Research Statement

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My research interests are in the area of topology and geometry. My dissertation establishes that for any 4-dimensional infra-solvmanifold M, there is a compact 5-dimensional manifold W with $\partial W = M$. It also completes the classification of crystallographic groups of the solvable 4-dimensional geometries which satisfy Bieberbach's First Theorem. Curvature properties of the 4-dimensional solvable geometries are also investigated.

The final section of this document demonstrates how undergraduates could get involved in my research program.

1. Definitions and Background

For our purposes, we will assume all Lie groups to be connected and simply connected. Let G be a solvable Lie group. The group of *affine diffeomorphisms* of G is:

$$\mathrm{Aff}(G) := G \rtimes \mathrm{Aut}(G),$$

which has group operation $(a, A)(b, B) = (aA(b), AB)$. There is an action of $\mathrm{Aff}(G)$ on G. For $(a, A) \in \mathrm{Aff}(G)$ and $g \in G$,

$$(a, A).g := aA(g).$$

Under this action, the subgroup $G \cong \{(g, \mathrm{id}) \mid g \in G\} \subset \mathrm{Aff}(G)$ acts on G as left translations.

Definition 1.1. Let K be a maximal compact subgroup of $\mathrm{Aut}(G)$. A discrete subgroup Π of $G \rtimes K \subset \mathrm{Aff}(G)$ is called a *crystallographic group* of G when

(1) The quotient $\Pi \backslash G$ is compact, and
(2) the translation subgroup $\Gamma := \Pi \cap G$ is of finite index in Π.

The translation subgroup Γ is normal in Π, and we refer to the finite group $\Phi := \Pi/\Gamma$ as the *holonomy group* of Π. Thus, a crystallographic group Π fits the short exact sequence

$$1 \to \Gamma \to \Pi \to \Phi \to 1.$$

Condition (2) of Definition 1.1 implies that Γ is a discrete subgroup of G and that $\Gamma\backslash G$ is compact. That is, Γ is a *cocompact lattice* of G. Since Γ acts as left translations on G, it acts freely on G, and we say that the quotient $\Gamma\backslash G$ is a *solvmanifold* when G is solvable, and a *nilmanifold* when G is nilpotent.

The quotient $\Pi\backslash G$ is a closed manifold precisely when Π acts freely on G. This is equivalent to Π being torsion free.

Definition 1.2. Let Π be a torsion free crystallographic group of G. The quotient $\Pi\backslash G$ is an *infra-solvmanifold* when G is solvable, and an *infra-nilmanifold* when G is nilpotent.

Note that with the Euclidean metric, $\mathrm{Isom}(\mathbb{R}^n) = \mathbb{R}^n \rtimes O(n, \mathbb{R})$, so an infra-solvmanifold for $G = \mathbb{R}^n$ and $K = O(n, \mathbb{R})$ is simply a closed flat manifold.

Topologically, Condition (2) of Definition 1.1 states that an infra-solvmanifold $\Pi\backslash G$ is finitely covered by the solvmanifold $\Gamma\backslash G$ with covering transformation group Φ. This explains the prefix "infra" and generalizes the classical result that any closed flat n-manifold is finitely covered by a flat torus $\mathbb{Z}^n\backslash\mathbb{R}^n$. Thus, in our context, infra-solvmanifolds generalize closed flat manifolds, and solvmanifolds generalize flat tori.

When G is nilpotent, Auslander proved that Condition (1) of Definition 1.1 actually implies Condition (2). When G is abelian, this is due to Bieberbach.

Theorem 1.3. [19, Bieberbach's First Theorem in the nilpotent case, Theorem 8.3.2] *Let*

$$\Pi \subset G \rtimes K$$

be discrete and such that $\Pi \backslash G$ *is compact. Then the translation subgroup* $\Gamma := \Pi \cap G$ *is of finite index in* G*, and the quotient* $\Gamma \backslash G$ *is compact.*

When G is solvable, Condition (2) in Definition 1.1 is needed, because there are examples of solvable G and discrete subgroups $\Pi \subset G \rtimes K$ with $\Pi \backslash G$ compact, for which $\Pi \cap G$ is *not* of finite index in Π. In other words, Bieberbach's First Theorem does *not* extend to all solvable Lie groups.

2. Results

It is a remarkable theorem of Hamrick and Royster that for every closed flat n-manifold, M, there is an $(n+1)$-dimensional compact manifold W with $\partial W = M$ [10]. That is, M *bounds.* Thus, it seems natural to conjecture that all infra-nilmanifolds bound.

