  on the vertices of  $\tilde{X}$ .

An edge in  $\tilde{X}$  is either a subarc of  $\tilde{\Gamma}$  or a vertical boundary component of a band or a 1-cell in  $\tilde{X}$ . Up to the action of  $\mathbb{F}_7$ , there are only finitely many edges. Using Lemma 4.8, we can find  $i > 0$  such that for every edge  $e$  in  $\tilde{X}$ , the edge path in  $\tilde{\tau}'_i$  joining the two vertices of  $\tilde{\psi}_i \dots \tilde{\psi}_1 f_0(\partial e)$  is legal. Now extend  $\tilde{\psi}_i \dots \tilde{\psi}_1 f_0$  to an equivariant map  $f_i : \tilde{X} \rightarrow \tilde{\tau}'_i$  that sends edges to legal paths (or points) and is constant on the leaves. Thus  $f_i$  is a resolution of  $\tilde{\tau}'_i$ .

$$\begin{array}{c}
 \tilde{X} \xrightarrow{f_0} \tilde{\tau}'_0 \xrightarrow{\psi_1} \tilde{\tau}'_1 \xrightarrow{\psi_2} \dots \xrightarrow{\psi_i} \tilde{\tau}'_i \xrightarrow{\psi_{i+1}} \dots \xrightarrow{h_i} T \\
 \begin{array}{l}
 \text{---} \xrightarrow{f} \text{---} \\
 \text{---} \xrightarrow{f_i} \text{---} \\
 \text{---} \xrightarrow{h_i} \text{---}
 \end{array}
 \end{array}$$

This yields a factorization

$$\tilde{X} \xrightarrow{f_i} \tilde{\tau}'_i \xrightarrow{h_i} T$$

but  $h_i$  is not an isometry. This shows  $T$  is non-geometric.  $\square$

## 6 Non-uniquely ergodic unfolding sequence

The goal of this section is to show that if a sequence  $(r_i)_{i \geq 1}$  grows sufficiently fast, then the set of currents supported on the legal lamination  $\Lambda$  of the unfolding sequence  $(\tau_i)_{i \geq 0}$  is a 1-simplex in  $\mathbb{P}\text{Curr}_7$ .

Recall that  $M_r$  is a  $7 \times 7$  matrix of the block form

$$\begin{pmatrix} 0 & I \\ B^r & 0 \end{pmatrix}$$

where  $I$  is the  $4 \times 4$  identity matrix, and  $B$  is the transition matrix of  $\theta$ ; all that matters is that some positive power of  $B$  has all entries positive. Let  $\lambda_B$  be the Perron-Frobenius eigenvalue of  $B$ . Recall the constant  $\kappa_B > 0$  from [Lemma 4.5](#). Given a sequence  $(r_i)_i$ , define for each  $i \geq 1$

$$P_i = \frac{1}{\kappa_B \lambda_B^{r_{i+1}}} M_{r_i} M_{r_{i+1}}.$$

Let  $\{e_k : k = 1, \dots, 7\}$  be the standard basis for  $\mathbb{R}^7$ . Denote by  $\mathbb{P}\mathbb{R}_{\geq 0}^7$  the projectivization of  $\mathbb{R}_{\geq 0}^7$ , and the projective class of a vector  $v$  by  $[v]$ . Fix a metric  $d$  on  $\mathbb{P}\mathbb{R}_{\geq 0}^7$ . We view  $M_r$  as a projective transformation  $\mathbb{P}\mathbb{R}_{\geq 0}^7 \rightarrow \mathbb{P}\mathbb{R}_{\geq 0}^7$ . For a sequence  $(r_i)_{i \geq 1}$  and for  $i < j$  denote by  $S_{i,j} \subset \mathbb{P}\mathbb{R}_{\geq 0}^7$  the image of the composition

$$M_{ij} := M_{r_i} M_{r_{i+1}} \cdots M_{r_j}$$

and by  $S_i = \bigcap_{j > i} S_{i,j}$ . We denote by  $v_B$  a positive Perron-Frobenius eigenvector of  $B$ , and by  $v_B^{234}$  (resp.  $v_B^{567}$ ) the vector in  $\mathbb{R}^7$  which is  $v_B$  in coordinates 2, 3, 4 (resp. 5, 6, 7) and 0 in all other coordinates. The main result of this section is the following.

**Proposition 6.1.** *Let  $(r_i)_{i \geq 1}$  be a sequence of positive integers with  $r_{i+1} - r_i \geq i$ . Then for all  $i$  the set  $S_i$  is a 1-simplex, i.e. it is the convex hull of two distinct points  $p_i, q_i \in \mathbb{P}\mathbb{R}_{\geq 0}^7$ . Moreover, as  $i \rightarrow \infty$ ,  $\{p_i, q_i\}$  converges (as a set) to  $\{[v_B^{234}], [v_B^{567}]\}$ .*

Before we give a technical proof of [Proposition 6.1](#), we will give a simpler, more intuitive proof where the sequence  $r_1 < r_2 < \dots$  is chosen inductively so that  $r_1$  is sufficiently large and each  $r_i$  is sufficiently large depending on  $r_1, r_2, \dots, r_{i-1}$ . Later, we do a more careful analysis where we can control the growth of the sequence.

*Proof idea of Proposition 6.1.* For  $\epsilon > 0$  we will write  $x \stackrel{\epsilon}{\approx} y$  if  $d(x, y) < \epsilon$  in  $\mathbb{P}\mathbb{R}_{\geq 0}^7$ . Each  $S_{i,j}$  is the convex hull of the  $M_{i,j}$ -images of the vectors  $e_i, i = 1, \dots, 7$ . The proof consists of computing these images using the Perron-Frobenius dynamics. We first observe that there is a sequence  $\epsilon_r \rightarrow 0$  such that:

- $M_r(e_7) = e_4, M_r(e_6) = e_3, M_r(e_5) = e_2, M_r(e_4) = e_1,$
- $M_r(e_i) \stackrel{\epsilon_r}{\approx} v_B^{567}, i = 1, 2, 3,$
- $M_r(v_B^{567}) = v_B^{234}, M_r(v_B^{234}) \stackrel{\epsilon_r}{\approx} v_B^{567}.$

Next, we consider the composition  $M_s M_r$  for  $r \gg s$ . The third bullet uses uniform continuity of  $M_s$  and the assumption that  $r$  is sufficiently large compared to  $s$ .

- $M_s M_r(e_7) = e_1,$
- $M_s M_r(e_i) \stackrel{\epsilon_s}{\approx} v_B^{567}, i = 4, 5, 6,$
- $M_s M_r(e_i) \stackrel{\epsilon_s}{\approx} v_B^{234}, i = 1, 2, 3.$

Finally, for  $r \gg s \gg t$  we see similarly:

- $M_t M_s M_r(e_7) \stackrel{\epsilon_t}{\approx} v_B^{567},$
- $M_t M_s M_r(e_i) \stackrel{\epsilon_t}{\approx} v_B^{234}, i = 4, 5, 6,$

- $M_t M_s M_r(e_i) \stackrel{\epsilon t}{\cong} v_B^{567}$ ,  $i = 1, 2, 3$ .

