ALL FOUR-DIMENSIONAL INFRA-SOLVMANIFOLDS ARE BOUNDARIES

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ABSTRACT. Infra-solvmanifolds are a certain class of aspherical manifolds which generalize both flat manifolds and almost flat manifolds (i.e., infra-nilmanifolds). Every 4-dimensional infra-solvmanifold is diffeomorphic to a geometric 4-manifold with geometry of solvable Lie type.

There were questions about whether or not all 4-dimensional infra-solvmanifolds bound. We answer this affirmatively. On each infra-solvmanifold M admitting $\mathrm{Nil}^3 \times \mathbb{R}$, Nil^4, $\mathrm{Sol}^3 \times \mathbb{R}$, or ${\mathrm{Sol}_1}^4$ geometry, an isometric involution with 2-dimensional fixed set is constructed. The Stiefel-Whitney number $\omega_1^4(M)$ vanishes by a result of R.E. Stong and from this it follows that all Stiefel-Whitney numbers vanish.

We say that a closed n-manifold M *bounds* if there is a compact $(n+1)$-dimensional manifold W with $\partial W = M$. The only 2-dimensional infra-solvmanifolds are the torus and Klein bottle, and both are boundaries. Also, it is well known that all 3-dimensional closed manifolds bound. So 4 is the first dimension of interest. Given a Lie group G with left invariant metric, if Π is a cocompact discrete subgroup of $\mathrm{Isom}(G)$ acting freely and properly discontinuously on G, we say that $\Pi\backslash G$ is a *compact form* of G. By the work of Hillman, all 4-dimensional infra-solvmanifolds admit a geometry of solvable Lie type [9, Theorem 8]; any 4-dimensional infra-solvmanifold M is diffeomorphic to a compact form of a solvable 4-dimensional geometry G. Therefore, to show that all 4-dimensional infra-solvmanifolds bound, it suffices to show that all compact forms of the solvable 4-dimensional geometries bound.

See [20] for a classification of the 4-dimensional geometries in the sense of Thurston. Of these, the 4-dimensional solvable geometries are $\mathbb{R}^4, \mathrm{Nil}^3 \times \mathbb{R}, \mathrm{Nil}^4, \mathrm{Sol}^3 \times \mathbb{R}, {\mathrm{Sol}_0}^4, \mathrm{Sol}^4_{m,n}$, and ${\mathrm{Sol}_1}^4$. It is a remarkable theorem of Hamrick and Royster that all closed flat n-manifolds bound [8]. Furthermore, as all compact forms of $\mathrm{Sol}^4_{m,n}$ $(m \neq n)$ and ${\mathrm{Sol}_0}^4$ are mapping tori of linear self-diffeomorphisms of T^3 ([10, Corollary 8.5.1] and [13, Theorem 3.5,

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Theorem 4.2]), they can be shown to bound easily. The infra-solvmanifolds with Sol_1^4 geometry were classified and shown to bound in [15]. Hillman has classified infra-solvmanifolds with $\mathrm{Sol}^3 \times \mathbb{R}$ geometry and has shown that they bound [11].

In the first section, we recall the definitions of infra-solvmanifolds and the solvable 4-dimensional geometries. In the second section, we show how to define an involution, induced by left translation, on certain infra-solvmanifolds. In the third section, we study the maximal compact subgroups of $\mathrm{Aut}(G)$ where G is one of $\mathrm{Nil}^3 \times \mathbb{R}, \mathrm{Nil}^4, \mathrm{Sol}^3 \times \mathbb{R}$, or Sol_1^4. In the last section we show that any infra-solvmanifold with $\mathrm{Nil}^3 \times \mathbb{R}, \mathrm{Nil}^4, \mathrm{Sol}^3 \times \mathbb{R}$, or Sol_1^4 geometry admits an involution with 2-dimensional fixed set. The Stiefel-Whitney number $\omega_1^4(M)$ vanishes by a result of R.E. Stong and from this it follows that all Stiefel-Whitney numbers vanish. This extends the argument in [15] and establishes that all 4-dimensional infra-solvmanifolds bound.

1. Infra-solvmanifolds and 4-dimensional geometries

Let G be a simply connected solvable Lie group and let K be a maximal compact subgroup of $\mathrm{Aut}(G)$. Let $\mathrm{Aff}(G) = G \rtimes \mathrm{Aut}(G)$ denote the affine group with group operation

$$(a, A)(b, B) = (aA(b), AB).$$

$\mathrm{Aff}(G)$ acts on G by

$$(a, A)g = aA(g).$$

Suppose we have a discrete subgroup

$$\Pi \subset G \rtimes K \subset \mathrm{Aff}(G)$$

such that Π acts freely on G with compact quotient $\Pi\backslash G$. If, in addition the translation subgroup $\Gamma := \Pi \cap G$ is a cocompact lattice of G and of finite index in Π, we say that $\Pi\backslash G$ is an *infra-solvmanifold* of G. For simply connected solvable Lie groups, a result of Mostow [17, Theorem 6.2] implies that $\Gamma\backslash G$ has finite volume precisely when $\Gamma\backslash G$ is compact. So, the terms "lattice" and "cocompact lattice" are equivalent for our purposes.

The condition that Π act freely on G is equivalent to Π being torsion free. The translation subgroup Γ is normal in Π and we refer to $\Phi := \Pi/\Gamma$ as the *holonomy group* of $\Pi\backslash G$. It is a finite subgroup of K. We have the diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Gamma & \longrightarrow & \Pi & \longrightarrow & \Phi & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & G & \longrightarrow & G \rtimes K & \longrightarrow & K & \longrightarrow & 1 \end{array}$$

By definition, an infra-solvmanifold is finitely covered by the solvmanifold $\Gamma\backslash G$ with Φ as the group of covering transformations, hence the prefix "infra". In this paper, a *solvmanifold* is a quotient of G by a lattice of itself. In [12] , the various definitions of infra-solvmanifold appearing in the literature are shown to all be equivalent. Here we have adopted Definition 1 in [12] as above.

Recall that the 3-dimensional geometry Nil^3 is the group of upper triangular matrices

$$\begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}$$

and the 3-dimensional geometry Sol^3 is the semidirect product $\mathbb{R}^2 \rtimes_{\phi(u)} \mathbb{R}$ where $\phi(u) = \begin{bmatrix} e^{-u} & 0 \\ 0 & e^u \end{bmatrix}$.