Conjecture 2.1. [6, Conjecture 1] If M is an n-dimensional infra-nilmanifold, then there exists a compact $(n+1)$-dimensional manifold W with $\partial W = M$.

Some partial results are known [30, 3]. Jonathan Hillman asked if all 4-dimensional infra-solvmanifolds bound [13]. My dissertation answers this affirmatively.

Theorem 2.2. [27, 26, **T**] *If* M *is a 4-dimensional infra-solvmanifold, then* M *bounds. That is, there is a compact 5-dimensional manifold* W *with* $\partial W = M$*.*

Wall gives a complete list of the 4-dimensional geometries in [32]. This is analogous to Thurston's eight 3-dimensional geometries [29, 23]. Seven of the 4-dimensional geometries are solvable Lie groups: $\mathbb{R}^4$, $\mathrm{Nil}^3 \times \mathbb{R}$, Nil^4, $\mathrm{Sol}^3 \times \mathbb{R}$, $\mathrm{Sol}^4_{m,n}$, Sol_1^4, and Sol_0^4. Bieberbach's First Theorem (Theorem 1.3) holds for all except Sol_0^4.

By the work of Hillman, any 4-dimensional infra-solvmanifold M is diffeomorphic to a compact isometric quotient of one of the 4-dimensional solvable geometries. Namely, any of the 4-dimensional solvable geometries admits a left invariant metric with

$$\mathrm{Isom}(G) = G \rtimes K \subset \mathrm{Aff}(G),$$

for a maximal compact subgroup $K \subset \mathrm{Aut}(G)$. Any 4-dimensional infra-solvmanifold M is diffeomorphic to $\Pi \backslash G$, for some 4-dimensional solvable geometry G, and some $\Pi \subset \mathrm{Isom}(G)$ [11, Theorem 8]. Therefore, the crystallographic groups of the seven 4-dimensional solvable geometries are of particular interest.

The crystallographic groups of $\mathbb{R}^4$ are classified in [1], while Dekimpe has classified those of $\mathrm{Nil}^3 \times \mathbb{R}$ and Nil^4 [4]. The classification of crystallographic groups of Sol_1^4 is joint with K.B. Lee [20]. The torsion free crystallographic groups of $\mathrm{Sol}^4_{m,n}$ and $\mathrm{Sol}^3 \times \mathbb{R}$ have been classified by Hillman [12, 13].

In my dissertation, a complete classification of the crystallographic groups of $\mathrm{Sol}^4_{m,n}$ and $\mathrm{Sol}^3 \times \mathbb{R}$, up to isomorphism, is given. This classfication includes the crystallographic groups with torsion, and it was inspired by my joint work with K.B. Lee [20]. Easily checked criteria for such groups to be torsion free are also provided. As corollaries of these classifications,

Corollary 2.3. [27, **T**] *Let Π be a crystallographic group of* $\mathrm{Sol}^4_{m,n}$. *The possible holonomy groups of Π are the subgroups of $\mathbb{Z}_2 \times \mathbb{Z}_2$. If Π is torsion free, so that $\Pi\backslash G$ is an infra-solvmanifold, Π must have holonomy $\{e\}$ or $\mathbb{Z}_2$.*

Corollary 2.4. [27, **T**] *Let Π be a crystallographic group of* $\mathrm{Sol}^3 \times \mathbb{R}$. *Any subgroup of $D_4 \times \mathbb{Z}_2$ can be the holonomy group of Π, where D_4 is the dihedral group of 8 elements. If Π is torsion free, so that $\Pi\backslash G$ is an infra-solvmanifold, Π must have holonomy $\{e\}$, $\mathbb{Z}_2$, $\mathbb{Z}_2^2$, $\mathbb{Z}_2^3$, $\mathbb{Z}_4$, or D_4.*

We now consider curvature of the 4-dimensional solvable geometries. Let G be a 4-dimensional solvable geometry. With arbitrary left invariant metric on G,

$$\mathrm{Isom}(G) \subseteq G \rtimes K \subset \mathrm{Aff}(G),$$

for a maximal compact subgroup $K \subset \mathrm{Aut}(G)$ [7]. Depending on choice of left invariant metric, a 4-dimensional solvable geometry can have different isometry groups and Ricci signatures. Ricci signature is a measure of curvature of the metric. Here are the possible Ricci signatures on some of the 4-dimensional solvable geometries [16]:

