1. The first operator $\text{grad}$ takes a function $f = f(x, y)$ and returns the vector field $\text{grad}(f) = \nabla f = \langle f_x, f_y \rangle$.

2. The second operator $\text{diff}$ takes a vector field $\mathbf{F} = \langle P(x, y), Q(x, y) \rangle$ and returns the “difference” of partial derivatives $\text{diff}(\mathbf{F}) = \text{diff}(\langle P, Q \rangle) = Q_x - P_y$.

Note that the partial derivatives are just functions of $(x, y)$, and the difference of two partial derivatives is a function of $(x, y)$.

3. It is clear that the composition of these two operators gives the zero map.

$$\text{diff} \circ \text{grad}(f) = \text{diff}(\text{grad}(f)) = \text{diff}(\langle f_x, f_y \rangle) = (f_y)_x - (f_x)_y = 0.$$ 

4. **Question.** Which vector fields $\mathbf{F} = \langle P, Q \rangle$ are the gradient vector fields of functions?

(a) We know that one condition is that $\text{diff}(\mathbf{F}) = 0$. This gives a good test; if $\text{diff}(\mathbf{F}) \neq 0$, then we conclude that $\mathbf{F}$ is not the gradient vector field of a function.

(b) However, we have seen that the vector field 

$$\mathbf{A} = \frac{\langle -y, x \rangle}{x^2 + y^2}$$

defined on $\mathbb{R}^2 - \{(0, 0)\}$ satisfies $\text{diff}(\mathbf{A}) = 0$ but $\mathbf{A}$ is not the gradient of a function.

(c) Indeed, we saw in class that $\mathbf{A}$ is locally the gradient of the “angle function” $f(x, y) = \tan^{-1}(y/x) + c$. However, the “angle function” is not well defined on $\mathbb{R}^2 - \{(0, 0)\}$; one has to add $2\pi$ every time one travels counterclockwise around a circle which encloses $(0, 0)$.

(d) Suppose $\mathbf{F}$ is a vector field defined on $\mathbb{R}^2 - \{(0, 0)\}$ which satisfies: (i) $\text{diff}(\mathbf{F}) = 0$; and, (ii) $\oint_{S^1} \mathbf{F} \cdot d\mathbf{r} \neq 0$ where $S^1$ is the unit circle centered at $(0, 0)$ with the standard counterclockwise orientation.

i. Use Green’s theorem to show that

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_{S^1} \mathbf{F} \cdot d\mathbf{r}$$

for ANY counter clockwise oriented circle $C$ which encircles once around $(0, 0)$. 
ii. Use Green’s theorem to show that

\[ \mathbf{F} = \frac{\int_{S_1} \mathbf{F} \cdot d\mathbf{r}}{2\pi} \mathbf{A} \]

is a conservative vector field. Here \( \mathbf{A} \) is the special vector field introduced in (b) above.

iii. Conclude that if \( \mathbf{F} \) defined on \( \mathbb{R}^2 - \{(0,0)\} \) satisfies \( \text{diff}(\mathbf{F}) = 0 \), then

\[ \mathbf{F} = \nabla f + \frac{\int_{S_1} \mathbf{F} \cdot d\mathbf{r}}{2\pi} \mathbf{A} \]

for some scalar field (function) \( f(x, y) \) on \( \mathbb{R}^2 - \{(0,0)\} \). That is, \( \mathbf{F} \) is a gradient plus a constant multiple of \( \mathbf{A} \).

(e) More generally, suppose that \( \mathbf{F} \) is a vector field defined on \( \mathbb{R}^2 - \{(0,0), (p,q)\} \) which satisfies \( \text{diff}(\mathbf{F}) = 0 \), then

\[ \mathbf{F} = \nabla f + \frac{\int_{C_1} \mathbf{F} \cdot d\mathbf{r}}{2\pi} \frac{(-y,x)}{x^2 + y^2} + \frac{\int_{C_2} \mathbf{F} \cdot d\mathbf{r}}{2\pi} \frac{(-(y-q),(x-p))}{(x-p)^2 + (y-q)^2} \]

for some scalar field (function) \( f(x, y) \) on \( \mathbb{R}^2 - \{(0,0), (p,q)\} \), and where \( C_1 \) and \( C_2 \) are small circles (bounding disjoint disks) about \((0,0)\) and \((p,q)\) respectively. That is, \( \mathbf{F} \) is a gradient plus a constant multiple of \( \mathbf{A} \) and a constant multiple of \( \frac{(-y-q),(x-p)}{(x-p)^2 + (y-q)^2} \).

(f) Generalize the result above to the case of \( \mathbb{R}^2 \) with \( N \) points removed.

5. **Remark.** The sets of functions and vector fields above are actually vector spaces (from your linear algebra class). Just like with regular vectors in 3-dimensions, one can add functions (or vector fields) or multiply them by constants to get new functions (or vector fields). The differential operators \( \nabla \) and \( \text{diff} \) respect sums and constant multiples (these are just versions of the standard “rules” of differentiation), and so are examples of linear maps.

(a) The *kernel* of \( \text{diff} \) is the set of all vector fields \( \mathbf{F} \) defined on the domain \( D \) such that \( \text{diff}(\mathbf{F}) = 0 \).

\[ \ker(\text{diff}) = \{ \mathbf{F} | \mathbf{F} \text{ a vector field on } D \text{ such that } \text{diff}(\mathbf{F}) = 0 \} \]

This is a vector subspace of the space of vector fields.

(b) The *image* of \( \nabla \) is the set of all vector fields \( \mathbf{F} \) of the form \( \mathbf{F} = \nabla f \) for some function \( f \) defined on the domain \( D \).

\[ \text{Im}(\nabla) = \{ \nabla f | f(x,y) \text{ a function on } D \} \]

This is a vector subspace of the space of vector fields.

(c) Now \( \text{diff} \circ \nabla = 0 \) implies that one of these spaces is a subspace of the other:

\[ \text{Im}(\nabla) \subset \ker(\text{diff}) \]

Furthermore, the Green’s theorem applications above tell us that the extra dimensions needed to pass from \( \text{Im}(\nabla) \) to \( \ker(\text{diff}) \) is equal to the number of “holes” in the domain \( D \).