In dimension 4, except for $\mathrm{Sol_1}^4 = \mathrm{Nil}^3 \rtimes \mathbb{R}$, all solvable geometries are of the form $\mathbb{R}^3 \rtimes_{\phi(u)} \mathbb{R}$ for $\phi : \mathbb{R} \to \mathrm{GL}(3, \mathbb{R})$.

$$\mathbb{R}^4: \quad \phi(u) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathrm{Nil}^3 \times \mathbb{R}: \quad \phi(u) = \begin{bmatrix} 1 & u & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathrm{Nil}^4: \quad \phi(u) = \begin{bmatrix} 1 & u & \frac{1}{2}u^2 \\ 0 & 1 & u \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathrm{Sol}^3 \times \mathbb{R}: \quad \phi(u) = \begin{bmatrix} e^{-u} & 0 & 0 \\ 0 & e^u & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathrm{Sol_0}^4: \quad \phi(u) = \begin{bmatrix} e^u & 0 & 0 \\ 0 & e^u & 0 \\ 0 & 0 & e^{-2u} \end{bmatrix} \qquad \mathrm{Sol}^4_{m,n}: \quad \phi(u) = \begin{bmatrix} e^{\lambda u} & 0 & 0 \\ 0 & e^u & 0 \\ 0 & 0 & e^{-(1+\lambda)u} \end{bmatrix}$$

$$\mathrm{Sol_0}^{4\prime}: \quad \phi(u) = \begin{bmatrix} e^u & ue^u & 0 \\ 0 & e^u & 0 \\ 0 & 0 & e^{-2u} \end{bmatrix}$$

For $\mathrm{Sol}^4_{m,n}$, $\lambda > 0$ is such that $\phi(u)$ is conjugate to an element of $\mathrm{GL}(3, \mathbb{Z})$. This guarantees that $\mathrm{Sol}^4_{m,n}$ has a lattice [13]. The characteristic polynomial of $\phi(u)$ is $x^3 - mx^2 + nx - 1$ for $m, n \in \mathbb{Z}$. It is known that $\mathrm{Sol_0}^{4\prime}$ has no compact forms [13] and therefore does not appear in the list of 4-dimensional geometries in [20].

$\mathrm{Sol_1}^4$ can be described as the multiplicative group of matrices

$$\begin{bmatrix} 1 & e^u x & z \\ 0 & e^u & y \\ 0 & 0 & 1 \end{bmatrix},$$

which splits as the semidirect product $\mathrm{Nil}^3 \rtimes \mathbb{R}$. It has 1-dimensional center ($u = x = y = 0$ in the above matrix). The quotient of $\mathrm{Sol_1}^4$ by its center is Sol^3; we have the short exact sequence

$$1 \to \mathcal{Z}(\mathrm{Sol_1}^4) \to \mathrm{Sol_1}^4 \to \mathrm{Sol}^3 \to 1.$$

All 4-dimensional solvable geometries are type (R) and unimodular. Therefore, with a left invariant metric on G,

$$\mathrm{Isom}(G) \subset G \rtimes K \subset \mathrm{Aff}(G),$$

where K is a maximal compact subgroup of $\mathrm{Aut}(G)$ [6]. In fact, all of the 4-dimensional solvable geometries admit a left invariant metric so that

$$\mathrm{Isom}(G) = G \rtimes K.$$

All 4-dimensional solvable geometries, except Sol_0^4, satisfy the generalized first Bieberbach theorem [4]. Consequently, if $\Pi \subset \mathrm{Isom}(G)$ ($G \neq \mathrm{Sol}_0^4$) is a discrete subgroup acting freely and properly discontinuously on G, then the translation subgroup $\Gamma := \Pi \cap G$ is a lattice of G and $\Phi := \Pi/\Gamma$ is finite. Therefore, the compact forms of a solvable 4-dimensional geometry G, excluding Sol_0^4, are indeed infra-solvmanifolds of G. Note that the holonomy group Φ acts freely and isometrically on the solvmanifold $\Gamma\backslash G$ with quotient the infra-solvmanifold $\Pi\backslash G$. The compact forms of Sol_0^4 are not infra-solvmanifolds of Sol_0^4; however, they can be realized as infra-solvmanifolds of a different simply connected solvable Lie group [21].

For the rest of this paper, given an infra-solvmanifold $\Pi\backslash G$, we shall always let Γ denote the translation subgroup and Φ denote the holonomy group.

2. Translational Involution

We show how to define an involution on an infra-solvmanifold $M = \Pi\backslash G$ when the center of G, $\mathcal{Z}(G)$, is non-trivial. The involution is induced by left translation. This technique was used to show closed flat n-manifolds bound [8, 7].

Lemma 2.1. *Let $M = \Pi\backslash G$ be an infra-solvmanifold with $\mathcal{Z}(G)$ non-trivial. Note $\Gamma \cap \mathcal{Z}(G)$ is a lattice of $\mathcal{Z}(G)$. Let t be a free generator of $\Gamma \cap \mathcal{Z}(G)$ and set $s = t^{\frac{1}{2}}$. Translation by s induces an involution on M if and only if $A(s) = s$ modulo $\Gamma \cap \mathcal{Z}(G)$, for all $A \in \Phi$. That is, translation by s commutes with the action of Φ on $\Gamma\backslash G$.*

Proof. Since s commutes with Γ, translation by s defines a free involution on the solvmanifold $\Gamma\backslash G$. To induce an involution on M, translation by s must normalize the action of Φ on $\Gamma\backslash G$. For any $(a, A) \in \Pi$, we have

$$\begin{aligned}(s, \mathrm{id})(a, A)(-s, \mathrm{id}) &= (s \cdot a \cdot As^{-1}, A)\\ &= ((I - A)s \cdot a, A) \text{ (since } s \in \mathcal{Z}(G)).\end{aligned}$$

Therefore, s induces an involution on M when $(I - A)s \in \Gamma \cap \mathcal{Z}(G)$; that is, $A(s) = s$ modulo $\Gamma \cap \mathcal{Z}(G)$, for all $A \in \Phi$. □

Let $\hat{M}$ denote the solvmanifold $\Gamma\backslash G$. We have the coverings

$$G \xrightarrow{q} \hat{M} \xrightarrow{p} M.$$

We refer to the involutions induced by translation by s as *translational involutions*. Let $\hat{i}_s : \hat{M} \to \hat{M}$ denote the induced involution on $\hat{M}$, $i_s : M \to M$ denote the induced involution on M, and F denote the fixed set of i_s on M.