It follows that if we make suitably large choices for the  $r_i$ 's, the set  $S_{i,i+3}$  will be contained in the  $\epsilon_{r_i}$ -neighborhood of the 1-simplex  $[v_B^{567}, v_B^{234}]$ . Moreover, given any  $\epsilon > 0$  and  $j > i + 3$  we can choose  $r_j$  large (depending on uniform continuity constants of  $M_{ij}$ ) to ensure that  $S_{i,j+3} = M_{ij}(S_{j,j+3})$  is contained in the  $\epsilon$ -neighborhood of the 1-simplex with endpoints  $M_{ij}(v_B^{567})$  and  $M_{ij}(v_B^{234})$ . Thus each  $S_i$  is the nested intersection of simplices of dimension  $\leq 6$  such that for all  $\epsilon > 0$  they are eventually all contained in the  $\epsilon$ -neighborhood of a 1-simplex with definite distance between the endpoints. This proves the Proposition.  $\square$

We now present a more detailed proof of [Proposition 6.1](#).

For a sequence of integers  $(r_i)_{i \geq 1}$  such that  $r_i, r_{i+1} - r_i \rightarrow \infty$ , by [Lemma 4.5](#)  $(P_i)_i$  converges to an idempotent matrix  $Y$ . Let  $\Delta_i = Y - P_i$  and let  $\|Y\|$  be the operator norm.

**Lemma 6.2.** *Let  $(r_i)_{i \geq 1}$  be a sequence of positive integers such that  $r_{i+1} - r_i \geq i$ . Then there exists an  $I \geq 1$ , such that for all  $i \geq I$ ,  $\|\Delta_i\| \leq 1/(2 \cdot 2^i)$ .*

*Proof.* Let  $\lambda_B, \mu_B, \mu'_B$  be the modulus of the three eigenvalues of  $B$ ; we have  $\lambda_B \sim 1.46$  and  $\mu_B = \mu'_B \sim 0.826$ . Then

$$\|\Delta_i\| = \|P_i - Y\| \leq \max \left( \frac{\mu^{r_{i+1}}}{\lambda^{r_{i+1}}}, \frac{\lambda^{r_i}}{\lambda^{r_{i+1}}} \right) \leq \frac{\lambda^{r_i}}{\lambda^{r_{i+1}}}$$

where the two terms comes from the two blocks in  $P_i$ . For the last inequality, note that  $\mu < 1 < \lambda$  and  $r_i$  are positive integers. Therefore,  $\mu^{r_{i+1}} < 1 < \lambda^{r_i}$ .

Now we claim that there exists an  $I \geq 1$ , such that for all  $i \geq I$ ,

$$\frac{\lambda^{r_i}}{\lambda^{r_{i+1}}} \leq \frac{1}{2^{i+1}} \quad \text{equivalently,} \quad 2 \leq \lambda^{\frac{r_{i+1} - r_i}{i+1}}$$

We only need to show that the sequence  $\frac{r_{i+1} - r_i}{i+1}$  is eventually increasing. Indeed, by assumption,  $r_{i+1} - r_i \geq i$ , so

$$\begin{aligned} i &\leq r_{i+1} - r_i \\ \frac{i}{i+1} &\leq \frac{r_{i+1} - r_i}{i+1} \end{aligned}$$

Since  $i/(i+1)$  is an increasing sequence, it follows that our sequence is also increasing.  $\square$

The following lemma is a consequence of [Lemma 4.5](#) and [Lemma 11.1](#).

**Lemma 6.3.** *Let  $(r_i)_{i \geq 1}$  be a sequence of positive integers such that  $r_{i+1} - r_i \geq i$ ,  $Y$  be the idempotent matrix of [Lemma 4.5](#) and  $M_\infty = \lim_{r \rightarrow \infty} M_r / \lambda_B^r$ . Then the following statements hold.*

- (1) *For all  $i \geq 1$ , the sequence of matrices  $\{P_i P_{i+2} \cdots P_{i+2k}\}_{k=1}^\infty$  converges to a matrix  $Y_i$ . Furthermore, for all sufficiently large  $i$ ,*

$$\|Y_i - Y\| \leq \frac{2}{2^i} (\|Y\| + \|Y\|^2)$$

- (2) *The kernel of  $Y$  is a subspace of the kernel of  $Y_i$  for all  $i \geq 1$ .*
- (3) *For all  $i \geq 1$ ,  $Y_i(e_1) \neq 0$  with non-negative entries and  $Y_i(e_2)$  and  $Y_i(e_3)$  are positive multiples of  $Y_i(e_1)$ .*
- (4) *For all  $i \geq 1$ ,  $M_{r_i} Y_{i+1}(e_1) \neq 0$  with non-negative entries, and  $M_{r_i} Y_{i+1}(e_1)$  and  $Y_i(e_1)$  are not scalar multiples of each other.*
- (5) *Projectively,  $[Y_i(e_1)] \rightarrow [Y(e_1)]$  and  $[M_{r_i} Y_{i+1}(e_1)] \rightarrow [M_\infty Y(e_1)]$  as  $i \rightarrow \infty$ .*

*Proof.* For (1), it suffices to show convergence for all  $i$  greater than some  $I$ . Indeed, if such  $I$  exists and  $i < I$ , then let  $i_0 \geq I$  be such that  $i = i_0 \pmod{2}$  and observe that

$$\{P_i P_{i+2} \cdots P_{i+2k}\}_{k=\frac{i_0-i}{2}}^\infty = P_i P_{i+2} \cdots P_{i_0-2} \{P_{i_0} P_{i_0+2} \cdots P_{i_0+2k}\}_{k=0}^\infty.$$

By assumption  $\{P_{i_0} P_{i_0+2} \cdots P_{i_0+2k}\}_{k=0}^\infty$  converges. Since matrix multiplication is continuous, the sequence  $\{P_i P_{i+2} \cdots P_{i+2k}\}_{k=0}^\infty$  also converges.

For each  $i$ , let

$$\Delta_i = P_i - Y.$$

By [Lemma 6.2](#), there exists  $I \geq 1$  such that for all  $i \geq I$ ,  $\|\Delta_i\| \leq \frac{1}{2 \cdot 2^i}$ . Also, choose  $I$  sufficiently large so that  $\frac{1}{2^I} \|Y\| \leq 1/2$ . Then, by [Lemma 11.1](#), for all  $i \geq I$ , the sequence  $\{P_i P_{i+2} \cdots P_{i+2k}\}_{k=0}^\infty$  converges to some matrix  $Y_i$ , with

$$\|Y_i - Y\| \leq \frac{2}{2^i} \left( \|Y\| + \|Y\|^2 \right). \quad (3)$$

For (2), it again suffices to show the statement is true for all sufficiently large  $i$ , and the statement holds for all  $i \geq I$  by [Lemma 11.1](#).

For (3), first note that since all the matrices involved are non-negative, the resulting vectors are all also non-negative. So we only need to show that they are not the zero vector. It suffices to check that  $Y_i(e_1) \neq 0$  for all sufficiently large  $i$ , since each  $P_i$  is non-negative and has full rank. For large  $i$ , the statement follows because  $Y(e_1)$  is not equal to 0 and  $\|Y_i(e_1) - Y(e_1)\| \leq \|Y_i - Y\|$  can be made arbitrarily small. For the second statement, we know that  $Y(e_2)$  and  $Y(e_3)$  are positive multiples of  $Y(e_1)$ , so there are  $s, t > 0$  such that  $se_2 - e_1$  and  $te_3 - e_1$  are in the kernel of  $Y$ . Then  $Y_i(se_2 - e_1) = Y_i(te_3 - e_1) = 0$  for all  $i$  by (2).

