Abstract Linear Algebra, Fall 2011 - Solutions to Problems II

1. Suppose $v_1, \ldots, v_n$ is a basis of a vector space $V$. Show that $v_1, v_1 + v_2, \ldots, v_1 + v_2 + \cdots + v_n$ is also a basis of $V$.

Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be scalars such that

$$\alpha_1 v_1 + \alpha_2 (v_1 + v_2) + \cdots + \alpha_n (v_1 + v_2 + \cdots + v_n) = 0.$$ 

Rearranging, we have

$$(\alpha_1 + \alpha_2 + \cdots + \alpha_n)v_1 + (\alpha_2 + \cdots + \alpha_n)v_2 + \cdots + \alpha_n v_n = 0.$$ 

By linear independence of $v_1, \ldots, v_n$,

$$\alpha_1 + \alpha_2 + \cdots + \alpha_n = 0, \alpha_2 + \cdots + \alpha_n = 0, \ldots, \alpha_n = 0.$$ 

It follows that $\alpha_1 = 0, \alpha_2 = 0, \ldots, \alpha_n = 0$, so $v_1, v_1 + v_2, \ldots, v_1 + v_2 + \cdots + v_n$ are linearly independent. Now $\dim V = n$ and we know from class that any $n$ linearly independent vectors in a vector space of dimension $n$ form a basis. Thus $v_1, v_1 + v_2, \ldots, v_1 + v_2 + \cdots + v_n$ is a basis of $V$. [Alternatively, one can show directly that $v_1, v_1 + v_2, \ldots, v_1 + v_2 + \cdots + v_n$ generate $V$. Indeed, the subspace generated by $v_1, v_1 + v_2, \ldots, v_1 + v_2 + \cdots + v_n$ certainly contains $v_1$. It also contains $v_2$ (take the difference of the second and first vectors in the list). Similarly, it contains $v_3$ (take the difference of the third and second vectors in the list). Continuing in this way, we see that it contains $v_1, v_2, \ldots, v_n$. Since every vector in $V$ is a linear combination of $v_1, \ldots, v_n$, it follows that $v_1, v_1 + v_2, \ldots, v_1 + v_2 + \cdots + v_n$ generate $V$.]

2. Let $V$ be an $n$-dimensional vector space and suppose the vectors $v_1, \ldots, v_n$ generate (or span) $V$. Show that $v_1, \ldots, v_n$ is a basis of $V$.

From class, any maximal subset of linearly independent elements in $\{v_1, \ldots, v_n\}$ gives a basis of $V$. Since $\dim V = n$, any basis of $V$ has $n$ elements. Hence a maximal subset of linearly independent elements in $\{v_1, \ldots, v_n\}$ must have $n$ elements and so must contain $v_1, \ldots, v_n$. Therefore $v_1, \ldots, v_n$ is a basis of $V$.

3. Let $A = (a_{ij})$ be an $n \times n$ matrix over a field $F$. Define $\text{tr}(A)$, the trace of $A$, to be the sum of the diagonal elements of $A$:

$$\text{tr}(A) = \sum_{i=1}^{n} a_{ii}.$$ 

a. Show that $A \mapsto \text{tr}(A) : M_n(F) \to F$ is a linear map.

b. Let $B$ be an $m \times n$ matrix and $C$ be an $n \times m$ matrix (both over $F$). Show that $\text{tr}(BC) = \text{tr}(CB)$.

c. Show that, for any invertible $n \times n$ matrix $B$ (over $F$), $\text{tr}(BAB^{-1}) = \text{tr}(A)$. 
a. Let $A = (a_{ij})$ and $B = (b_{ij})$. Then $A + B = (a_{ij} + b_{ij})$ and

$$\text{tr}(A + B) = \sum_{i=1}^{n} (a_{ij} + b_{ij})$$
$$= \sum_{i=1}^{n} a_{ij} + \sum_{i=1}^{n} b_{ij}$$
$$= \text{tr}(A) + \text{tr}(B).$$