Lemma 2.2. *The preimage of F in $\hat{M}$ is a finite disjoint union of closed, connected, submanifolds. We can write*

$$p^{-1}(F) = \bigcup_{\eta} E_\eta,$$

where the union is over all possible injective homomorphisms $\eta : \langle \hat{i}_s \rangle \to \Phi$ and

$$E_\eta = \{\hat{x} \in \hat{M} \mid s(\hat{x}) = \eta(\hat{i}_s)(\hat{x})\}.$$

Each E_η is a finite disjoint union of components of F.

Proof. The fixed set F of the translational involution must be a finite disjoint union of closed connected submanifolds [2, p. 72]. Since p is a finite sheeted covering, $p^{-1}(F)$ also admits the structure of a finite disjoint union of closed connected submanifolds.

If $\hat{x} \in p^{-1}(F)$, then $s(\hat{x}) = (a, A)(\hat{x})$ for some unique $A \in \Phi$ where $(a, A) \in \Pi$. Thus,

$$s^2(\hat{x}) = (a, A)^2(\hat{x}).$$

Since the deck transformation group acts freely, $(a, A)^2 = s^2 \in \Gamma$, and thus $A^2 = I$. So $\eta(\hat{i}_s) = A$ defines an injective homomorphism $\eta : \mathbb{Z}_2 \to \Phi$. We warn the reader the action of $\eta(\hat{i}_s) = A \in \Phi$ on $\hat{M}$ is induced not just by the automorphism A, but rather by the affine transformation (a, A). The preimage of F in $\hat{M}$ is indexed by all possible injective homomorphisms $\eta : \mathbb{Z}_2 \to \Phi$. That is,

$$p^{-1}(F) = \bigcup_{\eta} E_\eta.$$

Note that $E_{\eta_1} = E_{\eta_2}$ when $\eta_1 = \eta_2$ and $E_{\eta_1} \cap E_{\eta_2} = \emptyset$ otherwise.

The actions of s and $\eta(\hat{i}_s)$ commute on $\hat{M}$ by Lemma 2.1. By definition, E_η is the fixed set of the involution $\eta(\hat{i}_s)^{-1} \circ \hat{i}_s = \eta(\hat{i}_s) \circ \hat{i}_s$ on $\hat{M}$. So it must be a finite disjoint union of closed connected submanifolds, and therefore must be a finite disjoint union of components of $p^{-1}(F)$. $\square$

When $\exp : \mathfrak{g} \to G$ is a diffeomorphism, for $A \in \mathrm{Aut}(G)$, we have $\mathrm{Fix}(A) = \exp(\mathrm{Fix}(A_*))$, where A_* is the automorphism of $\mathfrak{g}$ induced from A. When G is a 4-dimensional solvable geometry, G is type (E) and exp is a diffeomorphism. So $\mathrm{Fix}(A)$ is always diffeomorphic to $\mathbb{R}^n$. We have the diagram of coverings, where the vertical arrows are inclusions.

$$\begin{tikzcd}
G \arrow[r, "q"] & \hat{M} \arrow[r, "p"] & M \\
\bigcup_\eta q^{-1}(E_\eta) \arrow[u] \arrow[r, "q"] & \bigcup_\eta E_\eta \arrow[u] \arrow[r, "p"] & F \arrow[u]
\end{tikzcd}$$

Now we analyze $q^{-1}(E_\eta)$.

Lemma 2.3. *Assume that* $\exp : \mathfrak{g} \to G$ *is a diffeomorphism. The preimage of* E_η *in* G *under* $q : G \to \hat{M}$ *is a disjoint union of submanifolds of* G*. In fact, if* $\eta(\hat{i}_s) = A \in \Phi$ *with* $(a, A) \in \mathit{\Pi}$*, then any component of* $q^{-1}(E_\eta)$ *is* $\mathrm{Fix}(\gamma s^{-1}a, A)$ *for some* $\gamma \in \Gamma$*. Consequently, the preimage of* E_η *in* G *is*

$$q^{-1}(E_\eta) = \bigcup_{\gamma\in\Gamma} \mathrm{Fix}(\gamma s^{-1}a, A).$$

Further, each $\mathrm{Fix}(\gamma s^{-1}a, A)$ *is a left translate of the connected subgroup* $\mathrm{Fix}(A)$ *of* G *and is diffeomorphic to* $\mathbb{R}^n$*, where* $n = \dim(\mathrm{Fix}(A)) = \dim(\mathrm{Fix}(A_*))$.

Proof. Because E_η is a disjoint union of closed submanifolds and q is a covering, $q^{-1}(E_\eta)$ is a (possibly not connected) submanifold of G without boundary.

Let $A = \eta(\hat{i}_s)$ and let $(a, A) \in \mathit{\Pi}$. An element $\tilde{x} \in G$ projects to $\hat{x} \in E_\eta$ if and only if there exists $\gamma \in \Gamma$ such that $s(\tilde{x}) = \gamma(a, A)(\tilde{x})$, or equivalently,

$$\tilde{x} = (\gamma s^{-1}a, A)(\tilde{x}).$$

That is, $\tilde{x}$ must be in the fixed set of the affine transformation $(\gamma s^{-1}a, A)$. Consequently, the preimage of E_η in G is

$$q^{-1}(E_\eta) = \bigcup_{\gamma\in\Gamma} \mathrm{Fix}(\gamma s^{-1}a, A).$$

Some sets in the above union may be empty. The fixed set of an affine transformation, if non-empty, is just a translation of the fixed subgroup of its automorphism part; that is, if $x_0 \in \mathrm{Fix}(b, B)$, then

$$\mathrm{Fix}(b, B) = x_0\mathrm{Fix}(B).$$

Any two left translates of $\mathrm{Fix}(A)$ are either disjoint or equal. Since exp is a diffeomorphism, any left translate of $\mathrm{Fix}(A)$ is a submanifold of G diffeomorphic to $\mathbb{R}^n$, where $n = \dim(\mathrm{Fix}(A))$. Since Γ is countable, $q^{-1}(E_\eta)$ is expressed as a countable union of submanifolds of G, each of which has dimension $\dim(\mathrm{Fix}(A))$. This forces each component of the submanifold $q^{-1}(E_\eta)$ to have dimension equal to that of $\mathrm{Fix}(A)$.