For (4),  $M_{r_i} Y_{i+1}(e_1) \neq 0$  with non-negative entries since  $Y_{i+1}(e_1)$  is so by (3). To see that  $M_{r_i} Y_{i+1}(e_1)$  and  $Y_i(e_1)$  are projectively distinct, it is enough to do this for all sufficiently large  $i$ . Let  $M_\infty = \lim_{r \rightarrow \infty} M_r / \lambda_B^r$ . By [Lemma 4.5](#) and [Lemma 4.7](#),  $Y(e_1)$  and  $M_\infty Y(e_1)$  are orthogonal. Since  $r_i \rightarrow \infty$ , we can make  $\frac{M_{r_i}}{\lambda_B^{r_i}} Y_{i+1}(e_1)$  arbitrarily close to  $M_\infty Y(e_1)$ , and  $Y_i(e_1)$  close to  $Y(e_1)$ . This means  $M_{r_i} Y_{i+1}(e_1)$  and  $Y_i(e_1)$  are near orthogonal, so they can't be scalar multiples of each other.

Statement (5) is clear. □

*Proof of Proposition 6.1.* By [Lemma 4.5](#) and [Lemma 4.7](#),  $[v_B^{234}] = [Y(e_1)]$  and  $[v_B^{567}] = [M_\infty Y(e_1)]$ . Using notation from [Lemma 6.3](#), set

$$p_i = [Y_i(e_1)] \quad \text{and} \quad q_i = [M_{r_i} Y_{i+1}(e_1)].$$

By [Lemma 6.3](#) (3) - (5),

- $p_i$  and  $q_i$  are well defined and distinct.
- $p_i = [Y_i(e_k)]$ , and  $q_i = [M_{r_i} Y_{i+1}(e_k)]$ , for  $k = 1, 2, 3$ .
- $p_i \rightarrow [v_B^{234}]$  and  $q_i \rightarrow [v_B^{567}]$ .
- $[M_{r_i}(p_{i+1})] = q_i$  and  $[M_{r_i}(q_{i+1})] = p_i$ .

Our goal is to show  $S_i$  is the 1-simplex spanned by  $p_i$  and  $q_i$ . To do this, we consider  $S_{ij}$ , which is the convex hull of the  $M_{ij}$ -images of the vectors  $e_k$ ,  $k = 1, \dots, 7$ . That is, we have to show that  $[M_{ij}(e_k)]$  is close to either  $p_i$  or  $q_i$  for each  $k$ . We first observe that for all  $r, s > 0$ :

- $M_r(e_4) = e_1$ ,  $M_r(e_5) = e_2$ ,  $M_r(e_6) = e_3$ ,  $M_r(e_7) = e_4$ ,
- $M_r M_s(e_7) = e_1$ .

We may assume that  $j - 1 = i + 2m$ , so  $M_{ij}$  breaks up into pairs, i.e. for all  $k$ ,

$$[M_{ij}(e_k)] = [P_i \cdots P_{j-1}(e_k)].$$

Let  $\epsilon > 0$  be arbitrary. Choose  $\delta > 0$  such that for any vector  $u \in \mathbb{R}_+^7$  and any  $v \in \left\{ Y_i(e_k), \frac{M_{r_i}}{\lambda_B^{r_i}} Y_{i+1}(e_k) : k = 1, 2, 3 \right\}$ , if  $\|u - v\| \leq \delta$ , then  $d([u], [v]) \leq \epsilon$ . Now by [Lemma 6.3](#), we can choose  $J$  sufficiently large so that whenever  $i + 2m \geq J$ , then

- $\|P_i \cdots P_{i+2m} - Y_i\| \leq \delta$
- $\|P_{i+1} \cdots P_{i+2m+1} - Y_{i+1}\| \leq \frac{\delta}{\|M_{r_i}/\lambda_B^{r_i}\|}$ .

Now we may assume that  $j - 3 \geq J$ . Then,

- For  $k = 1, 2, 3$ , we have

$$\|P_i \cdots P_{j-1}(e_k) - Y_i(e_k)\| \leq \delta \implies d([M_{ij}(e_k)], p_i) \leq \epsilon.$$

- For  $k = 7$ , we have  $M_{ij}(e_7) = M_{i,j-2}(e_1)$ , so  $[M_{ij}(e_7)] = [P_i \cdots P_{j-3}(e_1)]$  is  $\epsilon$ -close to  $[p_i]$  by the same reasoning as the previous bullet point.
- For  $k = 4, 5, 6$ ,  $M_{ij}(e_k) = M_{i,j-1}(e_{k-3})$ . In this case, we consider  $\frac{M_{r_i}}{\lambda^{r_i}} P_{i+1} \cdots P_{j-2}(e_{k-3})$  and approximate it by  $\frac{M_{r_i}}{\lambda^{r_i}} Y_{i+1}(e_{k-3})$ , as follows:

$$\begin{aligned} & \left\| \frac{M_{r_i}}{\lambda^{r_i}} P_{i+1} \cdots P_{j-2}(e_{k-3}) - \frac{M_{r_i}}{\lambda^{r_i}} Y_{i+1}(e_{k-3}) \right\| \\ & \leq \left\| \frac{M_{r_i}}{\lambda^{r_i}} \right\| \|P_{i+1} \cdots P_{j-2} - Y_{i+1}\| \\ & \leq \delta. \end{aligned}$$

Thus, for  $k = 4, 5, 6$ ,  $d([M_{ij}(e_k)], q_i) \leq \epsilon$ .

We have shown that for any  $\epsilon$ , the vertices of the simplex  $S_{i,j}$  come  $\epsilon$ -close to  $p_i$  and  $q_i$  for all sufficiently large  $j$ . Since  $S_{i,j+1} \subset S_{i,j}$  and  $S_i = \bigcap_{j>i} S_{i,j}$ , it follows that  $S_i$  must be the 1-simplex spanned by  $p_i$  and  $q_i$ . This proves the Proposition.  $\square$

Recall the unfolding sequence  $(\tau_i)_{i \geq 0}$  where  $M_{r_i}$  is the transition matrix of the train track map  $\phi_{r_i} : \tau_i \rightarrow \tau_{i-1}$ . Let  $\Lambda$  be the legal lamination of  $(\tau_i)_{i \geq 0}$ .

**Corollary 6.4.** *If  $(r_i)_{i \geq 1}$  is a positive sequence with  $r_{i+1} - r_i \geq i$ , then  $\mathbb{PC}(\Lambda)$  is a 1-simplex.*

*Proof.* In light of [Theorem 3.3](#), it is enough to show  $\mathbb{PC}((\tau_i)_i)$  is a 1-simplex. For each  $i \geq 0$ , we have a well-defined projection

$$p_i: \mathbb{PC}((\tau_i)_i) \rightarrow \mathbb{PR}_+^7 \quad \text{given by} \quad p_i([\mu_i]) = [\mu_i].$$

The image of the projection is  $S_{i+1}$ , which is always a 1-simplex by [Proposition 6.1](#). Therefore,  $\mathbb{PC}((\tau_i)_i)$  is a 1-simplex.  $\square$

## 7 Non-uniquely ergometric tree

The goal of this section is to show that if a sequence  $(r_i)_{i \geq 1}$  grows sufficiently fast, then the set of projectivized length measures  $\mathbb{PD}((\tau'_i)_i)$  on the folding sequence  $(\tau'_i)_i$  is a 1-simplex. By [Proposition 3.1](#), if  $(\tau'_i)_i$  converges to an arational tree  $T$ , then  $\mathbb{PD}(T)$  is also a 1-simplex in  $\partial\text{CV}_7$ .

