1. Suppose that \( u(x, y) \) is a function defined on \( D(0, 1) \) such that \( u \) is \( C^2 \) on \( D(0, 1) \) and \( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \) for all \((x, y) \in D(0, 1)\). Prove that \( u \) is \( C^\infty \) on \( D(0, 1) \). You may use any result from class, but be clear about which results you are using.

2. Let \( f(z) = \frac{e^z - e^{-z}}{z^4} \).
   
   a. Give a Laurent series for \( f(z) \) in powers of \( z \).
   
   b. For which values of \( z \) does the Laurent series converge? Justify your answer.
   
   c. Find \( \int_\gamma f(z) \, dz \), where \( \gamma \) is the curve shown in the diagram:

3. Find \( \int_C \frac{1}{z^2(z + 1)^2} \, dz \), where \( C \) is the circle \( \{|z| = 1/2\} \), with positive orientation. Justify your answer.

4. Suppose that \( U \subseteq \mathbb{C} \) is open and \( f \) and \( g \) are holomorphic functions on \( U \) such that, for all \( z \in U \), \( f(z)g(z) = 0 \). Show that either \( f(z) = 0 \) for all \( z \in U \), or \( g(z) = 0 \) for all \( z \in U \).

5. Suppose that \( f(z) \) and \( g(z) \) are holomorphic functions on \( D(z_0, r) \), and \( f(z_0) \neq 0 \). Suppose \( f(z)/g(z) \) has a pole of order \( k \) at \( z_0 \). Show that \( g(z) \) has a zero of order \( k \) at \( z_0 \).

6. Use residues to find the value of \( \int_0^\infty \frac{x^2 \, dx}{(x^2 + 1)(x^2 + 9)} \).

7. Suppose \( U \) is an open subset of \( \mathbb{C} \) and \( S \subseteq U \) is discrete: that is, for every \( z \in S \) there exists \( \epsilon > 0 \) such that there are no points of \( S \) in \( D(z, \epsilon) \) besides \( z \). Suppose \( f \) is holomorphic on \( U \setminus S \) and \( f \) has a pole at every point of \( S \). (You may assume \( S \neq \emptyset \).)
   
   a. Show that \( 1/f \) has a removable singularity at every point of \( S \).
   
   b. Show that if \( w \in U \) and \( f(w) = 0 \), then \( 1/f \) has a pole at \( w \).
   
   c. Show that \( S \) has no accumulation points in \( U \).

8. Suppose \( p(z) \) is a polynomial of degree \( n \), and \( R > 0 \) is such that \( p(z) \neq 0 \) for \( |z| \geq R \). Find \( \int_{|z|=R} \frac{p'(z)}{p(z)} \, dz \).
We have that \( u \) is harmonic on \( D(0,1) \), so from a result in class we know there exists a holomorphic function \( f \) on \( D(0,1) \) such that \( u = \text{Re} f \) on \( D(0,1) \). We also know that every holomorphic function is \( C^\infty \) on its domain, so \( f \), and therefore also \( u \), are \( C^\infty \) on \( D(0,1) \).

\[
\begin{align*}
e^z &= 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \ldots, \\
e^{-z} &= 1 - z + \frac{z^2}{2!} - \frac{z^3}{3!} + \frac{z^4}{4!} - \ldots,
\end{align*}
\]
and
\[
\frac{e^z - e^{-z}}{2} = \frac{2}{3!} z + \frac{2}{3!} \frac{1}{2!} z^2 + \frac{2}{3!} \frac{1}{3!} z^3 + \frac{2}{3!} \frac{1}{4!} z^4 + \ldots.
\]

The function \( f(z) = \frac{e^z - e^{-z}}{2} \) is holomorphic on \( C \setminus \{0\} \), so its Laurent series in powers of \( z \) converges on \( C \setminus \{0\} \).

\[
\int_C f(z) \, dz = \text{Res}_f(0) \cdot \text{Ind}_f(0) = \frac{2}{3!} \cdot 2 = \frac{2}{3}.
\]

The only pole of \( f \) is \( z = 0 \), and since \( g(z) = \frac{1}{(z+1)^2} \) has \( g(0) = 1 \) and \( g'(0) = -2 \), then \( g(z) = \sum_{k=0}^{\infty} a_k z^k \) where \( a_0 = 1 \) and \( a_1 = -2 \). Therefore

\[
g(z) = \frac{1}{z^2} \left( \sum_{k=0}^{\infty} a_k z^k \right) = \frac{a_0}{z^2} + \frac{a_1}{z} + a_2 z + a_3 z^2 + \ldots
\]
where \( a_0 = 1 \) and \( a_1 = -2 \), so \( \text{Res}_f(0) = -2 \). Therefore, by the Residue Theorem,

\[
\int_C f(z) \, dz = 2\pi i \cdot (-2) = -4\pi i.
\]
4) Suppose $f \neq 0$ on $U$, we'll show $g \equiv 0$ on $U$. Since $f \neq 0$, then there exists $z_0 \in U$ such that $f(z_0) \neq 0$. By continuity of $f$ and openness of $U$, there exists $\varepsilon > 0$ such that $B(z_0, \varepsilon) \subseteq U$ and $f(z) \neq 0$ for all $z \in B(z_0, \varepsilon)$. Since $f(z)g(z) = 0$ for all $z \in U$, it follows that $g(z) = 0$ for all $z \in B(z_0, \varepsilon)$, since $U$ is connected, it then follows from a result in class that $g \equiv 0$ on $U$.

