Math 4853
Review Problems for Exam 1 — Answers

NOTE: The concepts in problems 6 and 7 are not covered on Exam 1.

PROBLEM 1. Let $f : X \to Y$ be a function.
(a) Give an explicit example that shows that if $A_1$ and $A_2$ are subsets of $X$ then $f(A_1 \cap A_2) \neq f(A_1) \cap f(A_2)$.
(b) Using basic definitions, prove that $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$ where $B_1$ and $B_2$ are subsets of $Y$.

Answer:
(a) There are lots of possible examples (as long as $f : X \to Y$ is not one-to-one). For instance, let $f : \mathbb{R} \to \mathbb{R}$ be the constant function $f(x) = 0$ for all $x \in \mathbb{R}$, and take $A_1 = [0,1]$ and $A_2 = [3,5]$. Then $f(A_1 \cap A_2) = f(\emptyset) = \emptyset$ but $f(A_1) \cap f(A_2) = \emptyset \cap \emptyset = \emptyset \neq \emptyset$.
(b) Let $f : X \to Y$ be a function and let $B_1$ and $B_2$ be subsets of $Y$. Then using the definitions of inverse image and intersection

$$ f^{-1}(B_1 \cap B_2) = \{ x \in X \mid f(x) \in B_1 \cap B_2 \} $$
$$ = \{ x \in X \mid f(x) \in B_1 \text{ and } f(x) \in B_2 \} $$
$$ = \{ x \in X \mid f(x) \in B_1 \} \cap \{ x \in X \mid f(x) \in B_2 \} $$
$$ = f^{-1}(B_1) \cap f^{-1}(B_2). $$

PROBLEM 2. In class we mentioned that the discrete topology on any set $X$ is finer than the trivial topology on $X$, however it is not strictly finer for some sets $X$. State and prove a theorem which precisely identifies those sets $X$ for which the discrete topology is strictly finer than the trivial topology.

Answer:

**Theorem 1.** The discrete topology $T_{\text{discrete}}$ on a set $X$ is strictly finer than the trivial topology $T_{\text{trivial}}$ if and only if $X$ contains more than one element.

**Proof.** Let $X$ be a set. Recall that the trivial topology on $X$ is $T_{\text{trivial}} = \{ \emptyset, X \}$ and that the discrete topology on $X$ is $T_{\text{discrete}} = \{ \{ \} \cup \mathcal{P}(X) \}$. Clearly $T_{\text{trivial}}$ is a subset of $T_{\text{discrete}}$ so the discrete topology on $X$ is always finer than the trivial topology.

Suppose that $X$ contains more than one element. Let $x_0$ be an element of $X$. Then the singleton set $\{ x_0 \}$ is not empty and not equal to $X$ (since $X$ has at least one element different than $x_0$), and so it is not open in the trivial topology. However $\{ x_0 \}$ is open in the discrete topology (since it is a subset of $X$). Therefore $\{ x_0 \} \in T_{\text{discrete}}$ but $\{ x_0 \} \notin T_{\text{trivial}}$, which shows that the discrete topology is strictly finer than the trivial topology.

Suppose that the discrete topology on $X$ is strictly finer than the trivial topology on $X$. Then there is a subset $U \subseteq X$ which is open in the discrete topology but not open in the trivial topology. By the definition of $T_{\text{trivial}}$, this means that $U$ is nonempty and not equal to $X$. It follows that both $U$ and $X - U$ are nonempty. Choose $x_1 \in U$ and $x_2 \in X - U$. Then $x_1$ and $x_2$ are two distinct elements of $X$, and this shows that $X$ contains more than one element.

PROBLEM 3. (a) Using the definition of closed set, prove that if $C$ is a closed subset of a topological space $(X,T)$ then its complement $X - C$ is an open set.
(b) Show that under the discrete topology every subset of $X$ is open.


**Answer:**

(a) Let \((X, T)\) be a topological space. Suppose that \(C\) is a closed subset of \(X\). By definition of ‘closed set’, this means that \(X - C\) is an open set. So part (a) is just the definition of closed set! [ASIDE: Notice that that the complement of an open set is a closed set: If \(U\) is an open set and \(C = X - U\) then \(X - C = X - (X - U) = U\) and since \(U\) is open \(C\) must be closed.]

(b) Consider the discrete topology \(T_{\text{discrete}} = \{U \mid U \subseteq X\} = \mathcal{P}(X)\) (the power set of \(X\)) on a set \(X\). Let \(A\) be a subset of \(X\). Then \(X - A\) is a subset of \(X\) so it is an open set in the discrete topology. By definition this shows that \(A\) is a closed set.

**Problem 4.** Let \(X\) be the 3-element set \(X = \{a, b, c\}\).

(a) Is there any topology \(T\) on \(X\) that has precisely three elements in it? If so, then list all of the possibilities.

(b) Is there any topology \(T\) on \(X\) that has precisely seven elements in it? If so, then list all of the possibilities.