Similarly, for any scalar $\alpha$, we have $\alpha A = (\alpha a_{ij})$ and

$$\text{tr}(\alpha A) = \sum_{i=1}^{n} \alpha a_{ij}$$
$$= \alpha \sum_{i=1}^{n} a_{ij}$$
$$= \alpha \text{tr}(A).$$

b. Write $B = (b_{ij})$, $C = (c_{kl})$. Then $BC$ is an $m \times m$ matrix with $ij$-entry $\sum_{j=1}^{n} b_{ij} c_{ji}$. Similarly, $CB$ is an $n \times n$ matrix with $kj$-entry $\sum_{l=1}^{m} c_{kl} b_{lj}$. Hence

$$\text{tr}(BC) = \sum_{i=1}^{m} \sum_{j=1}^{n} b_{ij} c_{ji}$$
$$= \sum_{j=1}^{n} \sum_{i=1}^{m} c_{ji} b_{ij}$$
$$= \text{tr}(CB).$$

c. We use the preceding problem with $C = AB^{-1}$.

$$\text{tr}(BAB^{-1}) = \text{tr}(B(AB^{-1}))$$
$$= \text{tr}((AB^{-1})B)$$
$$= \text{tr}(AB^{-1}B)$$
$$= \text{tr}(A).$$

4. Let $v_1, \ldots, v_m$ and $w_1, \ldots, w_n$ be bases of a vector space $V$ (over a field $F$). We know from class that $m = n$ (the number of vectors in a basis of $V$ is uniquely determined). This problem outlines another way to see this (at least for certain fields $F$). Writing the vectors in each basis as linear combinations of vectors in the other, we see that there are scalars $\alpha_{ij}$ and $\beta_{kl}$ such that

$$v_i = \sum_{j=1}^{n} \alpha_{ij} w_j, \quad w_k = \sum_{l=1}^{m} \beta_{kl} v_l.$$

Put $A = (\alpha_{ij})$, $B = (\beta_{kl})$ (so $A$ is an $m \times n$ matrix over $F$ and $B$ is an $n \times m$ matrix over $F$).

a. Show that $AB = I_m$ and $BA = I_n$. 
b. Suppose $F$ contains $Q$. Use $3b$ to show that $m = n$. [In a previous life, you probably studied the row echelon form of a matrix. You can use this to see that $m = n$ without the assumption that $F$ contains $Q$.]

a. We prove only that $AB = I_m$ (the proof that $BA = I_n$ is effectively identical). First recall from class that, by linear independence of $v_1, \ldots, v_m$,

$$\sum_{j=1}^{n} \alpha_j v_j = \sum_{j=1}^{n} \beta_j v_j \quad \text{(for scalars } \alpha_j, \beta_j) \implies \alpha_j = \beta_j, \ j = 1, \ldots, m.$$  

We have $v_i = \sum_{j=1}^{n} \alpha_{ij} w_j$ and $w_k = \sum_{l=1}^{m} \beta_{kl} v_l$. Substituting the second equations in the first, we obtain

$$v_i = \sum_{j=1}^{n} \alpha_{ij} w_j$$

$$= \sum_{j=1}^{n} \alpha_{ij} \left( \sum_{l=1}^{m} \beta_{jl} v_l \right)$$

$$= \sum_{j=1}^{n} \sum_{l=1}^{m} \alpha_{ij} \beta_{jl} v_l$$

$$= \sum_{l=1}^{m} \left( \sum_{j=1}^{n} \alpha_{ij} \beta_{jl} \right) v_l.$$  

We also have $v_i = \sum_{l=1}^{m} \delta_{il} v_l$ where $\delta_{il}$ is Kronecker’s delta symbol, i.e.,

$$\delta_{il} = \begin{cases} 1, & \text{if } i = l, \\ 0, & \text{if } i \neq l. \end{cases}$$  

Thus, by the fundamental fact recalled above,

$$\sum_{j=1}^{n} \alpha_{ij} \beta_{jl} = \delta_{il}, \quad 1 \leq i \leq m, \ 1 \leq l \leq m.$$  

This set of scalar equations is exactly the single matrix equation $AB = I_m$.

b. Since $AB = I_m$, $\text{tr}(AB) = 1 + \cdots + 1$ ($m$ times) and so $\text{tr}(AB) = m$. Similarly, $\text{tr}(BA) = n$. Using $\text{tr}(AB) = \text{tr}(BA)$, we conclude that $m = n$.