In fact, we claim a component $\tilde{E}_\eta$ of $q^{-1}(E_\eta)$ is equal to $\mathrm{Fix}(\gamma s^{-1}a, A)$ for some $\gamma \in \Gamma$. The argument above shows that $\tilde{x} \in \tilde{E}_\eta$ belongs to $\mathrm{Fix}(\gamma s^{-1}a, A)$ for some $\gamma \in \Gamma$. Since $\mathrm{Fix}(\gamma s^{-1}a, A)$ is connected,

$$\mathrm{Fix}(\gamma s^{-1}a, A) \subset \tilde{E}_\eta.$$

Also, $\mathrm{Fix}(\gamma s^{-1}a, A)$ is closed in $\tilde{E}_\eta$, since it is closed in G. Note that the inclusion $\mathrm{Fix}(\gamma s^{-1}a, A) \hookrightarrow \tilde{E}_\eta$ is open by invariance of domain, as both manifolds have the same dimension. Consequently, $\mathrm{Fix}(\gamma s^{-1}a, A) = \tilde{E}_\eta$. $\square$

An important consequence of Lemma 2.3 is that all components of F lifting to E_η must have the same dimension equal to that of $\mathrm{Fix}(\eta(\hat{i}_s))$, where $\eta(\hat{i}_s) \in \Phi$ is the unique automorphism of G coming from η.

Lemma 2.4. *Let $\varPi\backslash G$ be an infra-solvmanifold with translational involution induced by s. Suppose that $A \in \Phi$ has order 2 and $A(s) = -s$. Let $\eta : \mathbb{Z}_2 \to \Phi$ be the homomorphism $\eta(\hat{i}_s) = A$. Then $E_\eta = \emptyset$.*

Proof. Let $\alpha = (a, A) \in \varPi$ and define $\varPi' = \langle \Gamma, \alpha\rangle$. Note that $\varPi'\backslash G$ is an infra-solvmanifold with $\mathbb{Z}_2$ holonomy and translational involution i'_s induced by s.

We claim that the group generated by $\varPi'$ and s, $\langle \varPi', s\rangle$, is torsion free. A general element of $\langle \varPi', s\rangle$ with holonomy A is of the form $(s\gamma a, A)$, where $\gamma \in \Gamma$. Now

$$\begin{aligned}(s\gamma a, A)^2 &= (s\gamma a A(s)A(\gamma)A(a), \mathrm{id})\\ &= (\gamma a A(\gamma a), \mathrm{id}), \quad (\text{since } s \in \mathcal{Z}(G))\\ &= (\gamma a, A)^2 \neq (e, \mathrm{id}),\end{aligned}$$

where the last inequality follows since $\varPi$ is torsion free.

Consequently, $\langle \varPi', s\rangle$ is torsion free and hence $\langle \varPi', s\rangle$ acts freely on G. Therefore s acts as a free involution on the infra-solvmanifold $\varPi'\backslash G$. Note that the preimage of $\mathrm{Fix}(i'_s) \subset \varPi'\backslash G$ in $\Gamma\backslash G$ under the double covering $\Gamma\backslash G \to \varPi'\backslash G$ is precisely E_η. Hence E_η must be empty. □

3. Maximal Compact Subgroups of $\mathrm{Aut}(G)$

Given a 4-dimensional infra-solvmanifold $\varPi\backslash G$ with translational involution i_s induced by translation by s as defined in Lemma 2.1, the fixed set F will be a disjoint union of submanifolds. We will need to compute the dimension of $\mathrm{Fix}(i_s)$. By Lemma 2.3, a component of F lifts to G as a left translate of $\mathrm{Fix}(\eta(\hat{i}_s))$, where $\eta(\hat{i}_s)$ is an involution in $\mathrm{Aut}(G)$. Every involution in $\mathrm{Aut}(G)$ belongs to a maximal compact subgroup K of $\mathrm{Aut}(G)$. When G is one of the 4-dimensional solvable geometries, $\mathrm{Aut}(G)$ has finitely many components. In this case, a result of Mostow [16, Theorem 3.1] implies that all maximal compact subgroups of $\mathrm{Aut}(G)$ are conjugate. Therefore, we can fix a maximal compact subgroup K and compute $\dim(\mathrm{Fix}(A))$ for each involution A in K.

Lemma 3.1. $\left[\mathrm{Nil}^3 \times \mathbb{R}\right]$ *(1) A maximal compact subgroup of* $\mathrm{Aut}(\mathrm{Nil}^3 \times \mathbb{R})$ *is*

$$\mathrm{O}(2, \mathbb{R}) \times \mathbb{Z}_2.$$

(2) If $A \in \mathrm{O}(2, \mathbb{R}) \times \mathbb{Z}_2$ restricts to the identity on $\mathbb{R}$, then

$$\dim(\mathrm{Fix}(A)) = 2.$$

Proof. A maximal compact subgroup of $\mathrm{Aut}(\mathrm{Nil}^3)$ is $\mathrm{O}(2, \mathbb{R})$ and acts as

$$A = \begin{bmatrix} a & b\\ c & d\end{bmatrix} : \begin{bmatrix} 1 & x & z\\ 0 & 1 & y\\ 0 & 0 & 1\end{bmatrix} \longmapsto \begin{bmatrix} 1 & (ax+by) & \frac{1}{2}(abx^2 + 2bcxy + cdy^2 + 2\det(A)z)\\ 0 & 1 & (cx+dy)\\ 0 & 0 & 1\end{bmatrix}.$$

The induced action of $A \in \mathrm{O}(2, \mathbb{R})$ on $\mathcal{Z}(\mathrm{Nil}^3)$ is multiplication by $\det(A)$. Therefore, $\mathrm{Fix}(A)$ is 1-dimensional for all $A \in \mathrm{O}(2, \mathbb{R})$. It follows that a maximal compact subgroup of $\mathrm{Aut}(\mathrm{Nil}^3 \times \mathbb{R})$ is $\mathrm{O}(2,\mathbb{R}) \times \mathbb{Z}_2$, where $\mathrm{O}(2,\mathbb{R}) \subset \mathrm{Aut}(\mathrm{Nil}^3)$ and $\mathbb{Z}_2$ acts as a reflection on $\mathbb{R}$. Thus, if $A \in \mathrm{O}(2,\mathbb{R}) \times \mathbb{Z}_2$ restricts to the identity on $\mathbb{R}$, then

$$\dim(\mathrm{Fix}(A)) = 2.$$

□

Lemma 3.2. $[\mathrm{Nil}^4]$ *(1) A maximal compact subgroup of* $\mathrm{Aut}(\mathrm{Nil}^4)$ *is* $\mathbb{Z}_2 \times \mathbb{Z}_2$.