Recall that  $N_r$  is a  $7 \times 7$  matrix of the block form

$$\begin{pmatrix} 0 & C^r \\ I & 0 \end{pmatrix}$$

where  $I$  is the  $4 \times 4$  identity matrix, and  $C$  is the transition matrix of  $\vartheta$ . The transpose of  $N_r$  has the same shape as  $M_r$ . Therefore, the same theory from [Section 6](#) holds true. For brevity, we record only the essential statements that will be used later and omit all proofs from this section.

Let  $\lambda_C$  be the Perron-Frobenius eigenvalue of  $C$ . Let  $\kappa_C$  be the constants of [Lemma 4.6](#). Given a sequence  $(r_i)_i$ , define for each  $i \geq 1$

$$Q_i = \frac{1}{\kappa_C \lambda_C^{r_{i+1}}} N_{r_{i+1}} N_{r_i}.$$

**Lemma 7.1.** *Given a sequence  $(r_i)_{i \geq 1}$  of positive integers such that  $r_{i+1} - r_i \geq i$ . Then for all  $i \geq 1$ , the sequence of matrices  $\{Q_{i+2k} \cdots Q_{i+2} Q_i\}_{k=0}^{\infty}$  converges to a matrix  $Z_i$ . Furthermore,  $\lim_{i \rightarrow \infty} Z_i = Z$ , where  $Z$  is the idempotent matrix of [Lemma 4.6](#).*

**Corollary 7.2.** *If  $(r_i)_{i \geq 1}$  is a positive sequence with  $r_{i+1} - r_i \geq i$ . Then  $\mathbb{P}\mathcal{D}((\tau'_i)_i)$ , and hence  $\mathbb{P}\mathcal{D}(T)$ , is a 1-simplex.*

## 8 Non-uniquely ergodic tree

In this section we relate the legal lamination  $\Lambda$  associated to the unfolding sequence  $(\tau_i)_i$  defined in [Section 6](#) and the limiting tree  $T$  of the folding sequence  $(\tau'_i)_i$  defined in [Section 5](#), to show that  $T$  is not uniquely ergodic.

Recall the automorphism  $\Phi_i = \phi_{r_1} \circ \cdots \circ \phi_{r_i}$ , with  $\Phi_0 = \text{id}$ . We also use  $\Phi_i$  to denote the induced graph map from  $\tau_i$  to  $\tau_0$ . If each  $\tau_i$  and  $\tau'_i$  as a marked graph is the rose labeled by  $\{a_i, b_i, c_i, d_i, e_i, f_i, g_i\}$ , then  $x_i$  is represented by  $\Phi_i(x)$  for  $x \in \{a, b, c, d, e, f, g\}$  as a word in  $\mathbb{F}_7 = \langle a, b, c, d, e, f, g \rangle = \pi_1(\tau_0) = \pi_1(\tau'_0)$ . We denote  $x_0$  as above simply by  $x$ .

**Lemma 8.1.** *If  $(r_i)_{i \geq 1}$  is positive, then for any length measure  $(\lambda_i)_i \in \mathcal{D}((\tau'_i)_i)$ , the  $\lambda_i$ -volume of  $\tau'_i$  goes to 0 as  $i \rightarrow \infty$ .*

*Proof.* The composition  $\psi_{r_i} \psi_{r_{i-1}} \psi_{r_{i-2}} : \tau'_{i-3} \rightarrow \tau'_i$  has the property that the preimage of every point of  $\tau'_i$  consists of at least two (in fact, many more) points of  $\tau'_{i-3}$  and so the  $\lambda_i$ -volume of  $\tau'_i$  is at most half of the  $\lambda_{i-3}$ -volume of  $\tau'_{i-3}$ .  $\square$

**Lemma 8.2.** *Suppose  $(r_i)_{i \geq 1}$  is positive. Let  $\Lambda$  be the legal lamination of the unfolding sequence  $(\tau_i)_i$ . Then every leaf in  $\Lambda$  is obtained as a limit of a sequence  $\{\Phi_i(w)\}_i$ , where  $w$  is a legal word in  $\tau_0$  of length at most two in  $\{a, b, c, d, e, f, g\}$  and their inverses. Moreover,  $w$  can be closed up to a legal loop which is a cyclic word of length  $\leq 3$ .*

*Proof.* Let  $l$  be a leaf of  $\Lambda$  realized as a bi-infinite line in  $\tau_0$  and let  $s$  be any subsegment of  $l$ , with combinatorial edge length  $\ell_s > 0$  in  $\tau_0$ . By definition, for every  $i$  there is a bi-infinite legal path  $l_i$  in  $\tau_i$  such that  $l = \Phi_i(l_i)$ . Let  $i = i(s) \geq 0$  such that the edge length of  $x_i$  in  $\tau_0$  under the graph map  $\Phi_i$  is  $\geq \ell_s$  for all  $x \in \{a, b, c, d, e, f, g\}$ . Thus, there is a segment  $s_i$  of  $l_i$  of combinatorial length at most two in  $\{a_i, b_i, c_i, d_i, e_i, f_i, g_i\}$ , such that  $s \subset \Phi_i(s_i)$  (here  $\Phi_i$  is a graph map). Now if  $s_i = x_i y_i$  for  $x, y \in \{a, b, c, d, e, f, g\}$ , take  $w = xy$ . Thus we see that  $\Phi_i(w)$  (here  $\Phi_i$  is an automorphism) covers  $s$  in  $\tau_0$ . Since this is true for any segment of  $l$ , we conclude the lemma by taking a nested sequence of subsegments of  $l$  with edge length in  $\tau_0$  going to infinity. The fact that legal paths of length  $\leq 2$  can be closed up to legal loops of length  $\leq 3$  follows from the description of the train track in [Lemma 4.3](#).  $\square$

Recall that if  $(\tau'_i)_i$  converges to an arational tree  $T$ , then we can identify  $\mathcal{D}((\tau'_i)_i)$  with  $\mathcal{D}(T)$  by [Proposition 3.1](#).

**Lemma 8.3.** *Suppose  $(r_i)_{i \geq 1}$  is positive and that the folding sequence  $(\tau'_i)_i$  converges to an arational tree  $T$ . Let  $w$  be any conjugacy class in  $\mathbb{F}_7$  represented by a cyclic word in  $\{a, b, c, d, e, f, g\}$  and their inverses and let  $\lambda \in \mathcal{D}(T)$  correspond to a length measure  $(\lambda_i)_i \in \mathcal{D}((\tau'_i)_i)$ . Then*

$$\lim_{i \rightarrow \infty} \|\Phi_i(w)\|_{(T, \lambda)} = 0.$$

*Proof.* Under the isomorphism from  $\mathcal{D}((\tau'_i)_i) \rightarrow \mathcal{D}(T)$  that maps  $(\lambda_i)_i \mapsto \lambda$ , the sequence  $(\tau'_i, \lambda_i) \subset \text{cv}_7$  also converges to  $(T, \lambda) \in \partial \text{cv}_7$ . Thus, for any  $x \in \mathbb{F}_7$ ,

$$\|x\|_{(T, \lambda)} = \lim_{i \rightarrow \infty} \|x\|_{(\tau'_i, \lambda_i)}.$$

In fact, the sequence  $\|x\|_{(\tau'_i, \lambda_i)}$  is monotonically non-increasing. Recall that  $\tau'_0$  as a marked graph is the rose labeled by  $\{a, b, c, d, e, f, g\}$ . Represent  $w$  by a loop  $c_w$  in  $\tau'_0$ . The graph  $\tau'_i$  is the

rose labeled by  $\{\Phi_i(a), \dots, \Phi_i(g)\}$ . Thus, the loop  $c_w$  in  $\tau'_i$  represents the conjugacy class  $\Phi_i(w)$ . This shows

$$\|\Phi_i(w)\|_{(T, \lambda)} \leq \|\Phi_i(w)\|_{(\tau'_i, \lambda_i)} \leq \|w\|_{\text{word}} \text{vol}(\tau'_i, \lambda_i)$$

where  $\|w\|_{\text{word}}$  is the word length of  $w$ . By [Lemma 8.1](#) the last term goes to 0.  $\square$

We now come to the main statement of this section.