G	Possible Ricci signatures
$\mathbb{R}^4$	$(0,0,0,0)$
$\mathrm{Nil}^3 \times \mathbb{R}$	$(0,+,-,-)$
Nil^4	$(0,+,-,-), (+,+,-,-), (+,-,-,-)$
$\mathrm{Sol}^3 \times \mathbb{R}$	$(0,0,0,-), (0,+,-,-), (+,+,-,-), (+,-,-,-)$
$\mathrm{Sol}^4_{m,n}$	$(0,0,0,-), (0,+,-,-), (+,+,-,-), (+,-,-,-)$
Sol_0^4	$(0,0,0,-), (0,+,-,-)$

Suppose we only consider left invariant metrics on G that induce a particular Ricci signature. The compact isometric quotients of G may not account for all infra-solvmanifolds of G. The reason is that among all left invariant metrics with prescribed Ricci signature, $\mathrm{Isom}(G)$ may be a *proper* subgroup of $G \rtimes K$, for a maximal compact subgroup K of $\mathrm{Aut}(G)$.

Given an infra-solvmanifold M of G, a Ricci signature can be *realized* on M if there is a left invariant metric on G with prescribed Ricci signature such that M is diffeomorphic to $\Pi\backslash G$, for some $\Pi \subset \mathrm{Isom}(G)$.

Theorem 2.5. [27, **T**] (1) *If M is an infra-nilmanifold of* Nil^4, *then any of the three Ricci signatures $(0,+,-,-)$, $(+,+,-,-)$, $(+,-,-,-)$ can be realized on M.*

(2) *If M is an infra-solvmanifold of* $\mathrm{Sol}^3 \times \mathbb{R}$ *which has an order 4 element in its holonomy, then only $(0,0,0,-)$ can be realized on M.*

(3) *Every infra-solvmanifold of* $\mathrm{Sol}^4_{m,n}$ *is the mapping torus of a linear self diffeomorphism $\mathcal{S}$ of T^3; $\mathcal{S}$ has three distinct real eigenvalues. If all three eigenvalues are positive or all three eigenvalues are negative, any of $(0,0,0,-)$, $(0,+,-,-)$, $(+,+,-,-)$, $(+,-,-,-)$ can be realized on M. Else, only $(0,0,0,-)$ and $(0,+,-,-)$ can be realized on M.*

(4) *If M is a compact isometric quotient of* Sol_0^4, *then M is the mapping torus of a linear self diffeomorphism $\mathcal{S}$ of T^3; $\mathcal{S}$ has two complex eigenvalues $z, \bar{z}$, and one real eigenvalue α. The signature $(0,+,-,-)$ cannot be realized on M when $Re(z) > 0$ and $\alpha < 0$, or when $Re(z) < 0$ and $\alpha > 0$.*

3. Plan for Future Research

I would like to continue exploring the topology and geometry of 4-dimensional infra-solvmanfolds (Problems 3.1 and 3.2 below). I am also planning to work on Conjecture 2.1, and attempt to extend known partial results for Conjecture 2.1 to certain infra-solvmanifolds (Problems 3.5 and 3.4 below). Another direction for my research is to characterize which solvable Lie groups satisfy the First Bieberbach Theorem (Problem 3.6 below).

Geometric description of 4-dimensional infra-solvmanifolds

Hillman provides a geometric classification of infra-solvmanifolds of $\mathrm{Sol}^3 \times \mathbb{R}$ in [13] as Seifert fiberings over 2-dimensional flat orbifolds. We have a complete classification of the crystallographic groups of Sol_1^4 in [20], and those of $\mathrm{Sol}^3 \times \mathbb{R}$ in my dissertation. The torsion free crystallographic groups $\mathit{\Pi}$ correspond to infra-solvmanifolds $\mathit{\Pi}\backslash G$. However, we do not give a correspondence between our classification and Hillman's classification.

Problem 3.1. Find the correspondence between the torsion free crystallographic groups of $\mathrm{Sol}^3 \times \mathbb{R}$ in my dissertation and the classification Hillman provides in [13]. Also describe the infra-solvmanifolds of Sol_1^4, as Hillman does for $\mathrm{Sol}^3 \times \mathbb{R}$ in [13].

$Pin^{\pm}$ structures on 4-dimensional infra-solvmanifolds

Hillman begins to explore which 4-dimensional infra-solvmanifolds admit Pin^+ and Pin^- structures in [14]. A Riemannian manifold M has a Pin^+ structure if and only if the Stiefel-Whitney class $\omega_2(M)$ vanishes, and has a Pin^- structure if and only if $\omega_2(M) + \omega_1^2(M)$ vanishes. An orientation together with a Pin^+ (or Pin^-) structure is equivalent to a *Spin* structure.