(2) Let $A \in \mathrm{Aut}(\mathrm{Nil}^4)$ *have order 2. If* A *restricts to the identity on* $\mathcal{Z}(\mathrm{Nil}^4)$*, then*

$$\dim(\mathrm{Fix}(A)) = 2.$$

Proof. Recall the splitting of Nil^4 as the semidirect product $\mathbb{R}^3 \rtimes \mathbb{R}$. Letting $\mathfrak{g}$ denote the Lie algebra of Nil^4, we have $\mathfrak{g} \cong \mathbb{R}^3 \rtimes \mathbb{R}$, where $\mathbb{R}$ acts by the matrix

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

With standard bases $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ of $\mathbb{R}^3$ and $\mathbf{e}_4$ of $\mathbb{R}$, $\mathfrak{g}$ has relations

$$[\mathbf{e}_4, \mathbf{e}_2] = \mathbf{e}_1, [\mathbf{e}_4, \mathbf{e}_3] = \mathbf{e}_2.$$

For $A \in \mathrm{Aut}(\mathfrak{g})$, A induces an action on the quotient $\mathfrak{g}/[\mathfrak{g},\mathfrak{g}] \cong \langle \mathbf{e}_3, \mathbf{e}_4 \rangle$, denote this action by

$$B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

We will see that B determines the action of A on $\mathbf{e}_1$ and $\mathbf{e}_2$. The relation $[\mathbf{e}_4, \mathbf{e}_3] = \mathbf{e}_2$, implies $A(\mathbf{e}_2) = \det(B)\mathbf{e}_2$. Compactness forces $\det(B) = \pm 1$. We also compute

$$\begin{aligned} A(\mathbf{e}_1) = A([\mathbf{e}_4, \mathbf{e}_2]) &= [A(\mathbf{e}_4), A(\mathbf{e}_2)] \\ &= [b\mathbf{e}_3 + d\mathbf{e}_4, \det(B)\mathbf{e}_2] = \det(B) d \mathbf{e}_1 \end{aligned}$$

Again, compactness implies $d = \pm 1$. Since $[\mathbf{e}_3, \mathbf{e}_2]$ vanishes, we have

$$\begin{aligned} 0 = A([\mathbf{e}_3, \mathbf{e}_2]) &= [A(\mathbf{e}_3), A(\mathbf{e}_2)] \\ &= [a\mathbf{e}_3 + c\mathbf{e}_4, \det(B)\mathbf{e}_2] = \det(B) c \mathbf{e}_1 \end{aligned}$$

Thus c vanishes and B must be upper triangular of the form

$$B = \begin{bmatrix} \pm 1 & b \\ 0 & \pm 1 \end{bmatrix}.$$

By conjugation we can set $b = 0$. Thus, a maximal compact subgroup of $\mathrm{Aut}(\mathrm{Nil}^4)$ cannot be larger than $\mathbb{Z}_2 \times \mathbb{Z}_2$. Conversely, we see that

$$\mathbb{Z}_2 \times \mathbb{Z}_2 = \{(\pm I_3, 1), (\pm J, -1)\} \subset \mathrm{Aut}(\mathbb{R}^3 \rtimes \mathbb{R}),$$

where

$$J = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

defines a subgroup of $\mathrm{Aut}(\mathrm{Nil}^4)$. It now follows that a maximal compact subgroup of $\mathrm{Aut}(\mathrm{Nil}^4)$ is $\{(\pm I_3, 1), (\pm J, -1)\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

Note that $(-I_3, 1)$ and $(-J, -1)$ do not act as the identity on $\mathcal{Z}(\mathrm{Nil}^4)$. The remaining involution $(J, -1)$ restricts to the identity on $\mathcal{Z}(\mathrm{Nil}^4)$ and has 2-dimensional fixed subgroup. $\square$

Lemma 3.3. $\left[\mathrm{Sol}_1^4\right]$ *(1) A maximal compact subgroup of* $\mathrm{Aut}(\mathrm{Sol}_1^4)$ *is* D_4.

(2) Let $A \in \mathrm{Aut}(\mathrm{Sol}_1^4)$ *have order 2. If* A *restricts to the identity on* $\mathcal{Z}(\mathrm{Sol}_1^4)$*, then*

$$\dim(\mathrm{Fix}(A)) = 2.$$

Proof. A maximal compact subgroup of both $\mathrm{Aut}(\mathrm{Sol}^3)$ and $\mathrm{Aut}(\mathrm{Sol}_1^4)$ is the dihedral group [15]

$$D_4 = \left\langle \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\rangle = \mathbb{Z}_4 \rtimes \mathbb{Z}_2.$$

For $A \in D_4$, let $\bar{A}$ be $+1$ if A is diagonal, and -1 if A is off-diagonal. Then A acts on $\mathrm{Sol}^3 = \mathbb{R}^2 \rtimes \mathbb{R}$ as

$$A : \left(\begin{bmatrix} x \\ y \end{bmatrix}, u \right) \longmapsto \left(A \begin{bmatrix} x \\ y \end{bmatrix}, \bar{A}u \right),$$

and on Sol_1^4 as

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} : \begin{bmatrix} 1 & e^u x & z \\ 0 & e^u & y \\ 0 & 0 & 1 \end{bmatrix} \longmapsto \begin{bmatrix} 1 & e^{\bar{A}u}(ax + by) & \frac{1}{2}(abx^2 + 2bcxy + cdy^2 + 2\det(A)z) \\ 0 & e^{\bar{A}u} & (cx + dy) \\ 0 & 0 & 1 \end{bmatrix}.$$