**Proposition 8.4.** *Suppose  $(r_i)_{i \geq 1}$  is positive and that the folding sequence  $(\tau'_i)_i$  converges to an arational tree  $T$ . Let  $\tilde{\Lambda}$  be the lamination corresponding to the legal lamination  $\Lambda$  of the unfolding sequence  $(\tau_i)_i$  and let  $L(T)$  be the lamination dual to  $T$ . Then  $\tilde{\Lambda} \subseteq L(T)$ . In particular, if  $T$  is non-geometric, then  $\mathcal{C}(\Lambda) = \mathcal{C}(T)$ .*

*Proof.* Recall the lamination dual to an arational tree is independent of the length measure on the tree ([Theorem 2.2](#)). So fix an arbitrary length measure  $\lambda \in \mathcal{D}(T)$  on  $T$ .

Let  $W_3$  be the set of legal loops of length at most three in  $\{a, b, c, d, e, f, g\}$  and their inverses. By [Lemma 8.3](#), for every  $\epsilon > 0$ , there exists  $I_\epsilon > 0$  such that for all  $i \geq I_\epsilon$ ,  $\|\Phi_i(w)_{(T, \lambda)}\| < \epsilon$ , for every  $w \in W_3$ . Then the bi-infinite line  $(\Phi_i(w)^{-\infty}, \Phi_i(w)^\infty)$  is in  $L_\epsilon(T)$  for all  $i \geq I_\epsilon$ . Therefore,

$$\bigcap_{\epsilon > 0} \overline{\bigcup_{\substack{w \in W_3 \\ i \geq I_\epsilon}} (\Phi_i(w)^{-\infty}, \Phi_i(w)^\infty)} \subseteq \bigcap_{\epsilon > 0} L_\epsilon(T).$$

By [Lemma 8.2](#), we conclude that  $\tilde{\Lambda} \subseteq L(T)$ .

If  $T$  is non-geometric and arational, then it is freely indecomposable by [\[Rey12\]](#). By [\[CHR15, Corollary 1.4\]](#),  $\mathcal{C}(\Lambda) = \mathcal{C}(T)$ .  $\square$

The following is the consequence of [Proposition 8.4](#) and [Corollary 6.4](#).

**Corollary 8.5.** *For a positive sequence  $(r_i)_{i \geq 1}$  of integers with  $r_{i+1} - r_i \geq i$ , if the folding sequence  $(\tau'_i)_i$  converges to a non-geometric arational tree  $T$ , then  $\mathbb{P}\mathcal{C}(T)$  is a 1-simplex. In particular,  $T$  is not uniquely ergodic.*

## 9 Non-convergence of unfolding sequence

In this section, fix a sequence  $(r_i)_{i \geq 1}$  such that  $r_{i+1} - r_i \geq i$ . We will show that the corresponding unfolding sequence  $(\tau_i)_i$  does not converge to a unique point in  $\partial\text{CV}_7$ . In fact, we will show in [Corollary 9.3](#) that it converges to a 1-simplex in  $\partial\text{CV}_7$ .

Recall the folding and unfolding sequences  $(\tau'_i)_i$  and  $(\tau_i)_i$ , respectively, from [Section 4](#).

$$\cdots \longrightarrow \tau_{i+1} \xrightarrow{M_{r_{i+1}}} \tau_i \xrightarrow{M_{r_i}} \tau_{r_{i-1}} \longrightarrow \cdots \xrightarrow{M_{r_3}} \tau_2 \xrightarrow{M_{r_2}} \tau_1 \xrightarrow{M_{r_1}} \tau_0,$$

and

$$\cdots \longleftarrow \tau'_{i+1} \xleftarrow{N_{r_{i+1}}} \tau'_i \xleftarrow{N_{r_i}} \tau'_{r_{i-1}} \longleftarrow \cdots \xleftarrow{N_{r_3}} \tau'_2 \xleftarrow{N_{r_2}} \tau'_1 \xleftarrow{N_{r_1}} \tau'_0.$$

Here  $\tau_i$  and  $\tau'_i$  as marked graphs belong to the same simplex in  $\text{CV}_7$ . Also, recall the matrices defined for all  $i \geq 0$

$$P_i = \frac{1}{\kappa_B \lambda_B^{r_{i+1}}} M_{r_i} M_{r_{i+1}} \quad \text{and} \quad Q_i = \frac{1}{\kappa_C \lambda_C^{r_{i+1}}} N_{r_{i+1}} N_{r_i},$$

and the existence of the limiting matrices from [Lemma 6.3](#) and [Lemma 7.1](#)

$$Y_i = \lim_{k \rightarrow \infty} P_i P_{i+2} \cdots P_{i+2k} \quad \text{and} \quad Z_i = \lim_{k \rightarrow \infty} Q_{i+2k} \cdots Q_{i+2} Q_i.$$

For all even  $2m \geq 0$ ,

$$c_{2m} = \left( \kappa_B^m \lambda_B^{r_2} \lambda_B^{r_4} \cdots \lambda_B^{r_{2m}} \right) \left( \kappa_C^m \lambda_C^{r_2} \lambda_C^{r_4} \cdots \lambda_C^{r_{2m}} \right).$$

Similarly, for all odd  $2m+1 \geq 1$ , set

$$c_{2m+1} = \left( \kappa_B^m \lambda_B^{r_1} \lambda_B^{r_3} \cdots \lambda_B^{r_{2m+1}} \right) \left( \kappa_C^m \lambda_C^{r_1} \lambda_C^{r_3} \cdots \lambda_C^{r_{2m+1}} \right).$$

Let  $\ell = \ell_0 \in \mathbb{R}^{|E\tau_0|}$  be a positive length vector on  $\tau_0$ . Then  $\ell$  determines a length vector  $\ell_i$  on each  $\tau_i$  given by  $\ell_i = M_{r_i}^T \cdots M_{r_1}^T \ell \in \mathbb{R}^{|E\tau_i|}$ . We set  $\ell_e^T = \ell^T Y_1$  and  $\ell_o^T = \ell^T \frac{M_{r_1}}{\lambda_B} Y_2$ . Note that both  $\ell_e$  and  $\ell_o$  are positive vectors. For  $\ell_e$ , this follows since  $\ell$  is a positive vector and  $Y_1$  is a non-negative matrix. Similarly,  $\ell^T M_{r_1}$  is positive and  $Y_2$  is non-negative, so  $\ell_o$  is also positive.