**Answer:** Let \(X = \{a, b, c\}\).

(a) A topology on \(X\) always contains \(\emptyset\) and \(X\) (by axioms (T1) and (T2)), and since \(X \neq \emptyset\) this are two distinct subsets that must be in \(T\). If \(A\) is any third subset of \(X\) (that is \(A \neq \emptyset\) and \(A \neq X\) then \(T = \{\emptyset, A, X\}\) can easily be seen to form a topology on \(X\). There are six such subsets \(A\) in \(X = \{a, b, c\}\), so this leads to six different topologies on \(X\) with precisely three elements. The possibilities for \(A\) are

\[
\{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \text{ and } \{b, c\}.
\]

(b) The power set \(\mathcal{P}(X)\) of \(X = \{a, b, c\}\) has eight elements. These are \(\emptyset, X\) and the six sets listed in part (a). A collection \(T\) of seven subsets is obtained by removing one of these sets from \(\mathcal{P}(X)\). If the removed set is \(\emptyset\) or \(X\) then \(T\) will not satisfy the (T1) or (T2) axiom. So the removed set must be one of the six sets listed in (a). If the removed set is a singleton then the (T4) axiom will fail (for instance, if the removed set is \(\{a\}\) then \(\{a, b\}\) and \(\{a, c\}\) are in \(T\) but their intersection is not). If the removed set is a doubleton then the (T3) axiom will fail (for instance, if the removed set is \(\{a, b\}\) then \(\{a\}\) and \(\{b\}\) are in \(T\) but their union is not). This shows that no matter which subset is removed to form \(T\), \(T\) will not be topology. From this we conclude that there is no topology on this set \(X\) which has precisely seven elements.

**Problem 5.** Let \((X, T_1), (Y, T_2)\) and \((Z, T_3)\) be topological spaces and suppose that \(f : X \to Y\) and \(g : Y \to Z\) are functions which are continuous with respect to these topologies. Prove that the composition function \(g \circ f : X \to Z\) is continuous.

**Answer:** Let \((X, T_1), (Y, T_2)\) and \((Z, T_3)\) be topological spaces and suppose that \(f : X \to Y\) and \(g : Y \to Z\) are functions which are continuous with respect to these topologies. Suppose \(U \in T_3\). Then

\[
g \circ f^{-1}(U) = \{x \in X \mid g \circ f(x) \in U\} = \{x \in X \mid g(f(x)) \in U\} = \{x \in X \mid f(x) \in g^{-1}(U)\} = f^{-1}(g^{-1}(U)).
\]

Since \(U \in T_3\) and \(g\) is continuous, \(g^{-1}(U) \in T_2\). Since \(g^{-1}(U) \in T_2\) and \(f\) is continuous, \(f^{-1}(g^{-1}(U)) \in T_1\). This shows that \(g \circ f^{-1}(U) \in T_1\) and that \(g \circ f\) is continuous.

**Problem 6.** Using only the definition of the Euclidean topology on \(\mathbb{R}\) and the definition of limit point, give complete justifications for each of the following:

(a) Show that 5 is not a limit point for the set \(A = (-1, 4]\) in the Euclidean line \((\mathbb{R}, T_{\text{euclid}})\).

(b) Show that \(-1\) is a limit point for the set \(A = (-1, 4]\) in the Euclidean line \((\mathbb{R}, T_{\text{euclid}})\).

**Answer:** Recall the definition of limit point: Let \((X, T)\) be a topological space, let \(A\) be a subset of \(X\), and let \(x\) be an element of \(X\). Then \(x\) is a limit point of \(A\) provided that every neighborhood of \(x\) contains an element of \(A\) other than \(x\) itself.
(a) In the Euclidean topology on the real line \( \mathbb{R} \), the open interval \((9/2, 11/2)\) is a neighborhood of 5 (because open intervals are open in the Euclidean topology and \(5 \in (9/2, 11/2)\)). However, this neighborhood contains no elements of \(A\) since \((-1, 4) \cap (9/2, 11/2) = \emptyset\). This shows that 5 is not a limit point of \((-1, 4]\).

(b) Let \(U\) be a neighborhood of \(-1\) in the Euclidean line. By the definition of the Euclidean topology, this means that there is a positive real number \(\epsilon\) such that \((-1-\epsilon, \neg 1+\epsilon) = B(-1, \epsilon) \subseteq U\). This open interval \((-1-\epsilon, \neg 1+\epsilon)\) must contain a real number \(t\) with \(-1 < t \leq 4\). Specifically, take \(t\) to be the smaller of the two numbers \(-1 + \epsilon/2\) and 4. Then \(t\) is an element of \(U\) which is contained in \((-1, 4]\) but not equal to \(-1\). Since we assumed \(U\) to be an arbitrary neighborhood of \(-1\) this verifies that \(-1\) is a limit point of \((-1, 4]\).

**Problem 7.** Consider the excluded point topology \(T = \{\mathbb{R}\} \cup \{U \subset \mathbb{R} \mid \emptyset \notin U\}\) on the set of real numbers \(\mathbb{R}\).