For both Sol^3 and Sol_1^4, $\bar{A}$ is the induced action of A on $\mathrm{Sol}^3/\mathbb{R}^2 \cong \mathrm{Sol}_1^4/\mathrm{Nil} \cong \mathbb{R}$. Note that multiplication by $\det(A)$ is the induced action of A on $\mathcal{Z}(\mathrm{Sol}_1^4)$. Thus, the only involution in D_4 restricting to the identity on $\mathcal{Z}(\mathrm{Sol}_1^4)$ is $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ and evidently it has 2-dimensional fixed subgroup. $\square$

Lemma 3.4. $\left[\mathrm{Sol}^3 \times \mathbb{R}\right]$ *(1) A maximal compact subgroup of* $\mathrm{Aut}(\mathrm{Sol}^3 \times \mathbb{R})$ *is*

$$D_4 \times \mathbb{Z}_2,$$

where $D_4 \subset \mathrm{Aut}(\mathrm{Sol}^3)$ *and* $\mathbb{Z}_2$ *acts as a reflection on* $\mathbb{R}$.

(2) Let $A \in D_4 \subset D_4 \times \mathbb{Z}_2 \subset \mathrm{Aut}(\mathrm{Sol}^3 \times \mathbb{R})$ *have order 2.*

If $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ *or* $\pm \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$*, then*

$$\dim(\mathrm{Fix}(A)) = 2.$$

If $A = \pm \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$*, then*

$$\dim(\mathrm{Fix}(A)) = 3.$$

Proof. The first statement is clear since a maximal compact subgroup of $\mathrm{Aut}(\mathrm{Sol}^3)$ is D_4. For the second statement, recall that the induced action of A on the quotient $\mathrm{Sol}^3/\mathbb{R}^2 \cong \mathbb{R}$ is $+1$ if A is diagonal and -1 otherwise. □

Whenever G is one of $\mathrm{Nil}^3 \times \mathbb{R}, \mathrm{Nil}^4, \mathrm{Sol}_1^4$ or $\mathrm{Sol}^3 \times \mathbb{R}$, we can equip G with left invariant metric so that

$$\mathrm{Isom}(G) = G \rtimes K,$$

where K is one of the maximal compact subgroups of $\mathrm{Aut}(G)$ described in Lemmas 3.1, 3.2, 3.3, 3.4.

4. Proof of Bounding

The following relations among Stiefel-Whitney classes of 4-manifolds are known.

Lemma 4.1. *For any 4-manifold M,*

(1) $\omega_1^2\omega_2 = \omega_1\omega_3 = 0$
(2) $\omega_2^2 = \omega_1^4 + \omega_4$

Therefore, M is a boundary if and only if the Stiefel-Whitney numbers $\omega_1^4(M)$ and $\omega_4(M)$ are 0.

A solvmanifold $\Gamma\backslash G$ is parallelizable since one can project a framing of left invariant vector fields from G to $\Gamma\backslash G$. Hence the Euler characteristic $\chi(\Gamma\backslash G)$ vanishes. Since any infra-solvmanifold $\Pi\backslash G$ is finitely covered by a solvmanifold, $\chi(\Pi\backslash G) = 0$. Therefore, the mod 2 Euler characteristic $\omega_4(M)$ vanishes. Hence the only Stiefel Whitney number to consider is $\omega_1^4(M)$. The following is crucial for our argument that 4-dimensional infra-solvmanifolds bound.

Proposition 4.2. [19, Proposition 9.2] *A manifold M^n is unoriented cobordant to a manifold M' with differentiable involution having a fixed set of dimension $n-2$ if and only if $\omega_1^n(M) = 0$.*

We will also need the following result on the discrete cocompact subgroups of $\mathrm{Isom}(\mathrm{Sol}^3) = \mathrm{Sol}^3 \rtimes D_4$, which are also known as *crystallographic groups* of Sol^3. Note that the nil-radical of Sol^3 is $\mathbb{R}^2$. Let $\Pi \subset \mathrm{Isom}(\mathrm{Sol}^3)$ be a crystallographic group with lattice Γ and holonomy Φ. Recall the action of D_4 as automorphisms of Sol^3 from Lemma 3.3. Let

$$\mathrm{pr}_1 : \mathrm{Sol}^3 \to \mathrm{Sol}^3/\mathbb{R}^2 \cong \mathbb{R}.$$

denote the quotient map. If Γ is a lattice of Sol^3, then Γ meets the nil-radical in a lattice $\Gamma \cap \mathbb{R}^2 \cong \mathbb{Z}^2$ and $\mathrm{pr}_1(\Gamma) \cong \mathbb{Z}$ is a lattice of $\mathbb{R}$.

Proposition 4.3. [15, Lemma 3.4] *Let $\Pi \subset \mathrm{Isom}(\mathrm{Sol}^3) = \mathrm{Sol}^3 \rtimes D_4$ be crystallographic and let v denote a generator of $\mathrm{pr}_1(\Gamma) \cong \mathbb{Z}$.*

If $(b, B) \in \Pi$ where $B = \pm \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \subset D_4$, then

$$\mathrm{pr}_1(b) = v^{\frac{1}{2}}.$$

For $\mathrm{Nil}^3 \times \mathbb{R}$ geometry manifolds, we need to study the holonomy representation

$$\rho : \Phi \to \mathrm{Aut}(\mathcal{Z}(\mathrm{Nil}^3 \times \mathbb{R})).$$

Lemma 4.4. *Let $M = \Pi\backslash G$ be an infra-solvmanifold with $G = \mathrm{Nil}^3 \times \mathbb{R}$. Then there is a set of generators t_1, t_2 for $\Gamma \cap \mathcal{Z}(G) \cong \mathbb{Z}^2$, $t_1 \in \mathcal{Z}(\mathrm{Nil}^3)$, so that with respect to the basis t_1, t_2,*

(1) $\rho(\Phi) \subset \left\langle \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right\rangle$, or

(2) $\rho(\Phi) \subset \left\langle \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right\rangle$

Proof. We have $\mathcal{Z}(G) = \mathcal{Z}(\mathrm{Nil}^3) \times \mathbb{R}$. Since

$$[G, G] = \mathcal{Z}(\mathrm{Nil}^3) = \mathbb{R},$$

$\mathcal{Z}(\mathrm{Nil}^3)$ is invariant under any automorphism of G. Further, $\Gamma \cap \mathcal{Z}(\mathrm{Nil}^3)$ is a lattice of $\mathcal{Z}(\mathrm{Nil}^3)$.

