9. We will prove the converse of the desired statement. That is, we assume there exists $N \in \mathbb{Z}$ such that for every sequence $z_n$ in $D(P, r) \setminus \{P\}$ with $\lim z_n = P$, there exists $n \in \mathbb{N}$ such that $|(z_n - P)^N f(z_n)| \leq N$; and we will show that $f$ cannot have an essential singularity at $P$.

From our assumption it follows that there exists some $r_0 \in (0, r)$ such that for every $z \in D(P, r_0) \setminus \{P\}$, $|(z - P)^N f(z)| \leq N$. For if this were not true, then for every $n \in \mathbb{N}$, there would exist $z = z_n \in D(P, 1/n) \setminus \{P\}$ such that $|(z_n - P)^N f(z_n)| > N$. Then $\{z_n\}$ is a sequence such that $\lim z_n = P$ and there is no $n \in \mathbb{N}$ such that $|(z_n - P)^N f(z_n)| \leq N$, violating our assumption.

Now define $g(z) = (z - P)^N f(z)$. From the preceding paragraph we know that $g(z)$ is bounded on $D(P, r_0) \setminus \{P\}$, so $g$ has a removable singularity at $P$. Therefore $g(z)$ has a power series expansion

$$g(z) = \sum_{n=0}^{\infty} a_k (z - P)^k$$
on $D(P, r_0) \setminus \{P\}$. Hence

$$f(z) = g(z)/(z - P)^N = \sum_{n=0}^{\infty} a_k (z - P)^{k-N}$$
on $D(P, r_0) \setminus \{P\}$. But this implies that $f$ has a pole of order $N$ at $P$, not an essential singularity at $P$.

23. Since $f$ has a pole of order $k$ at $P$, then $f$ has the Laurent expansion

$$f(z) = \sum_{n=-k}^{\infty} a_n (z - P)^n$$
for all $z$ in some punctured neighborhood $D(P, r) \setminus \{P\}$. Then $g(z) = (z - P)^k f(z)$ has the expansion

$$g(z) = \sum_{n=-k}^{\infty} a_n (z - P)^{n+k} = \sum_{n=0}^{\infty} a_{n-k} (z - P)^n$$
in $D(P, r) \setminus \{P\}$. So the coefficient of $(z - P)^n$ in the Taylor series expansion for $g$ is the same as the coefficient of $(z - P)^{n-k}$ in the Laurent series expansion for $f$.

34(a). We are integrating $f$ over the circle $C = \{z \mid |z| = 5\}$ with (presumably) the positive orientation. The poles of $f$ are at $z = -1$ and $z = -2i$, both of which are within $C$. The residue of $f$ at $z = -1$ is

$$\frac{-1}{-1 + 2i} = \frac{1 + 2i}{5},$$
and the residue of $f$ at $z = -2i$ is

$$\frac{-2i}{-2i + 1} = \frac{4 - 2i}{5}.$$ So by the residue theorem,

$$\frac{1}{2\pi i} \int_C f(z) \, dz = \frac{1 + 2i}{5} + \frac{4 - 2i}{5} = 1.$$

34(d). The poles of $f$ are at $0$, $-1$, and $-2$, all of which are within $\gamma$. We have

$$\text{Res}_f(0) = e^0/((1)(2)) = 1/2$$
$$\text{Res}_f(-1) = e^{-1}/((-1)(1)) = -1/e$$
$$\text{Res}_f(-2) = e^{-2}/((-2)(-1)) = 1/(2e^2)$$

Since $\gamma$ has the negative orientation, then the desired integral is equal to the negative of the sum of the residues, and is therefore equal to $-(e^2 - 2e + 1)/(2e^2)$.
34(i). We have \( f(z) = \frac{\sin z}{\cos z} \), and \( \sin z \) and \( \cos z \) are entire, so the only singularities of \( f \) are at the zeroes of \( \cos z \), which as we saw in class are all on the real line and are the same as the zeroes of the real cosine function, namely \( \{((2k+1)\pi)/2 : k \in \mathbb{Z}\} \). All these poles are simple (because \( \sin z \) is non-zero at each pole and the derivative of \( \cos z \) is non-zero at each pole) and the residues are

\[
\text{Res}_f \left( \frac{(2k+1)\pi}{2} \right) = \frac{\sin((2k+1)\pi/2)}{\sin((2k+1)\pi/2)} = 1.
\]

From the diagram of \( \gamma \) we see that the only poles of \( f \) about which \( \gamma \) has non-zero index are \(-3\pi/2\) and \(3\pi/2\), and \( \text{Ind}_\gamma(-3\pi/2) = -1 \) and \( \text{Ind}_\gamma(3\pi/2) = 1 \). Therefore

\[
\frac{1}{2\pi i} \int_\gamma \frac{\tan z}{dz} = \text{Res}_f \left( \frac{-3\pi}{2} \right) \text{Ind}_\gamma(-3\pi/2) + \text{Res}_f \left( \frac{3\pi}{2} \right) \text{Ind}_\gamma(3\pi/2) = -1 + 1 = 0.
\]