Additional problem (this is NOT an FFT problem!)

**Additional problem.** In this problem you will generalize the results about Gabriel’s horn (shown in Exercise 8.2/25). Consider the region

\[ \mathcal{R}_\alpha = \left\{ (x, y) \bigg| x \geq 1, \ 0 \leq y \leq \frac{1}{x^\alpha} \right\}, \]

in the \((x, y)\)-plane, where \(\alpha\) is a positive constant (not necessarily integer). Let \(\mathcal{D}_\alpha\) be the solid of revolution obtained by rotating \(\mathcal{R}_\alpha\) about the \(x\)-axis.

(a) Find the range of values of \(\alpha\) for which the volume of \(\mathcal{D}_\alpha\) is finite, by using the method of slicing \(\mathcal{D}_\alpha\) by parallel planes (as in Section 5.2). Find the volume of \(\mathcal{D}_\alpha\) for the values of \(\alpha\) for which this volume is finite.

(b) Find the range of values of \(\alpha\) for which the volume of \(\mathcal{D}_\alpha\) is finite, by using the method of cylindrical shells (as in Section 5.3). Find the volume of \(\mathcal{D}_\alpha\) for the values of \(\alpha\) for which this volume is finite. (Of course, the result should be the same as in part (a), but obtained in a different way.)

(c) Set up the integral for the (side) surface area of \(\mathcal{D}_\alpha\). Show that for \(\alpha > 1\) the (side) surface area is finite. Do not attempt to solve the integral, use the Comparison Theorem.

(d) Use the Comparison Theorem to show that for \(\alpha < 1\) the (side) surface area is infinite.