Ternary Expansions and the Cantor Set

**Geometric Series**: Recall a geometric series, which converges for all $|a| < 1$:

$$
\sum_{k=0}^{\infty} a^k = \frac{1}{1-a}
$$

This is a result of the telescoping formula

$$(1-a)(1+a+a^2+\cdots+a^{N-1}) = 1-a^N$$

When $|a| < 1$, taking the limit as $N$ goes to infinity gives the geometric series formula.

We consider the special case where $a = \frac{1}{p}$ for $p$ a natural number greater than or equal to 2. When $p = 2$ we just have the familiar series:

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 2$$

This series also has the cool property that every tail of the series sums to exactly the previous term before the tail:

$$
\begin{align*}
1 &= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots \\
\frac{1}{2} &= \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots \\
\frac{1}{2N} &= \frac{1}{2N+1} + \frac{1}{2N+2} + \frac{1}{2N+3} + \cdots
\end{align*}
$$

This property is not unique, however. Let $a = \frac{1}{p}$. Prove that

$$
\sum_{k=N+1}^{\infty} \frac{p-1}{p^k} = \frac{1}{p^N}
$$

for any $p \in \mathbb{N}, p \geq 2$.

Note that, in particular, $\sum_{k=1}^{\infty} \frac{p-1}{p^k} = 1$, regardless of the choice of $p$.

**Expansions of real numbers**

Given $p \geq 2$, take any series of the form

$$
\sum_{k=1}^{\infty} \frac{a_k}{p^k}, \quad a_k \in \{0, 1, \ldots, p-1\}.
$$

Prove that this series converges to a real number in $[0, 1]$. 
Consider the space of all sequences \( \{a_k\}_{k=1}^{\infty} \) where \( a_k \in \{0, 1, \ldots, p-1\} \). The map

\[
F: \{a_k\}_{k=1}^{\infty} \mapsto \sum_{k=1}^{\infty} \frac{a_k}{p^k}
\]

is therefore a well-defined map from the space of such sequences to \([0, 1]\). We observe that \( F \) is not quite injective. If \( x \in [0, 1] \) is of the form \( \frac{q}{p^N} \) where \( q \in \mathbb{N}, q < P^N \), show that there are two different sequences \( \{a_k\}_{k=1}^{\infty} \) that \( F \) maps to \( x \).

Next, we want to prove that \( F \) is surjective. Given any \( x \in [0, 1] \), prove (constructively) that there exists a sequence \( \{a_k\}_{k=1}^{\infty} \) such that \( a_k \in \{0, 1, \ldots, p-1\} \) for all \( k \) and

\[
\sum_{k=1}^{\infty} \frac{a_k}{p^k} = x.
\]

Hint: observe that since the series consists of positive terms, the partial sums form a strictly increasing sequence. Also, use Equation (1) above.

You can now describe a bijection between the interval \([0, 1]\) and the space of sequences \( \{a_k\}_{k=1}^{\infty}, a_k \in \{0, 1, \ldots, p-1\} \). When \( p = 10 \), the sequence is just exactly the decimal expansion of the number \( x \), for example,

\[
x = 0.141592... = \frac{1}{10} + \frac{4}{10^2} + \frac{1}{10^3} + \cdots;
\]

when \( p = 2 \) the sequence is called the binary expansion; when \( p = 3 \), it is the ternary expansion.

**The Cantor Set**

One way to view the Cantor ternary set is in terms of ternary expansions. Given \( x \in [0, 1] \), there is a sequence of integers \( \{a_k\}_{k=1}^{\infty}, a_k \in \{0, 1, 2\} \) such that the series

\[
\sum_{k=1}^{\infty} \frac{a_k}{3^k}
\]

converges to \( x \). In other words, we can associate \( x \) to the ternary sequence

\[
\{a_1, a_2, a_3 \ldots\}, a_k \in \{0, 1, 2\}.
\]

Prove that the Cantor ternary set is equal to the subset of \([0, 1]\) consisting of all \( x \) which have a ternary expansion for which \( a_k \in \{0, 2\} \) for all \( k \), i.e. the numbers which have an expansion with no 1’s. (Read this carefully in the cases where \( x \) has two possible expansions. If \( x \) has one ternary expansion which contains no 1’s, then it is in the Cantor set.)

Prove that the map we defined in class:

\[
\sum_{k=1}^{\infty} \frac{a_k}{3^k} \mapsto \sum_{k=1}^{\infty} \frac{b_k}{2^k}, \quad b_k = \frac{a_k}{2}
\]

maps the Cantor ternary set \( C \) onto \([0, 1]\), hence proving \( C \) is uncountable.

Prove that if \( a, b \in C \), the Cantor ternary set, with \( a < b \), then there exists a real number \( r \notin C \) such that \( a < r < b \). In other words, the Cantor set contains no intervals.