1. a. Consider the set $S$ of all binary sequences. (An element of $S$ is thus a sequence $s_1, s_2, s_3, \ldots$ in which each $s_i = 0$ or 1.) By using Cantor’s diagonal method, or otherwise, show that $S$ is uncountable.

b. Prove that the set of all functions $f : \mathbb{N} \to \mathbb{N}$ is uncountable.

c. Show that the set of all finite subsets of $\mathbb{N}$ is countable.

a. Let $s_0, s_1, s_2, \ldots$ be a sequence of elements of $S$. Following Cantor, we will construct an element $t$ of $S$ such that $t \neq s_i$ (for any non-negative integer $i$). Write $s_{ij}$ for the $j$-th term of $s_i$ so that $s_i$ is the sequence $s_{i0}, s_{i1}, s_{i2}, \ldots$. We define $t = t_0, t_1, t_2, \ldots$ by

$$t_i = \begin{cases} 0 & \text{if } s_{ii} = 1, \\ 1 & \text{if } s_{ii} = 0. \end{cases}$$

By construction, $t \neq s_i$ (for any $i$) as the sequences differ in the $i$-th position. Thus no sequence of elements of $S$ can include all elements of $S$, that is, $S$ is uncountable.

b. Write $F$ for the set of all functions $f : \mathbb{N} \to \mathbb{N}$.

Method 1. We use a variant of the argument in part a. Let $f_0, f_1, f_2, \ldots$ be any sequence of elements of $F$ and define $f \in F$ by

$$f(i) = \begin{cases} 0 & \text{if } f_i(i) = 1, \\ 1 & \text{if } f_i(i) \neq 1. \end{cases}$$

By construction, $f(i) \neq f_i(i)$ and so $f \neq f_i$ (for any $i \in \mathbb{N}$). Thus $F$ is uncountable.

Method 2. Consider the subset $B$ of $F$ consisting of all $f \in F$ such that $f(i) = 0$ or 1 (for all $i \in \mathbb{N}$). We can view $B$ as the set $S$ of all binary sequences. More formally, given $f \in B$ if we write $s_f$ for the binary sequence $f(0), f(1), f(2), \ldots$ then the map

$$f \mapsto s_f : B \to S$$

is a bijection. Thus $B$ is uncountable. We noted in class that any set with an uncountable subset is uncountable (equivalently, any subset of a countable set is countable). Hence $F$ is uncountable.

c. A subset of $\mathbb{N}$ is finite if and only if it is contained in $\{0, 1, \ldots, n\}$ for some $n \in \mathbb{N}$. Thus we may list the finite subsets of $\mathbb{N}$ as follows. First list the subsets of $\{0\}$ (namely $\emptyset$ and $\{0\}$), then the subsets of $\{0, 1\}$ excluding those already listed (namely $\{1\}$ and $\{0, 1\}$), next the subsets of $\{0, 1, 2\}$ again leaving out those already listed. Continuing in this way, we obtain a sequence that includes each finite subset of $\mathbb{N}$ exactly once. This gives a bijection between $\mathbb{N}$ and the set of finite subsets of $\mathbb{N}$ and so this set is countable.

2. a. State a form of the principle of mathematical induction.

b. Show that $4^n - 1$ is divisible by 3, for all positive integers $n$. 
c. Prove that, for every integer \( n \geq 1 \),
\[
1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} \leq 2\sqrt{n}.
\]
a. **Version 1.** Let \( S \) be a subset of \( \mathbb{Z}_+ \) such that (i) \( 1 \in S \) and (ii) for any positive integer \( k \), if \( k \in S \) then also \( k + 1 \in S \). Then \( S = \mathbb{Z}_+ \).

