2.3.3(a) We already explained in class that the heat equation with these boundary conditions has solutions given by

\[ u(x, t) = B \sin \left( \frac{n \pi x}{L} \right) e^{-k(n \pi/L)^2 t} \]

where \( B \) is an arbitrary constant and \( n \in \{1, 2, 3 \ldots \} \). So you can answer this question simply by observing that if we take \( B = 6 \) and \( n = 9 \) in this solution, then we get

\[ u(x, 0) = 6 \sin \left( \frac{9 \pi x}{L} \right) e^0 = 6 \sin \left( \frac{9 \pi x}{L} \right), \]

which is the correct initial condition. So the answer is

\[ u(x, t) = 6 \sin \left( \frac{9 \pi x}{L} \right) e^{-k(9 \pi/L)^2 t} \]

Alternatively, you could start from the fact, stated in class, that the general solution of the heat equation with the given boundary condition is a linear combination of the above solutions:

\[ u(x, t) = \sum_{n=1}^{\infty} B_n \sin \left( \frac{n \pi x}{L} \right) e^{-k(n \pi/L)^2 t}, \]

where each \( B_n \) is a constant. We also stated in class that in order to achieve the initial condition \( u(x, 0) = f(x) \), where \( f(x) \) is a given function, we should choose the \( B_n \) according to the formula

\[ B_n = \frac{2}{L} \int_0^L f(\bar{x}) \sin \left( \frac{n \pi \bar{x}}{L} \right) d\bar{x}, \]

for \( n \in \{1, 2, 3 \ldots \} \). Finally, you already know that for all \( m \) and \( n \) in \( \{1, 2, 3 \ldots \} \), we have

\[ \int_0^L \sin \left( \frac{m \pi \bar{x}}{L} \right) \sin \left( \frac{n \pi \bar{x}}{L} \right) d\bar{x} = \frac{L}{2} \delta_{mn}, \]

where \( \delta_{mn} \) is the Kronecker delta, defined to equal 1 if \( m = n \) and 0 if \( m \neq n \). Since in this case, we have

\[ f(\bar{x}) = 6 \sin \left( \frac{9 \pi \bar{x}}{L} \right), \]

we can evaluate the above formula for \( B_n \) to get

\[ B_n = \frac{2}{L} \int_0^L 6 \sin \left( \frac{9 \pi \bar{x}}{L} \right) d\bar{x} = \frac{2 \cdot L}{2} \delta_{9n} = 6 \delta_{9n}, \]

or in other words, \( B_n = 6 \) when \( n = 9 \) and \( B_n = 0 \) when \( n \neq 9 \). Then

\[ u(x, t) = \sum_{n=1}^{\infty} B_n \sin \left( \frac{n \pi x}{L} \right) e^{-k(n \pi/L)^2 t} = 6 \sin \left( \frac{9 \pi x}{L} \right) e^{-k(9 \pi/L)^2 t}. \]

This is a much longer way of arriving at the same answer!

2.3.3(d) Here you need to use the formula for \( B_n \), given above, with the function \( f(x) \) as specified. This gives

\[ B_n = \frac{2}{L} \int_0^L f(\bar{x}) \sin \left( \frac{n \pi \bar{x}}{L} \right) d\bar{x} \]

\[ = \frac{2}{L} \left( \int_0^{L/2} 1 \cdot \sin \left( \frac{n \pi \bar{x}}{L} \right) d\bar{x} + \int_{L/2}^L 2 \cdot \sin \left( \frac{n \pi \bar{x}}{L} \right) d\bar{x} \right) \]

\[ = \frac{2}{L} \left( \frac{-L}{n \pi} \left( \cos \left( \frac{n \pi}{2} \right) - 1 \right) + \frac{2L}{n \pi} \left( \cos (n \pi) - \cos \left( \frac{n \pi}{2} \right) \right) \right) \]

\[ = \frac{2}{n \pi} \left( -2 \cos(n \pi) + \cos \left( \frac{n \pi}{2} \right) + 1 \right). \]
2.3.5 This is straightforward. To get full credit, though, you should handle both the cases when \( m = n \) and when \( m \neq n \).

2.4.1(a) You do not need to repeat all the steps of the derivation of the solution of the heat equation with these boundary conditions, because I did them in class, and they are also written out in the text. To get full credit, you only need to mention that the solution is given by

\[
u(x, t) = A_0 + \sum_{n=1}^{\infty} A_n \cos \left( \frac{n\pi x}{L} \right) e^{-k(n\pi/L)^2 t},
\]

and give the correct formulas for \( A_0 \) and \( A_n \) (\( n \in \{1, 2, 3, \ldots\} \)).