5) We know
\[
\frac{f(z)}{g(z)} = \sum_{n=-k}^{\infty} a_n (z-z_0)^n
\]
where $a_{-k} \neq 0$, so for all $\varepsilon > 0$,
\[
\frac{f(z)}{g(z)} = (z-z_0)^{-k} h(z)
\] where
\[
h(z) = \sum_{n=-k}^{\infty} a_{n-k} (z-z_0)^n
\]
is holomorphic on $U(z_0, \varepsilon)$ and $h(z_0) \neq 0$. Then
\[
g(z) = (z-z_0)^k \frac{f(z)}{h(z)}
\] and since $h(z_0) \neq 0$, then
\[
\frac{f(z)}{h(z)}
\]
is holomorphic on a neighborhood of $z_0$.

and $\frac{f(z_0)}{h(z_0)} \neq 0$. So
\[
g(z) = (z-z_0)^k \sum_{n=0}^{\infty} b_n (z-z_0)^n
\]
where $b_0 \neq 0$.

and it follows that $g$ has a zero of order $k$ at $z_0$.

6) Integrate $f(z) = \frac{z^2}{(z^2+1)(z^2+9)}$ over the contour $\gamma = \gamma_1 + \gamma_2$ shown in the diagram. We have
\[
\left| \int_{\gamma_2} f(z) \, dz \right| \leq (\text{length } \gamma_2) \sup_{|z|=R} f(z)
\]
\[
\leq \pi R \left( \frac{R^2}{(R^2-1)(R^2-9)} \right) \sim \frac{\pi R^3}{R^2-10R^2+9} \to 0 \quad \text{as } R \to \infty,
\]
and
\[
\int_{\gamma_1} f(z) \, dz = \int_{-R}^{R} \frac{x^2}{(x^2+1)(x^2+9)} \, dx \to \int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+9)} \, dx = 2 \int_{0}^{\infty} \frac{x^2}{(x^2+1)(x^2+9)} \, dx
\]
as $R \to \infty$. buried in the dirt, not on the surface, but deep beneath, waiting to be discovered.
Also \( \int \eta_2 \, b(z) \, dz = 2\pi i \left[ \text{Res}_1 (i) + \text{Res}_{3i} (z) \right] \) by

The residue theorem, since \( i \) and \( 3i \) are the only poles of \( b \) within \( \eta_2^R \). Now \( \text{Res}_1 (i) = \frac{(e^2}{i^2 + 9}) = \frac{-1}{8} = \frac{i}{16} \) and

\[
\text{Res}_{3i} (z) = \frac{(3i)^2}{(3i)^2 + 1} = \frac{-9}{6i} = -3i, \quad \text{so}
\]

\[
\int \eta_2 \, b(z) \, dz = 2\pi i \left( \frac{i}{16} - \frac{3i}{16} \right) = \frac{\pi}{4}
\]

for all \( R \).

So \( \frac{\pi}{4} = \int \eta_2 \, b = \lim_{n \to \infty} \int \eta_2 \, b = \lim_{R \to \infty} \left( \int_{\eta_1} \, b + \int_{\eta_2} \, b \right) = 2 \int_0^\infty \frac{x^2}{(x^2 + 1)(x^2 + 9)} \, dx \)

and \( \int_0^\infty \frac{x^2}{(x^2 + 1)(x^2 + 9)} \, dx = \frac{\pi}{8} \).

7. a) If \( z \in S \) then \( \lim_{z \to 20} |b(z)| = \infty \), so \( \lim_{z \to 20} \frac{1}{b(z)} = 0 \).

So \( b \) is bounded on a neighborhood of \( 20 \), and therefore has a removable singularity at \( 20 \). Moreover, the holomorphic extension \( g \) of \( b \) must have \( g(20) = 0 \).

b) Since \( b(w) = 0 \) and the zeroes of \( b \) are isolated,

there exists \( \varepsilon > 0 \) such that \( b(z) \neq 0 \) for \( z \in D(w, \varepsilon) \setminus \{w\} \).

Then \( \frac{1}{b} \) is holomorphic on \( D(w, \varepsilon) \setminus \{w\} \), and \( \lim_{z \to w} \left| \frac{1}{b(z)} \right| = \infty \).

Since \( \lim_{z \to w} b(z) = \infty \), \( f(w) = 0 \). So \( w \) is a pole of \( \frac{1}{b} \).

* Otherwise \( b \) would be identically \( 0 \) on \( U \), contradicting the fact that \( b \) has poles at the element(s) of \( S \).
Let \( T = \sum w \in U : h(w) = 0 \). Then \( g = \frac{1}{h} \) is defined and holomorphic on \( U \setminus \{ T \cup S \} \), and by part (a) \( g \) can be extended to a holomorphic function on \( U \setminus T \) by defining \( g(z) = 0 \) for \( z \in S \). Since \( g \) is holomorphic and not identically zero on the open, connected set \( U \setminus T \), then The zeroes of \( g \) cannot accumulate, so \( S \) has no accumulation points in \( U \setminus T \). Also, \( S \) cannot have any accumulation points in \( T \), because if \( w \in T \) then \( |h(z)| \leq 1 \) in some neighborhood of \( w \), and hence \( h \) cannot have any poles in that neighborhood. Therefore \( S \) cannot have any accumulation points in \( U \).

We know from the fundamental theorem of algebra that a polynomial of degree \( n \) must have \( n \) zeroes in \( \mathbb{C} \), counting multiplicity. Since the multiplicity of a zero of a polynomial is the same as the order of the zero, then the sum of the orders of the zeroes of \( p \) in \( \mathbb{C} \) must be \( n \). But all these zeroes lie within \( D = \{ |z| < R \} \), so by the argument principle

\[
\oint_D \frac{p'(z)}{p(z)} \, dz = 2\pi i \cdot n
\]