We will show the sequence  $(\tau_i, \ell_i)_i \subset \text{CV}_7$ , up to rescaling, does not have a unique limit in  $\partial\text{CV}_7$ . We start by showing the even sequence and the odd sequence do converge, up to scaling. More precisely:

**Lemma 9.1.** *For any positive length vector  $\ell = \ell_0$  on  $\tau_0$ , the corresponding even sequence  $(\tau_{2m}, \frac{\ell_{2m}}{c_{2m}})$  and odd sequence  $(\tau_{2m+1}, \frac{\ell_{2m+1}}{c_{2m+1}})$  of metric graphs converge to two points  $T_e$  and  $T_o$  respectively in  $\partial\text{CV}_7$ . In fact, for any conjugacy class  $x \in \mathbb{F}_7$ , there exists an index  $i_x \geq 0$ , a vector  $v_x \in \mathbb{R}^{|E\tau'_{i_x}|}$ , and matrices  $Y_x^e$  and  $Y_x^o$ , such that*

$$\|x\|_{T_e} = \ell_e^T Y_x^e v_x \quad \text{and} \quad \|x\|_{T_o} = \ell_o^T Y_x^o v_x$$

*Proof.* Let  $x \in \mathbb{F}_7$  be a cyclically reduced representative of its conjugacy class. By Lemma 4.8, there exists  $i \geq 0$  such that  $x$  is legal in  $\tau'_i$ . Let  $i_x$  be the smallest index among such  $i$ . Then we can represent  $x$  by a vector  $v_x$  in  $\mathbb{R}^{|E\tau'_{i_x}|}$  and by the vector  $N_{r_i} \cdots N_{i_x+1} v_x$  in  $\mathbb{R}^{|E\tau_i|}$  for  $i \geq i_x$ . Thus, for all  $i \geq i_x$ , we have

$$\|x\|_{(\tau_i, \ell_i)} = \left( \ell^T M_{r_1} \cdots M_{r_i} \right) \left( N_{r_i} \cdots N_{i_x+1} v_x \right).$$

If  $i_x$  is even, then write  $i_x = 2m_x$ , and set

$$c_x^e = \kappa_C^{m_x} \lambda_C^{r_2} \lambda_C^{r_4} \cdots \lambda_C^{r_{i_x}} \quad \text{and} \quad c_x^o = \kappa_C^{m_x} \lambda_C^{r_1} \lambda_C^{r_3} \cdots \lambda_C^{r_{i_x-1}}.$$

If  $i_x$  is odd, then write  $i_x = 2m_x + 1$ , and set

$$c_x^e = \kappa_C^{m_x+1} \lambda_C^{r_2} \lambda_C^{r_4} \cdots \lambda_C^{r_{i_x-1}} \quad \text{and} \quad c_x^o = \kappa_C^{m_x} \lambda_C^{r_1} \lambda_C^{r_3} \cdots \lambda_C^{r_{i_x}}.$$

First suppose  $i_x$  is even. Then for all even  $2m \geq i_x$ , we have

$$\begin{aligned} \|x\|_{\left(\tau_{2m}, \frac{\ell_{2m}}{c_{2m}}\right)} &= \frac{\|x\|_{(\tau_{2m}, \ell_{2m})}}{c_{2m}} \\ &= \frac{\ell^T \left( P_1 P_3 \cdots P_{2m-1} \right) \left( Q_{2m-1} \cdots Q_{i_x+3} Q_{i_x+1} \right) v_x}{c_x^e} \\ &\xrightarrow{m \rightarrow \infty} \frac{\ell^T Y_1 Z_{i_x+1} v_x}{c_x^e} = \ell_e^T \left( \frac{Z_{i_x+1}}{c_x^e} \right) v_x \end{aligned}$$

and for odd  $2m+1 \geq i_x$ , we have

$$\begin{aligned} \|x\|_{\left(\tau_{2m+1}, \frac{\ell_{2m+1}}{c_{2m+1}}\right)} &= \frac{\|x\|_{(\tau_{2m+1}, \ell_{2m+1})}}{c_{2m+1}} \\ &= \frac{\ell^T \frac{M_{r_1}}{\lambda_B} \left( P_2 P_4 \cdots P_{2m} \right) \left( Q_{2m} \cdots Q_{i_x+4} Q_{i_x+2} \right) N_{i_x+1} v_x}{c_x^o \lambda_C^{r_{i_x+1}}} \\ &\xrightarrow{m \rightarrow \infty} \frac{\ell^T \frac{M_{r_1}}{\lambda_B} Y_2 Z_{i_x+2} N_{i_x+1} v_x}{c_x^o} = \ell_o^T \left( \frac{Z_{i_x+2}}{c_x^o} \frac{N_{i_x+1}}{\lambda_C^{i_x+1}} \right) v_x \end{aligned}$$



Now suppose  $i_x$  is odd. Then for all even  $2m \geq i_x$ , we have

$$\begin{aligned} \|x\|_{\left(\tau_{2m}, \frac{\ell_{2m}}{c_{2m}}\right)} &= \frac{\ell^T \left( P_1 P_3 \cdots P_{2m-1} \right) \left( Q_{2m-1} \cdots Q_{i_x+3} \right) N_{r_{i_x+1}} v_x}{c_x^e \lambda_C^{r_{i_x+1}}} \\ &\xrightarrow{m \rightarrow \infty} \ell_e^T \left( \frac{Z_{i_x+2} N_{r_{i_x+1}}}{c_x^e \lambda_C^{r_{i_x+1}}} \right) v_x. \end{aligned}$$

and for odd  $2m+1 \geq i_x$ , we have

$$\begin{aligned} \|x\|_{\left(\tau_{2m+1}, \frac{\ell_{2m+1}}{c_{2m+1}}\right)} &= \frac{\ell^T \frac{M_{r_1}}{\lambda_B^{r_1}} \left( P_2 P_4 \cdots P_{2m} \right) \left( Q_{2m} \cdots Q_{i_x+3} Q_{i_x+1} \right) v_x}{c_x^o} \\ &\xrightarrow{m \rightarrow \infty} \ell_o^T \left( \frac{Z_{i_x+1}}{c_x^o} \right) v_x. \end{aligned}$$

Either way, for any conjugacy class  $x$  in  $\mathbb{F}_7$ , both

$$\|x\|_{T_e} = \lim_{m \rightarrow \infty} \|x\|_{\left(\tau_{2m}, \frac{\ell_{2m}}{c_{2m}}\right)} \quad \text{and} \quad \|x\|_{T_o} = \lim_{m \rightarrow \infty} \|x\|_{\left(\tau_{2m+1}, \frac{\ell_{2m+1}}{c_{2m+1}}\right)}$$

are well-defined and have the desired form.  $\square$

We now want to show  $T_e$  and  $T_o$  are not scalar multiples of each other. In fact, the following lemma will allow us to show that  $T_e$  and  $T_o$  are the extreme points of the simplex  $\mathbb{P}\mathcal{D}(T)$ .

**Lemma 9.2.** *There exist two sequences  $\alpha_i$  and  $\beta_i$  of conjugacy classes of elements of  $\mathbb{F}_7$  such that the following holds. For any positive length vector  $\ell = \ell_0$  on  $\tau_0$ , let  $T_e$  and  $T_o$  be the respective limiting trees in  $\partial CV_7$  for  $\left(\tau_{2m}, \frac{\ell_{2m}}{c_{2m}}\right)$  and  $\left(\tau_{2m+1}, \frac{\ell_{2m+1}}{c_{2m+1}}\right)$ . Then*

$$\frac{\|\alpha_i\|_{T_o}}{\|\alpha_i\|_{T_e}} \xrightarrow{i \rightarrow \infty} \infty, \quad \text{and} \quad \frac{\|\beta_i\|_{T_o}}{\|\beta_i\|_{T_e}} \xrightarrow{i \rightarrow \infty} 0.$$

*Proof.* Take the letter  $e \in \mathbb{F}_7$  and recall the automorphisms  $\Phi_i$  used to define the folding and unfolding sequences. Set  $x_i = \Phi_i(e)$ . For each  $i$ ,  $x_i$  is legal in  $\tau'_i$  and is represented by the vector  $e_5 = (0, 0, 0, 0, 1, 0, 0)^T$  in  $\tau'_i$ .