5. Consider the real vector space $S$ of all infinite sequences of real numbers (under component-wise operations). The elements of $S$ are infinite tuples $(\alpha_0, \alpha_1, \alpha_2, \ldots)$ which we write simply as $(\alpha_n)$.

a. Show that $S$ is infinite-dimensional.

b. Give an example of a proper subspace of $S$ that is again infinite-dimensional.

c. Let $\mathcal{F}$ denote the set of sequences $(\alpha_n)$ in $S$ such that $\alpha_n = \alpha_{n-1} + \alpha_{n-2}$, for all $n \geq 2$. Put

$$\tau_1 = \frac{1 + \sqrt{5}}{2}, \quad \tau_2 = \frac{1 - \sqrt{5}}{2}.$$
We saw in class that \( w_1 = (\tau^n_1) \) and \( w_2 = (\tau^n_2) \) form a basis of the subspace \( \mathcal{F} \) of \( \mathcal{S} \). Consider the Lucas sequence \( l = (l_n) \) given by
\[
l_0 = 2, \quad l_1 = 1, \quad l_n = l_{n-1} + l_{n-2} \quad (n \geq 2).
\]
Show that \( l = w_1 + w_2 \) in \( \mathcal{F} \) and deduce that
\[
l_n = \tau^n_1 + \tau^n_2 \quad (n \geq 0).
\]

a. For \( k = 0, 1, \ldots \), we set \( e_k = (\delta_{kn}) \). Thus \( e_k \) has 0 in all positions except the \( k \)-th in which it has a 1. For any positive integer \( N \), the vectors \( e_0, e_1, \ldots e_N \) are linearly independent. Indeed, for any scalars \( \alpha_0, \alpha_1, \ldots, \alpha_N \), we have
\[
\alpha_0 e_0 + \alpha_1 e_1 + \cdots + \alpha_N e_N = (\alpha_0, \alpha_1, \ldots, \alpha_N, 0, \ldots),
\]
so that
\[
\alpha_0 e_0 + \alpha_1 e_1 + \cdots + \alpha_N e_N = 0 \text{ in } \mathcal{S} \implies \alpha_0 = 0, \alpha_1 = 0, \ldots, \alpha_N = 0.
\]
It follows that the dimension of \( \mathcal{S} \) cannot be less than \( N + 1 \) (otherwise, by Theorem 1 from class, any list of \( N + 1 \) vectors in \( \mathcal{S} \), in particular \( e_0, e_1, \ldots e_N \), would be linearly dependent). Since \( N \) is arbitrary, we conclude that \( \mathcal{S} \) must be infinite-dimensional.

b. Let \( \mathcal{S}_{\text{fin}} \) denote the set of all sequences in \( \mathcal{S} \) with only finitely many non-zero terms. In symbols,
\[
\mathcal{S}_{\text{fin}} = \{ s = (\alpha_n) \mid \exists N = N_s \text{ such that } \alpha_n = 0, \forall n > N \}.
\]
Check that this gives a subspace of \( \mathcal{S} \). Since \( \mathcal{S}_{\text{fin}} \) contains the vectors \( e_k \), for \( k = 0, 1, \ldots \), the argument of part a shows that \( \mathcal{S}_{\text{fin}} \) is infinite-dimensional.

c. Since \( w_1, w_2 \) is a basis of \( \mathcal{F} \), there exist unique real numbers \( \alpha \) and \( \beta \) such that
\[
l = \alpha w_1 + \beta w_2.
\]
Equivalently,
\[
l_n = \alpha \tau^n_1 + \beta \tau^n_2, \quad n = 0, 1, \ldots
\]
Taking \( n = 0 \) and \( n = 1 \),
\[
2 = \alpha + \beta,
\]
\[
1 = \alpha \tau_1 + \beta \tau_2.
\]
Solving for \( \alpha \) and \( \beta \), we obtain \( \alpha = 1, \beta = 1 \) (or simply note that this gives a solution of the equations). Hence \( l = w_1 + w_2 \) and so
\[
l_n = \tau^n_1 + \tau^n_2 \quad (n \geq 0).
\]