1) Writing it in the form

\[ I_1 = \iint \int f(x,y,z) \, dx \, dy \, dz = \int dx \int dy \int f(x,y,z) \, dz = f(x). \]

I will use the second form of writing the integral. Clearly, the "outside" integration goes for \( x \) from 0 to 1, so

\[ I_1 = \int_0^1 dx \int dy \int f(x,y,z) \, dz = f(x). \]

For a fixed value of \( x \in [0,1] \), the cross-section of the body in the
Figure with the plane \( x = (\text{the chosen value} \ x^*) \) looks like this: in 3-dimensional picture.

and projected onto the \((y=z)\)-plane,

using that \( y = \frac{1}{x} \)

(\text{check: When } x^* = 1 \text{ this triangle disappears; when } x^* = 0 \text{ we get the "big" triangle in the 2-dim picture).}

So the second integration (over \( y \)) goes from \( 0 \) to \( 1 \):

\[
I_1 = \int_0^1 dx \int_0^{1-x} dy \int_0^{1-x} dz \ f(x, y, z)
\]

Finally, for a fixed value of \( y \), the variable \( z \) is allowed to vary between 0 and \( 1-y \) (see the 2-dim picture above), so

\[
I_1 = \int_0^1 dx \int_0^{1-y} dy \int_0^{1-y} dz \ f(x, y, z)
\]

Here I omitted the star on the \( x \) which was used only temporarily to make things clearer; from now on I will not use such temporary notations.

2.) It is easy to use the above picture in order to write the integral in the form

\[
I_2 = \int_0^1 dx \int_0^{1-x} dy \int_0^{1-x} dz \ f(x, y, z)
\]

The integration over \( x \) will go again from 0 to 1 and for a fixed value of \( x \) in \([0,1]\), we will have the 2-dim picture above. For this fixed value of \( x \) the variable \( x^* \) may vary between 0 and \( 1-x \) (I am omitting the star), so the integral becomes
\[ I_2 = \int_1^1 x f(x) \, dx \]

For a given \( x \in [0, 1] \), the allowed range for \( y \) is \([x, 1-x]\), so the integral becomes:

\[ I_2 = \int_0^{1-x} \int_x^{1-x} dy \, dx \]

3. Now I will take \( z \) to be the outside variable, and will write the integral in the form:

\[ I_3 = \int_0^1 \int_0^{1-z} \int_x^{1-x} dy \, dx \, f(x) \]

The variable \( z \) takes all values between 0 and 1, so

\[ I_3 = \int_0^1 \int_0^{1-z} dx \, f(x) \]

For a fixed value of \( z \in [0, 1] \), we have for a fixed \( z \in [0, 1] \), for \( x \) varies from 0 to \( 1-z \), and \( y \) varies from 0 to \( y_{z2} \), so we get

\[ I_3 = \int_0^1 (1-z) \, f(x) \, dx \]

The 2-d projection onto the \((x,y)\) plane:

\[ y = f(x) \]

\[ z = 1 - y \]

\[ y = z \]

The following 3-d picture:
4) Using the same picture as in part 3), one can write

$$I_3 = \int_0^1 \int_0^{1-x} \int_0^y f(\cdots) \, dy \, dx \, dz.$$ 

Please draw a 2-d picture that clarifies the integrations over $x$ and $y$ as an exercise.

5) Now we want to write

$$I_5 = \int_0^1 \int_0^{1-x} \int_0^y f(\cdots) \, dy \, dx \, dz.$$ 

Cross-section with a plane $y = \text{const} \in [0,1]$

6) Finally, we reverse the order of the internal integration in $I_5$, which is easy because (the projection of the cross-section in part 5) onto the $x,z$-plane is a rectangle — what does this fact imply about the limits of integration with $y$ and $x$ at fixed $y$? The result is

$$I_6 = \int_0^1 \int_0^y \int_0^{1-x} f(\cdots) \, dx \, dy \, dz.$$