(a) Describe the closure \(cl(A)\) of any subset \(A \subseteq \mathbb{R}\) in this topology.

(b) Describe the set of limit points \(A'\) of any subset \(A \subseteq \mathbb{R}\) in this topology.

(c) Determine whether or not this topological space is connected and prove your assertion.

**Answer:**
(a) The closed sets in the excluded point topology on \(\mathbb{R}\) are the empty set and any subset of \(\mathbb{R}\) which contains 0. Therefore (since the closure of a set \(A\) is the smallest closed set containing \(A\)) we see that \(cl(A) = \emptyset\) when \(A = \emptyset\) and that \(cl(A) = A \cup \{0\}\) when \(A \neq \emptyset\).

(b) Suppose that \(A\) is a subset of \(\mathbb{R}\) which is nonempty and not equal to \(\{0\}\). Then 0 is a limit point of \(A\) because \(\mathbb{R}\) (which is the only neighborhood of 0 in this topology) contains an element of \(A\) other than 0. However no real number \(x\) different than 0 can be a limit point of \(A\) because the singleton \(\{x\}\) is a neighborhood of \(x\) which doesn’t contain an element of \(A\) different than \(x\). This shows that if \(A\) is neither empty nor the singleton subset \(\{0\}\) then \(A' = \{0\}\). On the other hand, if \(A = \emptyset\) or \(A = \{0\}\) then \(A' = \emptyset\).

**Problem 8.** Let \((X, T)\) be a topological space. We say \((X, T)\) is a \(T_1\)-space provided that every singleton set \(\{x\}\) where \(x_1 \in X\) is a closed set. We say \((X, T)\) is a Hausdorff space for each pair of distinct elements \(x_1, x_2 \in X\) there are neighborhoods \(U_1\) of \(x_1\) and \(U_2\) of \(x_2\) such that \(U_1 \cap U_2 = \emptyset\).

(a) Prove that every Hausdorff space is a \(T_1\)-space.

(b) Prove that if \((X, T)\) is a \(T_1\)-space where \(X\) is a finite set then \(T\) is the discrete topology.

(c) Show that the cofinite topology \(T_{cofinite}\) on the set of integers \(\mathbb{Z}\) is a \(T_1\)-space but not Hausdorff.

**Answer:**
(a) Let \((X, T)\) be a Hausdorff topological space, and let \(x\) be an element of \(X\). Suppose that \(y \in X - \{x\}\). Then \(y \neq x\), so there must be neighborhoods \(U_1\) of \(x\) and \(U_2\) of \(y\) with \(U_1 \cap U_2 = \emptyset\). In particular, \(x \notin U_2\). This shows that each element \(y\) in \(X - \{x\}\) has a neighborhood \(U_2\) such that \(U_2 \subseteq X - \{x\}\). By theorem 5.3 of the class notes, it follows that \(X - \{x\}\) is an open set. This means that \(\{x\}\) is a closed set. As \(x\) was an arbitrary element of \(X\), this shows that \((X, T)\) is a \(T_1\)-space.

(b) Suppose that \(X\) is finite and that \((X, T_1)\) is a \(T_1\)-space. Then any subset \(A\) of \(X\) is finite and can be expressed as a finite union of singleton subsets. Since singleton subsets are closed and a finite union of closed sets is closed in a topological space, this shows that \(A\) is closed. Therefore every subset of \(X\) is closed. From this it follows that every subset of \(X\) is open (since its complement is closed), and that \(T = T_{discrete}\).

(c) In the cofinite topology \(T_{cofinite}\) on the set of integers \(\mathbb{Z}\) the collection of closed subsets is

\[ C = \{\mathbb{Z}\} \cup \{F \subset \mathbb{Z} \mid F \text{ is finite}\} \]

In particular, each singleton subset of \(\mathbb{Z}\) is finite and therefore closed in \(T_{cofinite}\). This shows that \((\mathbb{Z}, T_{cofinite})\) is a \(T_1\)-space. To see that it is not Hausdorff, let \(x_1\) and \(x_2\) be distinct integers and let \(U_1\) be a neighborhood of \(x_1\) and let \(U_2\) be a neighborhood of \(x_2\). This means that \(U_1 = \mathbb{Z} - F_1\) for some finite subset \(F_1 \subset \mathbb{Z}\) and that \(U_2 = \mathbb{Z} - F_2\) for some finite subset \(F_2 \subset \mathbb{Z}\). Then

\[ U_1 \cap U_2 = (\mathbb{Z} - F_1) \cap (\mathbb{Z} - F_2) = \mathbb{Z} - (F_1 \cup F_2) \].
So if \( U_1 \cap U_2 = \emptyset \) then \( Z = F_1 \cup F_2 \) which is impossible because \( F_1 \cup F_2 \) is finite but \( Z \) is infinite. Therefore \( U_1 \cap U_2 \) can’t be the empty set, and this shows that \( (Z, T_{cofinite}) \) is not a Hausdorff space.