Problem 3.2. Determine which 4-dimensional infra-solvmanifolds admit Pin^+ and Pin^- structures.

Bounding problem for infra-solvmanifolds

Since all 4-dimensional infra-solvmanifolds bound, some n-dimensional infra-nilmanifolds bound, and all closed flat n-manifolds bound, it is reasonable to ask if n-dimensional infra-solvmanifolds bound.

Conjecture 3.3. *Let $M = \mathit{\Pi}\backslash G$ be an n-dimensional infra-solvmanifold. Then there exists a compact $(n+1)$-dimensional manifold W with $\partial W = M$.*

Hamrick and Royster used translational involutions to show that closed flat n-manifolds bound [10]. Recent work by Davis and Fang on infra-nilmanifolds also uses this technique [3]. When G is solvable and has non-trivial center, one can also use translational involutions, as I used to show that 4-dimensional infra-solvmanifolds bound. My dissertation gave me some insight into how to use translational involutions in the solvable case.

Problem 3.4. Let $M = \mathit{\Pi}\backslash G$ be an n-dimensional infra-solvmanifold. Assume that G has non-trivial center. Is there a compact $(n+1)$-dimensional manifold W with $\partial W = M$?

On the other hand, Conjecture 3.3 may be false.

Problem 3.5. Give an example of an infra-solvmanifold or infra-nilmanifold which does not bound.

Bieberbach's First Theorem on solvable Lie groups

It is not well understood when a solvable Lie group satisfies Bieberbach's First Theorem (Theorem 1.3). Even a 4-dimensional solvable geometry, Sol_0^4, does not satisfy it. See [5] for more examples of solvable Lie groups which do not satisfy Bieberbach's First Theorem.

Problem 3.6. Characterize the solvable Lie groups for which Bieberbach's First Theorem holds.

Good progress has been made by Dekimpe, Lee, and Raymond, who give a sufficient, but not necessary, condition for a solvable Lie group to satisfy Bieberbach's First Theorem [5]. In [2], Buser gives a geometric proof of Bieberbach's First Theorem for $\mathbb{R}^n$, inspired by Gromov's work on almost flat manifolds. Examining Buser's argument when $\mathbb{R}^n$ is replaced with a solvable Lie group should provide a simple characterization of solvable Lie groups satisfying Bieberbach's First Theorem.

4. Undergraduate Research Opportunities

My research area of crystallographic groups and infra-solvmanifolds can be made accessible to undergraduates. Most of the background material involves group actions and general topology. The study of crystallographic groups of $\mathbb{R}^n$ is especially accessible. Szczepanski has a list of open problems on crystallographic groups of $\mathbb{R}^n$ and flat manifolds in [24]. Many of these problems are computational, and undergraduates can solve special cases of these problems with the aid of a computer algebra system.

Of the problems listed here, Problems 3.2, 3.4, and 3.5 all involve the computation of Stiefel-Whitney cohomology classes or Stiefel-Whitney numbers. These problems are surprisingly difficult in the general case. For example, my dissertation establishes that a 4-dimensional infra-solvmanifold bounds not by direct computation of the Stiefel-Whitney numbers, but instead by a geometric argument. However, we can do computation in very specific cases, with the aid of a computer algebra system, and this is how I plan to get undergraduates involved. Advanced knowledge of cohomology theory would not be required. For example, a substantial project for a very advanced undergraduate could be to first learn the definition of infra-solvmanifold, select specific infra-solvmanifolds, and study Problems 3.2, 3.4, or 3.5, for those particular examples. The resulting project could provide valuable evidence for Problems 3.2 or 3.4. The project might even settle Problem 3.5, if the student is fortunate to stumble across a counterexample.

I also have secondary research interests in graph theory and combinatorial optimization. As an undergraduate, I studied the so-called *composite graph coloring problem*, which is a generalization of the standard graph coloring problem. There are many algorithms for finding an approximate solution to the standard graph coloring problem. For my senior thesis [28], I generalized a certain class of these algorithms to the case of composite graphs. They were found to be superior to known algorithms. In addition, I found an error in, and corrected, a well known exact algorithm for solving the composite graph coloring problem. There are many powerful algorithms for standard graph coloring which have not been generalized to composite graph coloring. Generalizing and implementing these algorithms would make for excellent undergraduate research projects which combine mathematics and computer science.

I would be glad to discuss any ideas given here in more detail.

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