Note that $\rho(\Phi)$ has $\mathcal{Z}(\mathrm{Nil}^3)$ as an invariant subspace. Because $\rho(\Phi)$ can be conjugated (over $\mathrm{GL}(2, \mathbb{R})$) into $\mathrm{O}(\mathcal{Z}(G))$, we can assume that it leaves the orthogonal complement of $\mathcal{Z}(\mathrm{Nil}^3)$ invariant as well. The maximal compact subgroup of $\mathrm{O}(\mathcal{Z}(G))$ leaving $\mathcal{Z}(\mathrm{Nil}^3)$ invariant is

$$\mathbb{Z}_2 \times \mathbb{Z}_2 = \left\langle \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right\rangle.$$

Thus, over $\mathrm{GL}(2, \mathbb{R})$, $\rho(\Phi)$ can be conjugated into this $\mathbb{Z}_2 \times \mathbb{Z}_2$. But over $\mathrm{GL}(2, \mathbb{Z})$, there is one more case.

Let t_1, t_2 be two generators of $\Pi \cap \mathcal{Z}(G) \cong \mathbb{Z}^2$, where t_1 generates $\Pi \cap \mathcal{Z}(\mathrm{Nil}^3)$. It is known [1] that an involution $A \in \mathrm{GL}(2, \mathbb{Z})$ with vanishing trace is $\mathrm{GL}(2, \mathbb{Z})$ conjugate to either $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ or $\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$. Using this, it is not hard to see that we can keep t_1 the same, but change t_2 to $t_2' = at_1 \pm t_2$ for $a \in \mathbb{Z}$, to put $\rho(\Phi)$ in the desired form. □

We are now ready to prove the main theorem.

Theorem 4.5. *All 4-dimensional infra-solvmanifolds are boundaries.*

Proof. Every 4-dimensional infra-solvmanifold is diffeomorphic to a compact form of a solvable 4-dimensional geometry [9, Theorem 8]. So it suffices to show these compact forms bound.

The flat 4-dimensional manifolds $M = \Pi\backslash\mathbb{R}^4$ are all boundaries by Hamrick-Royster [8].

When $G = \mathrm{Sol}^4_{m,n}$ or Sol_0^4, any compact form $M = \Pi\backslash G$ is a mapping torus of T^3, and is therefore a T^3 bundle over S^1. Because T^3 is orientable, $\omega_1(M)$ is induced from the base of the fibration. That is, let p denote the projection $p : M \to S^1$. Now $\omega_1(M) = p^*(c)$ for some class $c \in H^1(S^1; \mathbb{Z}_2) \cong \mathbb{Z}_2$. Then:

$$\omega_1(M)^4 = (p^*(c))^4 = p^*(c^4) = p^*(0) = 0.$$

Of course $\omega_4(M)$ vanishes as well. To see that ω_1 is induced from a class in the base of the fibration, note that

$$H^1(M;\mathbb{Z}_2) \cong \hom(H_1(M;\mathbb{Z}),\mathbb{Z}_2) \cong \hom(\pi_1(M),\mathbb{Z}_2)$$

It is known that $\omega_1(M)$, under the above isomorphism, is the cohomology class which assigns 1 to an element of $\gamma \in \pi_1(M)$ if the restriction of the tangent bundle to γ is non-orientable, and 0 if the restriction is orientable. Since the fiber of M is orientable, $\omega_1(M)$ must come from the base.

Case $G = \mathrm{Nil}^4, \mathrm{Sol}_1^4$:

For $M = \Pi\backslash G$ when G is Nil^4 or Sol_1^4, let $s = t^{\frac{1}{2}}$ where t is a generator of $\Gamma \cap \mathcal{Z}(G)$. We construct the translational involution defined in Lemma 2.1. Our explicit computation of maximal compact subgroups of $\mathrm{Aut}(\mathrm{Nil}^4)$ and $\mathrm{Aut}(\mathrm{Sol}_1^4)$ shows that $A(s) = \pm s$ for any holonomy $A \in \Phi$. This also follows since $\mathcal{Z}(G)$ is invariant under any automorphism and 1-dimensional. Let $\eta : \mathbb{Z}_2 = \langle \hat{i}_s \rangle \to \Phi$ be an injective homomorphism. If $\eta(\hat{i}_s)(s) = -s$, then $E_\eta = \emptyset$ by Lemma 2.4. If $\eta(\hat{i}_s)(s) = s$, then $\eta(\hat{i}_s)$ acts as the identity on $\mathcal{Z}(G)$ and $\eta(\hat{i}_s)$ has 2-dimensional fixed subgroup on G (Lemmas 3.2 and 3.3). By Lemma 2.3, E_η is 2-dimensional. Therefore, $\mathrm{Fix}(i_s)$ is 2-dimensional.

Case $G = \mathrm{Sol}^3 \times \mathbb{R}$:

Now consider a $\mathrm{Sol}^3 \times \mathbb{R}$ geometry manifold $\Pi\backslash G$. Let $s = t^{\frac{1}{2}}$ where t is a generator of $\Gamma \cap \mathcal{Z}(G)$. Since $A(s) = \pm s$ for all $A \in \Phi$, s defines an involution on $\Pi\backslash G$. Let $\eta : \mathbb{Z}_2 = \langle \hat{i}_s \rangle \to \Phi$ be an injective homomorphism. If $\eta(\hat{i}_s)(s) = -s$, then $E_\eta = \emptyset$ by Lemma 2.4.