**Version 2.** Let \( P_1, P_2, P_3, \ldots \) be a sequence of statements (one for each positive integer). Suppose (i) \( P_1 \) is true and (ii) for any positive integer \( k \), \( P_k \Rightarrow P_{k+1} \) (that is, if \( P_k \) is true then \( P_{k+1} \) is also true). Then \( P_n \) is true for all \( n \).

b. Let \( P(n) \) be the statement that \( 3 \mid 4^n - 1 \). We’ll use induction to show that \( P(n) \) holds for all positive integers \( n \). Clearly, \( 3 \mid 4^1 - 1 = 3 \), so \( P(1) \) is true. Now assume \( P(k) \) holds for some positive integer \( k \). We need to show that \( P(k+1) \) also holds, i.e., that \( 3 \mid 4^{k+1} - 1 \) (given that \( 3 \mid 4^k - 1 \)). Now
\[
4^{k+1} - 1 = 4 \times 4^k + 1 = (3 + 1) \times 4^k - 1 \equiv 3 \times 4^k + (4^k - 1).
\]

By hypothesis, \( 3 \mid 4^k - 1 \) and obviously \( 3 \mid 3 \times 4^k \). Hence \( 3 \mid 4^{k+1} - 1 \), so \( P(k+1) \) holds. By induction, \( 3 \mid 4^n - 1 \), for all positive integers \( n \).

c. Again we use induction. Clearly, \( 1 \leq 2\sqrt{1} = 2 \) (base case). We assume
\[
1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{k}} \leq 2\sqrt{k},
\]
for a positive integer \( k \) (the inductive hypothesis) and need to show that
\[
1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} \leq 2\sqrt{k+1}
\]
(the inductive step). By the inductive hypothesis,
\[
1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} \leq 2\sqrt{k} + \frac{1}{\sqrt{k+1}}.
\]
It suffices therefore to show that
\[
2\sqrt{k} + \frac{1}{\sqrt{k+1}} \leq 2\sqrt{k+1}. \tag{\ast}
\]
Clearing the denominator, we see that this holds if and only if
\[
2\sqrt{k}\sqrt{k+1} + 1 \leq 2(k+1) = 2k + 2,
\]
that is,
\[
2\sqrt{k}\sqrt{k+1} \leq 2k + 1.
\]
Since each side is positive, this inequality is equivalent to the one obtained by squaring, i.e.,
\[
4k(k+1) \leq (2k + 1)^2,
\]
or
\[
4k^2 + 4k \leq 4k^2 + 4k + 1
\]
which clearly holds for all positive integers \( k \). This proves (\ast) and so establishes the inductive step. Hence, by induction,
\[
1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} \leq 2\sqrt{n},
\]
for all positive integers $n$.

3. a. Consider the Fibonacci sequence $1, 1, 2, 3, 5, 8, \ldots$, given by

$$f_1 = 1,$$
$$f_2 = 1,$$
$$f_n = f_{n-1} + f_{n-2}, \text{ for } n > 2.$$  

i. Show that $f_{3n}$ is even, for all natural numbers $n$.

ii. Show that $(f_n, f_{n+1}) = 1$, for all natural numbers $n$.

b. i. Find the greatest common divisor $d$ of 131 and 122 and find integers $x$ and $y$ such that $d = 131x + 122y$.

ii. Find the greatest common divisor of 111111 and 1111.

a. i. We show by induction that $f_{3n}$ is even, for all $n$. Clearly, $f_3 = 2$ is even. Assume that $f_{3k}$ is even, for some $k$. Then

$$f_{3(k+1)} = f_{3k+3}$$
$$= f_{3k+2} + f_{3k+1}$$
$$= (f_{3k+1} + f_{3k}) + f_{3k+1}$$
$$= 2f_{3k+1} + f_{3k},$$

so that $f_{3(k+1)}$ is also even. Hence, by induction, $f_{3n}$ is even, for all $n$.

ii. Again we use induction to show that $(f_n, f_{n+1}) = 1$, for all positive integers $n$. Since $f_1 = f_2 = 1$, it’s clear that $(f_1, f_2) = 1$. Suppose $(f_k, f_{k+1}) = 1$, for some $k$. We have

$$f_{k+2} = f_k + f_{k+1} \text{ or } f_k = f_{k+2} - f_{k+1}.$$  

Thus if $d = (f_{k+1}, f_{k+2})$ then $d \mid f_k$. By definition, $d \mid f_{k+1}$. Hence $d \mid (f_k, f_{k+1}) = 1$ and $d = 1$. We conclude that $(f_n, f_{n+1}) = 1$, for all $n$.

