Math 2924-050
Fall 2014
Exam 1

Name: ____________________________

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1. (15 points) Find the area of the region bounded by the curves $y = \cos x$ and $y = \sin x$ with $-3\pi/4 \leq x \leq \pi/4$.

\[ \text{Area} = \int_{-3\pi/4}^{\pi/4} (\cos x - \sin x) \, dx = \left( \sin x + \cos x \right) \bigg|_{-3\pi/4}^{\pi/4} = 2\sqrt{2} \]

2. (10 points) Find the derivative of $y = x^{\cos x}$.

\[ \ln y = \cos x \ln x \]

\[ \frac{1}{y} \frac{dy}{dx} = -\sin x \ln x + \frac{\cos x}{x} \]

\[ \frac{dy}{dx} = x^{\cos x} \left( -\sin x \ln x + \frac{\cos x}{x} \right) \]
3. (20 points) Show that the volume of a sphere of radius \( r \) is \( \frac{4}{3} \pi r^3 \).

\[
\begin{align*}
  r(x) &= \sqrt{r^2 - x^2} \\
  \text{Area of cross-section} &= \pi r(x)^2 \\
  &= \pi (r^2 - x^2)
\end{align*}
\]

\[
\begin{align*}
  \text{Volume} &= \int_{-r}^{r} \pi (r^2 - x^2) \, dx = \pi \left[ x \left( r^2 x - \frac{1}{3} x^3 \right) \right]_{-r}^{r} \\
  &= 2 \pi \left( r^3 - \frac{1}{3} r^3 \right) = \frac{4}{3} \pi r^3
\end{align*}
\]
4. (15 points) Set up an integral to find the volume of the solid obtained by rotating the bounded region contained between the curves $y^2 = x$, $y = x^2$ about the line $x = -1$. (You do not need to evaluate the integral.)

\[
\text{volume of shell} = 2\pi (x+1) \int (\sqrt{x} - x^2) \, dx
\]

Total volume = \[ \int_{0}^{1} 2\pi (x+1) (\sqrt{x} - x^2) \, dx \]
5. (10 points) Newton’s Law of Universal Gravitation says that two objects with masses $m_1$ and $m_2$ of distance $r$ apart exert a gravitational force of magnitude

$$F = Gm_1m_2/r^2$$

on each other. $G$ is a constant. Assuming the position of the first mass is fixed, how much work is required to move the second mass from a distance of $r = a$ to a distance of $r = b$ (you may assume $0 < a < b$)?

$$W = \int_a^b F \, dr = Gm_1m_2 \int_a^b \frac{1}{r^2} \, dr$$

$$= Gm_1m_2 \left( \frac{1}{a} - \frac{1}{b} \right)$$

6. (10 points) Let $y = f(x)$ and $x = g(y)$ be functions which are inverses of each other. Suppose $f'(x) = 1/x$. Prove that $g'(y) = g(y)$ by differentiating the identity $g(f(x)) = x$.

$$g(f(x)) = x$$

$$\Rightarrow \quad g'(f(x)) \cdot f'(x) = 1 \quad (\text{chain rule})$$

$$g'(y) \cdot \frac{1}{x} = 1$$

$$g'(y) = x = g(y)$$
7. (20 points) Evaluate the integrals.

a) \[ \int \cot x \, dx = \int \frac{\cos x}{\sin x} \, dx = \int \frac{du}{u} = \ln |u| + C \]

\[ u = \sin x \]
\[ du = \cos x \, dx \]
\[ = \ln |\sin x| + C \]

b) \[ \int e^{\frac{x}{2x}} \, dx = \int e^{x(1-\ln 2)} \, dx = \frac{1}{1-\ln 2} \, e^{x(1-\ln 2)} + C \]