In the $\mathrm{Sol}^3 \times \mathbb{R}$ geometry case, not all involutions in Φ inducing the identity on $\mathcal{Z}(\mathrm{Sol}^3 \times \mathbb{R})$ have 2-dimensional fixed subgroup (Lemma 3.4). When $\eta(\hat{i}_s) = A$, where A is one of $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ or $\pm \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, E_η is 2-dimensional, since $\mathrm{Fix}(A)$ is 2-dimensional. But for

$$B = \pm \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

$\mathrm{Fix}(B)$ is 3-dimensional. However, we shall see that when $\eta(\hat{i}_s) = B$, E_η is empty. Note that the nil-radical of $\mathrm{Sol}^3 \times \mathbb{R}$ is $\mathbb{R}^3$ with quotient $\mathbb{R}$. Let

$$\mathrm{pr} : \mathrm{Sol}^3 \times \mathbb{R} \to \mathbb{R}$$

denote the quotient homomorphism. If we let $\mathrm{pr}_2 : \mathrm{Sol}^3 \times \mathbb{R} \to \mathrm{Sol}^3$ denote the quotient of $\mathrm{Sol}^3 \times \mathbb{R}$ by its center and let $\mathrm{pr}_1 : \mathrm{Sol}^3 \to \mathbb{R}$ denote the quotient of Sol^3 by its nil-radical, then pr factors as $\mathrm{pr}_1 \circ \mathrm{pr}_2$,

$$\mathrm{pr} : \mathrm{Sol}^3 \times \mathbb{R} \xrightarrow{\mathrm{pr}_2 : /\mathcal{Z}(\mathrm{Sol}^3 \times \mathbb{R})} \mathrm{Sol}^3 \xrightarrow{\mathrm{pr}_1 : /\mathbb{R}^2} \mathbb{R}.$$

Now $\mathrm{pr}(\Gamma)$ is a lattice of $\mathbb{R}$. Let v denote a generator of $\mathrm{pr}(\Gamma)$. By Lemma 2.3, the preimage of E_η in $\mathrm{Sol}^3 \times \mathbb{R}$ is given by, for $(b, B) \in \Pi$,

$$\bigcup_{\gamma \in \Gamma} \mathrm{Fix}(\gamma s^{-1} b, B)$$

However, all sets $\mathrm{Fix}(\gamma s^{-1} b, B)$ are empty for any $\gamma \in \Gamma$. To see this, suppose $x \in \mathrm{Sol}^3 \times \mathbb{R}$ satisfies

$$\gamma s^{-1} b B(x) = x.$$

We will apply $\mathrm{pr} = \mathrm{pr}_1 \circ \mathrm{pr}_2$ to both sides. Note that $\mathrm{pr}(b) = v^{\frac{1}{2}}$ by Proposition 4.3, $\mathrm{pr}(\gamma) = v^n$ for some $n \in \mathbb{Z}$, $\mathrm{pr}(s) = 0$, and

$$\mathrm{pr}(B(x)) = \bar{B}(\mathrm{pr}(x)) = \mathrm{pr}(x)$$

(since B is diagonal, $\bar{B} = +1$). Thus, application of pr yields

$$v^{n+\frac{1}{2}} + \mathrm{pr}(x) = \mathrm{pr}(x),$$

which is a contradiction. This shows that E_η is empty when $\eta(\hat{i}_s) = B$.

Therefore, $\mathrm{Fix}(i_s)$ has no 3-dimensional components and is 2-dimensional in the $\mathrm{Sol}^3 \times \mathbb{R}$ geometry case.

Case $G = \mathrm{Nil}^3 \times \mathbb{R}$:

Finally, consider a $\mathrm{Nil}^3 \times \mathbb{R}$ geometry manifold $\Pi \backslash G$. Now $\mathcal{Z}(\mathrm{Nil}^3 \times \mathbb{R}) = \mathcal{Z}(\mathrm{Nil}^3) \times \mathbb{R}$. We will use either $\mathcal{Z}(\mathrm{Nil}^3)$ or $\mathbb{R}$ to induce an involution on $\Pi \backslash G$ depending on which case of Lemma 4.4 occurs.

Suppose we can take t_1, t_2 with $t_1 \in \mathcal{Z}(\mathrm{Nil}^3)$, as a generating set of $\Gamma \cap \mathcal{Z}(G)$ so that $\rho(\Phi)$ is diagonal for this generating set (case (1) of Lemma 4.4). Take $s = t_2^{\frac{1}{2}}$ for our involution on M. Lemma 2.4 implies that E_η is non-empty only when $\eta(\hat{i}_s)(s) = s$. This conditions means that $\eta(\hat{i}_s)$ fixes the $\mathbb{R}$ factor in $\mathcal{Z}(\mathrm{Nil}^3 \times \mathbb{R}) = \mathcal{Z}(\mathrm{Nil}^3) \times \mathbb{R}$. But all such $\eta(\hat{i}_s)$ fixing $\mathbb{R}$ must have a 2-dimensional fixed set on $\mathrm{Nil}^3 \times \mathbb{R}$ by Lemma 3.1.

Now suppose case (2) of Lemma 4.4 occurs. This time, we must take $s = t_1^{\frac{1}{2}}$ for our involution. If $\eta(\hat{i}_s)$ acts as a reflection on $\mathcal{Z}(\mathrm{Nil}^3)$, Lemma 2.4 implies E_η is empty.

We claim further that E_η is empty when $\eta(\hat{i}_s) \in \Phi$ with $\rho(\eta(\hat{i}_s)) = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$. To see this, note that translation by $s = t_1^{\frac{1}{2}}$ is also induced from translation by $s' = t_1^{\frac{1}{2}} t_2^{-1}$. But $\eta(\hat{i}_s)(s') = s'^{-1}$. Hence Lemma 2.4 implies that E_η is empty in this case.

Thus we conclude that the only non-empty components of $\mathrm{Fix}(i_s)$ can arise from $\eta(\hat{i}_s) \in \Phi$ with $\rho(\eta(\hat{i}_s)) = \mathrm{id}$. But Lemma 3.1 implies that all such $\eta(\hat{i}_s)$ have 2-dimensional fixed set. Therefore, when $G = \mathrm{Nil}^3 \times \mathbb{R}$, in either case of Lemma 4.4, we have an involution with fixed set of constant dimension 2.

For any manifold with $\mathrm{Nil}^3 \times \mathbb{R}, \mathrm{Nil}^4, \mathrm{Sol}_1{}^4$, or $\mathrm{Sol}^3 \times \mathbb{R}$ geometry, we have constructed an involution with 2-dimensional fixed set. By Stong's result (Proposition 4.2), $\omega_1^4(M) = 0$. Thus, all Stiefel-Whitney numbers are zero and we have established that all 4-dimensional infra-solvmanifolds bound. $\square$

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