Questions about Existence of Vector and Scalar Potentials

Recall we had the following picture of the \textit{grad}, \textit{curl}, and \textit{div} differential operators.

\[
\begin{align*}
\{ \text{Functions} &\} \xrightarrow{\text{grad}} \{ \text{Vector fields} &\} \xrightarrow{\text{curl}} \{ \text{Vector fields} &\} \xrightarrow{\text{div}} \{ \text{Functions} &\} \\
\{ f(x,y,z) &\text{ on a domain } E \in \mathbb{R}^3. \} &\rightarrow \{ F = \langle P, Q, R \rangle &\text{ on the domain } E \in \mathbb{R}^3. \} &\rightarrow \{ F = \langle P, Q, R \rangle &\text{ on the domain } E \in \mathbb{R}^3. \} &\rightarrow \{ f(x,y,z) &\text{ on the domain } E \in \mathbb{R}^3. \}\end{align*}
\]

1. **Tests to see if a vector field has a scalar or vector potential.**

   (a) Suppose the vector field \( F \) is equal to \( \nabla f \) for some function \( f \) (we say that \( F \) is conservative, and that it has a scalar potential). Then \( \nabla \times F = \nabla \times \nabla f = 0 \).

   In particular, if \( F \) is a vector field for which \( \nabla \times F \neq 0 \), then you can conclude that \( F \) is NOT the gradient of some function \( f \).

   (b) Suppose the vector field \( F \) is equal to \( \nabla \times G \) for some vector field \( G \) (we say that \( F \) has a vector potential). Then \( \nabla \cdot F = \nabla \cdot \nabla \times G = 0 \).

   In particular, if \( F \) is a vector field for which \( \nabla \cdot F \neq 0 \), then you can conclude that \( F \) is NOT the curl of some vector field \( G \).

2. **Suppose the vector field \( F \) satisfies \( \nabla \times F = 0 \). Is it the case that \( F \) is the gradient of some function \( f \)?**

   (a) The answer can be “No.” Consider the following example.

   \[ B = \frac{\langle -y, x, 0 \rangle}{x^2 + y^2} \]

   - Note that the domain of \( B \) is all of \( \mathbb{R}^3 \) minus the \( z \)-axis. This domain has a \textit{one dimensional hole}; that is, a hole which prevents the one dimensional circle
     \[ C : \quad r(t) = (\cos(t), \sin(t), 0) \quad 0 \leq t \leq 2\pi \]
     from being the boundary of an oriented surface contained in the domain.
   - It is easy to verify that \( \nabla \times B = 0 \).
   - It is also easy to verify that the line integral \( \int_C B \cdot dr = 2\pi \).
   - Because the path integral about a closed path is non-zero, we conclude that \( B \) is not a gradient.
   - \textit{Key idea: It is a global problem, not a local problem.} We saw in class notes (2-dim version) that \( B \) is locally the gradient of a function; for example, the polar angle function
     \[ f(x,y,z) = \tan^{-1}(y/x) \]
     is one such function.

   The key problem is that there is no \textit{globally defined function} \( f \) whose gradient is \( B \). In particular, when one tries to extend the definition of the polar angle function above around the circle unit \( C \) in the \( xy \)-plane, it becomes multivalued (we end up being forced to conclude that values of \( f \) at some point is both \( \alpha \) and \( 2\pi + \alpha \)). Note that the circle \( C \) is one of the circles which is not the boundary of an oriented surface in \( \mathbb{R}^3 \) minus the \( z \)-axis.
(b) If the domain has no one dimensional holes, then every simple, closed loop \( C \) is the boundary of an oriented surface \( S \), and then Stokes' Theorem gives
\[
\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \iint_S 0 \cdot d\mathbf{S} = 0
\]
Thus Path integrals are independent of the chosen path, and we saw in class how to use these path integrals to build a globally defined function \( f \) with \( \nabla f = \mathbf{F} \). The negative of such an \( f \) is called a (scalar) potential for \( \mathbf{F} \).

3. Suppose the vector field \( \mathbf{F} \) satisfies \( \nabla \cdot \mathbf{F} = 0 \). Is it the case that \( \mathbf{F} \) is the curl of some vector field \( \mathbf{G} \)?