For these boundary conditions, the coefficients \( A_0 \) and \( A_n \) are given in terms of the initial data \( f(x) \) by

\[
A_0 = \frac{1}{L} \int_0^L f(\bar{x}) \, d\bar{x}
\]

and, for \( n \in \{1, 2, 3, \ldots\} \),

\[
A_n = \frac{2}{L} \int_0^L f(\bar{x}) \cos \left( \frac{n\pi \bar{x}}{L} \right) \, d\bar{x}.
\]

Here we have

\[
f(\bar{x}) = \begin{cases} 
0 & \bar{x} < L/2 \\
1 & \bar{x} > L/2
\end{cases}
\]

so

\[
A_0 = \frac{1}{L} \int_{L/2}^L \, d\bar{x} = 1/2,
\]

and, for \( n \in \{1, 2, 3, \ldots\} \),

\[
A_n = \frac{2}{L} \int_{L/2}^L \cos \left( \frac{n\pi \bar{x}}{L} \right) \, d\bar{x} = \frac{2}{n\pi} \left( \sin(n\pi) - \sin \left( \frac{n\pi}{2} \right) \right) = \left( \frac{-2}{n\pi} \right) \sin \left( \frac{n\pi}{2} \right).
\]

2.4.7(b) Here you have to work out the eigenfunctions for the problem

\[
\phi''(x) + \lambda \phi(x) = 0 \\
\phi(0) = 0 \\
\phi'(L) = 0,
\]

since this wasn’t done in the text. I would like you to be able to actually consider all three of the possibilities \( \lambda < 0 \), \( \lambda = 0 \), and \( \lambda > 0 \) separately, and show carefully that in case \( \lambda < 0 \) and \( \lambda = 0 \) there are no non-trivial solutions \( \phi(x) \) — so there are no negative eigenvalues, and 0 is not an eigenvalue. (I noted, however, that the problem statement in the text says you can assume \( \lambda > 0 \), so if we mistakenly took points off your assignment for not doing the cases \( \lambda = 0 \) and \( \lambda < 0 \), let me know and I’ll restore them.)

You should present a correct derivation of the eigenvalues \( \lambda = \left( \frac{n\pi}{2L} \right)^2 \), \( n \in 1, 3, 5, 7, \ldots \), and the corresponding eigenfunctions \( \sin \left( \frac{n\pi x}{2L} \right) \).

After you’ve found the eigenvalues and eigenfunctions, you should say that the solution \( u \) is given by

\[
u(x, t) = \sum_{n \in \{1, 3, 5, \ldots\}} B_n \sin \left( \frac{n\pi x}{2L} \right) e^{-k(n\pi/(2L))^2 t}
\]
(or any equivalent way of writing this sum), and derive the formula for the coefficients

\[ B_n = \frac{2}{L} \int_0^L f(\bar{x}) \sin \left( \frac{n\pi \bar{x}}{2L} \right) \, d\bar{x}. \]

You should at least give a little explanation of where the formula comes from: that is, we start from the equation

\[ f(x) = u(x, 0) = \sum_{n \in \{1, 3, 5, \ldots \}} B_n \sin \left( \frac{n\pi x}{2L} \right), \]

(anne call the eigenfunctions \( \phi_n(x) \) instead \( \sin \left( \frac{n\pi x}{2L} \right) \) if they like), then multiply both sides of the equation by \( \sin \left( \frac{m\pi x}{2L} \right) \) and integrate from \( x = 0 \) to \( x = L \). It was given in the hint to the problem that you can assume that

\[ \int_0^L \sin \left( \frac{m\pi x}{2L} \right) \sin \left( \frac{n\pi x}{2L} \right) \, dx = \frac{L}{2} \delta_{mn}, \]

so this gives the result

\[ \int_0^L f(\bar{x}) \sin \left( \frac{m\pi \bar{x}}{2L} \right) \, d\bar{x} = \sum_{n \in \{1, 3, 5, \ldots \}} B_n \frac{L}{2} \delta_{mn} = \frac{L}{2} B_m, \]

and then multiplying by \( 2/L \) gives

\[ B_m = \frac{2}{L} \int_0^L f(\bar{x}) \sin \left( \frac{m\pi \bar{x}}{2L} \right) \, d\bar{x}. \]

(By the way, I tried to always use \( \bar{x} \) as the variable of integration in class so as to not get it mixed up with the variable \( x \) in the formula for \( u(x, t) \).)