Using notation from [Lemma 9.1](#), set  $c_i^e = c_{x_i}^e$  and  $c_i^o = c_{x_i}^o$ . Note here  $i$  is the smallest index such that  $x_i$  is legal in  $\tau'_i$ . We compare the ratio of  $c_i^o$  and  $c_i^e$ . Since  $r_{i+1} - r_i \rightarrow \infty$ , we have

$$\frac{c_{2i}^e}{c_{2i}^o} = \frac{\lambda_C^{r_2} \cdots \lambda_C^{r_{2i}}}{\lambda_C^{r_1} \cdots \lambda_C^{r_{2i-1}}} \xrightarrow{i \rightarrow \infty} \infty, \quad \text{while} \quad \frac{c_{2i+1}^e}{c_{2i+1}^o} = \frac{\kappa_C}{\lambda_C^{r_1}} \frac{\lambda_C^{r_2} \cdots \lambda_C^{r_{2i}}}{\lambda_C^{r_3} \cdots \lambda_C^{r_{2i+1}}} \xrightarrow{i \rightarrow \infty} 0.$$

Recall that both  $\ell_e$  and  $\ell_o$  are positive and by [Lemma 7.1](#) the sequence  $Z_i$  converges to  $Z$ . Since  $Ze_5$  is the zero vector, by continuity of the dot product,

$$\lim_{i \rightarrow \infty} \ell_e^T Z_{2i+1} e_5 = \ell_e^T Z e_5 = 0 \quad \text{and} \quad \lim_{i \rightarrow \infty} \ell_o^T Z_{2i+1} e_5 = \ell_o^T Z e_5 = 0.$$

Next let  $N_\infty = \lim_{i \rightarrow \infty} \frac{N_{r_i}}{\lambda_C^{r_i}}$  and recall by [Lemma 4.7](#) that the vector  $ZN_\infty e_5 = (\star, \star, \star, 0, 0, 0, 0)$  is non-negative. Thus there are positive constants  $A$  and  $B$  such that,

$$\lim_{i \rightarrow \infty} \ell_e^T \left( Z_{2i+2} \frac{N_{r_{2i+1}}}{\lambda_C^{r_{2i+1}}} \right) e_5 = \ell_e^T Z N_\infty e_5 = A > 0,$$

and

$$\lim_{i \rightarrow \infty} \ell_o^T \left( Z_{2i+2} \frac{N_{r_{2i+1}}}{\lambda_C^{r_{2i+1}}} \right) e_5 = \ell_o^T Z N_\infty e_5 = B > 0.$$

Combining the above observations and the formulas for length of  $x_i$  in  $T_e$  and  $T_o$  obtained in [Lemma 9.1](#) we get:

$$\frac{\|x_{2i}\|_{T_o}}{\|x_{2i}\|_{T_e}} = \frac{\ell_o^T \left( Z_{2i+2} \frac{N_{r_{2i+1}}}{\lambda_C^{r_{2i+1}}} \right) e_5 c_{2i}^e}{\ell_e^T (Z_{2i+1}) e_5 c_{2i}^o} \xrightarrow{i \rightarrow \infty} \frac{A}{0} \cdot \infty$$

$$\frac{\|x_{2i+1}\|_{T_o}}{\|x_{2i+1}\|_{T_e}} = \frac{\ell_o^T (Z_{2i+1}) e_5 c_{2i+1}^e}{\ell_e^T \left( Z_{2i+2} \frac{N_{r_{2i+1}}}{\lambda_C^{r_{2i+1}}} \right) e_5 c_{2i+1}^o} \xrightarrow{i \rightarrow \infty} \frac{0}{B} \cdot 0$$

Setting  $\alpha_i = x_{2i}$  and  $\beta_i = x_{2i+1}$  finishes the proof.  $\square$

**Corollary 9.3.** *For a sequence  $(r_i)_{i \geq 1}$  with  $r_{i+1} - r_i \geq i$ , if the folding sequence  $(\tau'_i)_i$  converges to an arational tree  $T$ , then for any positive length vector  $\ell_0$  on  $\tau_0$ , the limit set in  $\partial CV_7$  of the rescaled unfolding sequence  $(\tau_i, \ell_i)$  is always the 1-simplex  $\mathbb{PD}(T)$ .*

*Proof.* Since the folding  $(\tau'_i)_i$  and the unfolding sequence  $(\tau_i)_i$  are equal as marked graphs for all  $i \geq 0$ , no matter the metric, they both visit the same sequence of simplices in  $CV_7$ . In particular, they both project to the same quasigeodesic in  $\mathcal{FF}_7$ . Thus, the two limiting trees  $T_e$  and  $T_o$  of the even and odd sequences of  $(\tau_i, \ell_i)$  are length measures on  $T$ .

Recall  $\mathbb{PD}(T)$  is a 1-simplex by [Corollary 7.2](#). If neither  $T_e$  nor  $T_o$  are the extreme points of this simplex, then there exist constants  $c, c' > 0$  such that any  $x \in \mathbb{F}_n$ ,

$$c' \leq \frac{\|x\|_{T_o}}{\|x\|_{T_e}} \leq c.$$

On the other hand, if one of them, say  $T_o$ , is an extreme point but  $T_e$  is not, then we have a constant  $c > 0$  such that for any  $x \in \mathbb{F}_n$ ,  $\frac{\|x\|_{T_o}}{\|x\|_{T_e}} \leq c$ . In both the cases, we get a contradiction to [Lemma 9.2](#).  $\square$

## 10 Conclusion

Recall  $\phi \in \text{Aut}(\mathbb{F}_7)$  is the automorphism:

$$a \mapsto b, b \mapsto c, c \mapsto ca, d \mapsto d, e \mapsto e, f \mapsto f, g \mapsto g$$

and  $\rho \in \text{Aut}(\mathbb{F}_7)$  is the rotation by 4 clicks:

$$a \mapsto e, b \mapsto f, c \mapsto g, d \mapsto a, e \mapsto b, f \mapsto c, g \mapsto d.$$

For any integer  $r$ , let  $\phi_r = \rho \phi^r$ . To each sequence  $(r_i)_{i \geq 0}$  of positive integers, we have an unfolding sequence  $(\tau_i)_i$  with train track map  $\phi_{r_i}: \tau_i \rightarrow \tau_{i-1}$ , and a folding sequence  $(\tau'_i)_i$  with train track map  $\phi_{r_i}^{-1}: \tau'_{i-1} \rightarrow \tau_i$ . By the limit set of the unfolding sequence  $(\tau_i)_i$  in  $\partial CV_n$  we mean the limit set of  $(\tau_i, \ell_i)$  with respect to some (any) positive length vector  $\ell_i$  on  $\tau_i$ .

**Main Theorem.** *Given a strictly increasing sequence  $(r_i)_{i \geq 1}$  satisfying  $r_i \equiv i \pmod{7}$  and  $r_i \equiv 0 \pmod{3}$ , then the folding sequence  $(\tau'_i)_i$  converges to a non-geometric arational tree  $T$ .*

*If  $(r_i)_i$  grows fast enough, that is, if  $r_{i+1} - r_i \geq i$ , then  $T$  is both non-uniquely ergometric and non-uniquely ergodic. Both  $\mathbb{PD}(T)$  and  $\mathbb{PC}(T)$  are 1-dimensional simplices.*

*Furthermore, the limit set in  $\partial CV_7$  of the unfolding sequence  $(\tau_i)_i$  is always the 1-simplex spanned by the two ergodic metrics on  $T$ .*

*Proof.* A sequence as in the statement exists by the Chinese remainder theorem. The first statement follows from [Corollary 5.8](#) and [Proposition 5.10](#). Non-unique ergometricity of  $T$  follows from [Proposition 3.1](#) and [Corollary 7.2](#). Non-unique ergodicity of  $T$  is [Corollary 8.5](#). Finally, the last statement is [Corollary 9.3](#).  $\square$

# 11 Appendix

## 11.1 Convergence Lemma

Let  $\|\cdot\|$  denote the operator norm. Thus  $\|Y\| \geq 1$  for a nontrivial idempotent matrix  $Y$ .