(a) The answer can be “No.” Consider the following example.
\[
\mathbf{E} = \frac{\langle x, y, z \rangle}{(x^2 + y^2 + z^2)^{3/2}}
\]
- Note that the domain of \( \mathbf{E} \) is all of \( \mathbb{R}^3 \) minus the origin \((0, 0, 0)\). This domain has a two dimensional hole; that is, a hole which prevents the two dimensional sphere \( S \) defined by \( x^2 + y^2 + z^2 = 1 \) from bounding a solid ball in the domain.
- It is easy to verify that \( \nabla \cdot \mathbf{E} = 0 \).
- It is also easy to verify that \( \iint_S \mathbf{E} \cdot d\mathbf{S} = 4\pi \).
- Because the surface integral of \( \mathbf{E} \) about the closed sphere \( S \) is non-zero, we conclude (by the result from the Stokes’ Theorem handout) that \( \mathbf{E} \) is not the curl of any vector field.
- **Key idea:** It is a global problem, not a local problem. Because \( \nabla \cdot \mathbf{E} = 0 \), it is possible to “integrate” and find locally defined vector fields \( \mathbf{G} \) whose curl equals \( \mathbf{E} \) (do this as an exercise; we did some examples of finding such vector fields in class).
  The problem is that there is no globally defined vector field \( \mathbf{G} \) on \( \mathbb{R}^3 \) minus \((0, 0, 0)\) whose curl is \( \mathbf{E} \). In particular, there is no vector field defined on all of the unit sphere \( S \): \( x^2 + y^2 + z^2 = 1 \) whose curl is equal to \( \mathbf{E} \) on \( S \). (It is a good exercise to try extending different candidates for \( \mathbf{G} \) over all of \( S \) and to think about what goes wrong.) Note that the sphere \( S \) does not bound a solid ball in \( \mathbb{R}^3 \) minus \((0, 0, 0)\).

(b) If the domain has no two dimensional holes, so that every sphere bounds a solid ball, and if \( \nabla \cdot \mathbf{F} = 0 \), then one can argue that \( \mathbf{F} \) is the curl of another, globally defined vector field. The argument involves some integration.

A vector field \( \mathbf{G} \) such that \( \nabla \times \mathbf{G} = \mathbf{F} \) is called a vector potential for \( \mathbf{F} \).

4. **Remark 1.** It can be shown that these are essentially the only examples that occur. Of course a space may have several one or two dimensional holes, but locally (near the holes) the examples will all look like \( \mathbf{B} \) or \( \mathbf{E} \).

5. **Remark 2.** The vector fields \( \mathbf{B} \) and \( \mathbf{E} \) are not esoteric mathematical examples. They occur in nature, and you will meet them in your physics and engineering courses.

- For example, the field \( \mathbf{B} \) is (up to an appropriate positive scalar multiple) the static magnetic field due to a constant electric current flowing up an infinite wire along the \( z \)-axis.
• The field $\mathbf{E}$ is the standard “inverse square law, central force” field. It could be (up to an appropriate negative scalar multiple) the gravitational field due to a mass $m$ at $(0, 0, 0)$. Alternatively, it could be (up to an appropriate positive/negative scalar multiple) the electrostatic field due to a positive/negative charge $q$ at $(0, 0, 0)$.


• You should check that $\nabla \cdot \mathbf{B} = 0$.
• Verify that $\mathbf{A} = (0, 0, -\frac{1}{2} \ln(x^2 + y^2))$ is a vector potential for $\mathbf{B}$; that is, $\nabla \times \mathbf{A} = \mathbf{B}$.
• (One can verify that the domain of $\mathbf{B}$ has no two dimensional holes! It is possible to fill spheres in this domain in with solid balls in the domain.)
• Now check that $\nabla \times \mathbf{E} = 0$.
• Verify that $f(x, y, z) = \frac{1}{\sqrt{x^2 + y^2 + z^2}}$ is a scalar potential for $\mathbf{E}$; that is, $-\nabla f = \mathbf{E}$.
• (One can verify that the domain of $\mathbf{E}$ has no one dimensional holes; every simple, closed loop in the domain is the boundary of some oriented surface in the domain.)
• Working with vector potentials for Magnetic fields $\mathbf{B}$ and scalar potentials for Electric fields $\mathbf{E}$ will be useful in your EM–class.