b. i. We apply Euclid’s algorithm:

$$131 = 1 \times 122 + 9$$
$$122 = 13 \times 9 + 5$$
$$9 = 1 \times 5 + 4$$
$$5 = 1 \times 4 + 1$$

and hence $(122, 131) = 1$. Working backwards,

$$1 = 5 - 4$$
$$= 5 - (9 - 5)$$
$$= 2 \times 5 - 9$$
$$= 2 \times (122 - 13 \times 9) - 9$$
$$= 2 \times 122 - 27 \times 9$$
$$= 2 \times 122 - 27 \times (131 - 122)$$
$$= 29 \times 122 - 27 \times 131.$$
ii. We have

\[
\begin{align*}
111111 &= 100 \times 1111 + 11 \\
1111 &= 101 \times 11
\end{align*}
\]

and thus \((111111, 1111) = 11.\)

4. a. What day of the week will it be in \(10^{1,000}\) days from today?

b. Show that for any integer \(N\) exactly one of the numbers \(N, N + 1, N + 2\) is divisible by 3.

c. Let \(n\) be an odd integer. Prove that \(n(n^2 - 1) \equiv 0 \pmod{24}.\)

\[a.\] We need to find the remainder when \(10^{1000}\) is divided by 7, that is, the unique integer \(r\) with \(0 \leq r < 7\) such that \(10^{1000} \equiv r \pmod{7}.\) We have \(10 \equiv 3 \pmod{7},\) so \(10^{1000} \equiv 3^{1000} \pmod{7}.\) Further \(3^{2} \equiv 2 \pmod{7}\) and thus \(3^{3} = 3^{2} \times 3 \equiv 2 \times 3 = 6 \equiv -1 \pmod{7}.\)

Using \(1000 = 3 \times 333 + 1,\) it follows that

\[
\begin{align*}
3^{1000} &\equiv (3^{3})^{333} 3 \pmod{7} \\
&\equiv (-1)^{333} 3 \pmod{7} \\
&\equiv -3 \pmod{7} \\
&\equiv 4 \pmod{7}.
\end{align*}
\]

So in \(10^{1000}\) days' time, it will be four days later in the week. Thus the answer is Monday if today is Thursday.

\[b.\] Let \(r\) be the unique integer with \(0 \leq r < 3\) such that \(N \equiv r \pmod{3}\) – in other words, \(r\) is the remainder of \(N\) on division by 3. If \(r = 0,\) then \(N + 1 \equiv 1 \pmod{3}\) and \(N + 2 \equiv 2 \pmod{3},\) so \(3 \mid N\) but \(3 \nmid N + 1\) and \(3 \nmid N + 2.\) Similarly, if \(r = 1\) then \(N + 2 \equiv 3 \equiv 0 \pmod{3},\) so \(3 \mid N + 2\) but \(3 \nmid N\) and \(3 \nmid N + 1.\)

By the same reasoning, \(r = 2\) implies \(3 \mid N + 1\) but \(3 \nmid N\) and \(3 \nmid N + 1.\)

\[c.\] We want to show that \(24 \mid n(n^2 - 1)\) for \(n\) odd. Since \(24 = 3 \times 8\) and \((3, 8) = 1,\) it suffices to show that \(3 \mid n(n^2 - 1)\) and \(8 \mid n(n^2 - 1).\) Now \(n(n^2 - 1) = n(n - 1)(n + 1)\) is a product of three consecutive integers, so it follows from part \(b\) that \(3 \mid n(n^2 - 1).\)

Note this holds for any integer \(n,\) odd or even: the oddness of \(n\) is relevant only to the proof that \(8 \mid n(n^2 - 1).\) Write \(n = 2m + 1\) for some integer \(m.\) Then

\[
\begin{align*}
n(n^2 - 1) &= n(n - 1)(n + 1) \\
&= (2m + 1)2m(2m + 2) \\
&= 4(2m + 1)m(m + 1).
\end{align*}
\]

Now \(m(m + 1)\) is even for any integer \(m\) and thus \(4 \times 2 = 8\) divides \(4(2m+1)m(m+1),\) as required.