**Lemma 11.1.** *Let  $Y$  be an idempotent matrix and  $\Delta_i$ ,  $i \geq 1$ , a sequence of matrices with  $\|\Delta_i\| \leq \frac{\epsilon}{2^i}$  for some  $\epsilon > 0$ . Assume also that  $\epsilon\|Y\| \leq 1/2$ . Then the infinite product*

$$\prod_{i=1}^{\infty} (Y + \Delta_i)$$

*converges to a matrix  $X$  with  $\|X - Y\| \leq 2\epsilon(\|Y\| + \|Y\|^2)$ . Moreover, the kernel of  $Y$  is contained in the kernel of  $X$ .*

*Proof.* Write

$$Y + \Sigma_k = \prod_{i=1}^k (Y + \Delta_i)$$

Then  $(Y + \Sigma_k)(Y + \Delta_{k+1}) = Y + \Sigma_{k+1}$  and since  $Y^2 = Y$  it follows that

$$\Sigma_{k+1} = Y\Delta_{k+1} + \Sigma_k(Y + \Delta_{k+1}) \quad (4)$$

Multiplying on the right by  $Y$  and using  $Y^2 = Y$  we get

$$\Sigma_{k+1}Y = Y\Delta_{k+1}Y + \Sigma_kY + \Sigma_k\Delta_{k+1}Y$$

and applying the norm:

$$\|\Sigma_{k+1}Y\| \leq \|\Sigma_kY\| + \frac{\epsilon}{2^{k+1}}\|Y\|^2 + \|\Sigma_k\|\frac{\epsilon}{2^{k+1}}\|Y\|$$

By adding these for  $k = 1, 2, \dots, m-1$  and using  $\Sigma_1 = \Delta_1$  we have

$$\begin{aligned} \|\Sigma_mY\| &\leq \|\Sigma_1Y\| + \epsilon\|Y\|^2 \left( \frac{1}{4} + \dots + \frac{1}{2^m} \right) + \epsilon\|Y\| \left( \frac{\|\Sigma_1\|}{4} + \dots + \frac{\|\Sigma_{m-1}\|}{2^m} \right) \\ &\leq \epsilon \left( \|Y\| + \|Y\|^2 \right) + \epsilon\|Y\| \left( \frac{\|\Sigma_1\|}{4} + \dots + \frac{\|\Sigma_{m-1}\|}{2^m} \right) \end{aligned}$$

So the norms of  $\Sigma_mY$  are bounded by norms of  $\Sigma_i$  with  $i < m$ . From [Equation \(4\)](#) we also see that the norm of  $\Sigma_{k+1}$  is bounded by the norms of  $\Sigma_kY$ . Putting this together we have

$$\begin{aligned} \|\Sigma_{k+1}\| &\leq \|Y\|\|\Delta_{k+1}\| + \|\Sigma_kY\| + \|\Sigma_k\|\|\Delta_{k+1}\| \\ &\leq \frac{\epsilon}{2^{k+1}}\|Y\| + \|\Sigma_kY\| + \frac{\epsilon}{2^{k+1}}\|\Sigma_k\| \\ &\leq \frac{\epsilon}{2^{k+1}}\|Y\| + \frac{\epsilon}{2} \left( \|Y\| + \|Y\|^2 \right) + \epsilon\|Y\| \left( \frac{\|\Sigma_1\|}{4} + \dots + \frac{\|\Sigma_{k-1}\|}{2^k} \right) + \frac{\epsilon}{2^{k+1}}\|\Sigma_k\| \\ &\leq \epsilon \left( \|Y\| + \|Y\|^2 \right) + \epsilon\|Y\| \left( \frac{\|\Sigma_1\|}{4} + \dots + \frac{\|\Sigma_{k-1}\|}{2^k} + \frac{\|\Sigma_k\|}{2^{k+1}} \right). \end{aligned}$$

Thus we have an inequality of the form

$$\|\Sigma_{k+1}\| \leq a + b \left( \frac{\|\Sigma_1\|}{4} + \dots + \frac{\|\Sigma_k\|}{2^{k+1}} \right)$$

for  $a = \epsilon \left( \|Y\| + \|Y\|^2 \right)$  and  $b = \epsilon\|Y\|$ .

Set  $c = 2\epsilon(\|Y\| + \|Y\|^2)$ . Then  $c \geq \epsilon$ ,  $a \leq c/2$ , and  $b \leq 1/2$  by assumption. Easy induction then shows for all  $k \geq 1$ ,

$$\|\Sigma_k\| \leq c. \quad (5)$$

This obtains the inequality  $\|X - Y\| \leq c$  from the statement, once we establish convergence.

To see convergence we argue that the sequence of partial products forms a Cauchy sequence. For  $1 < k < m$ ,

$$\prod_{i=1}^m (Y + \Delta_i) - \prod_{i=1}^k (Y + \Delta_i) = \prod_{i=1}^{k-1} (Y + \Delta_i) \left( \prod_{i=k}^m (Y + \Delta_i) - (Y + \Delta_k) \right)$$

By Equation (5), the norm of  $\prod_{i=1}^{k-1} (Y + \Delta_i) = Y + \Sigma_{k-1}$  is bounded by  $c + \|Y\|$ . We can apply the same estimate to the sequence starting with  $Y + \Delta_k$  and with  $\epsilon$  replaced with  $\frac{\epsilon}{2^{k-1}}$  to see that

$$\left\| \prod_{i=k}^m (Y + \Delta_i) - Y \right\| \leq \frac{2\epsilon(\|Y\| + \|Y\|^2)}{2^{k-1}} \leq \frac{c}{2^{k-1}}$$

and so

$$\left\| \prod_{i=k}^m (Y + \Delta_i) - (Y + \Delta_k) \right\| \leq \frac{c}{2^{k-1}} + \frac{1}{2^k}$$

which proves the sequence is Cauchy.

For the second statement, set  $X_k = \prod_{i=k}^{\infty} (Y + \Delta_i)$  for  $k \geq 1$ . By the same estimate as above with  $\epsilon$  replaced with  $\frac{\epsilon}{2^{k-1}}$ , we know that  $X_k$  exists and

$$\|X_k - Y\| \leq \frac{2\epsilon}{2^{k-1}} (\|Y\| + \|Y\|^2) = \frac{c}{2^{k-1}}.$$

By definition,  $X = (Y + \Sigma_k)X_{k+1}$ . Suppose  $v$  is a unit vector with  $Yv = 0$ . Then

$$\begin{aligned} \|Xv\| &\leq \|Y + \Sigma_k\| \|X_{k+1}v\| \\ &= \|Y + \Sigma_k\| \|X_{k+1}v - Yv\| \\ &\leq \|Y + \Sigma_k\| \|X_{k+1} - Y\| \\ &\leq (\|Y\| + c) \frac{c}{2^k}. \end{aligned}$$

Since this is true for all  $k \geq 0$ , letting  $k \rightarrow \infty$  yields  $Xv = 0$ . □

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