SQUARE ROOTS, CONTINUED FRACTIONS AND THE ORBIT OF $\frac{1}{0}$ ON $\partial \mathbf{H}^2$

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1. When is $\sqrt{k}$ irrational? A simple proof

Early in our mathematical training we learn the standard proof that $\sqrt{2}$ is irrational. The usual proof involves writing $\sqrt{2} = \frac{m}{n}$ and then using unique factorization of the integers to conclude that $\frac{m}{n}$ cannot be in lowest terms. Of course, many proofs exist of this fact and we present one more such proof which has the virtues of brevity and of not appealing to the fundamental theorem of arithmetic.

Proposition 1. *Let k be a positive integer. If $\lfloor \sqrt{k} \rfloor \neq \sqrt{k}$, then $\sqrt{k}$ is irrational.*

Proof. Consider the set of positive integers $\{l \mid l > 0 \text{ and } \frac{h}{l} = \sqrt{k} \text{ for some } h\}$. We are assuming, by way of contradiction, that the set is non-empty so that it has a minimum element n by the well-ordering of the natural numbers. If for some m we have $\sqrt{k} = m/n$, then for any integer d we have

$$\sqrt{k} = \frac{m}{n} = \frac{-dm + kn}{m - dn}$$

Letting $d = \lfloor \sqrt{k} \rfloor$ we get $d < m/n < d + 1$ which implies $0 < m - dn < n$, contradicting our choice of n. $\square$

Remark. Note that a similar proof works for $\sqrt{k} = m/n$, where we write $m/n = (\lceil \sqrt{k} \rceil m - kn)/(-m + \lceil \sqrt{k} \rceil n)$. Here we use a slightly non-standard definition of the ceiling function namely, $\lceil x \rceil = \lfloor x \rfloor + 1$. The usual definition, "the smallest integer greater than or equal to x", is the same as this one for $x \notin \mathbf{Z}$, but is off by 1 if $x \in \mathbf{Z}$.

2. Linear fractional transformations on $\partial\mathbf{H}^2$

Sometimes what is more interesting than the result of a proof is the technique used to obtain it. The proof above leads us to study the following linear fractional transformation on $\partial\mathbf{H}^2$ (here $\mathbf{H}^2$ is viewed as the upper half plane); let $d = \lfloor\sqrt{k}\rfloor$ and $d+1 = \lceil\sqrt{k}\rceil$.

$$\begin{aligned} P_k(z) &= \begin{pmatrix} d & k \\ 1 & d \end{pmatrix}\begin{pmatrix} z \\ 1 \end{pmatrix} = \frac{dz+k}{z+d} = \frac{\lfloor\sqrt{k}\rfloor z + k}{z + \lfloor\sqrt{k}\rfloor} \\ M_k(z) &= \begin{pmatrix} d+1 & k \\ 1 & d+1 \end{pmatrix}\begin{pmatrix} z \\ 1 \end{pmatrix} = \frac{(d+1)z+k}{z+(d+1)} = \frac{\lceil\sqrt{k}\rceil z + k}{z + \lceil\sqrt{k}\rceil} \end{aligned}$$

Note, first of all, that $\pm\sqrt{k}$ are the only fixed points of P_k, M_k. In fact, $-\sqrt{k}$ is a source and $\sqrt{k}$ is a sink as we iterate P_k, M_k and look at the orbits of points in $\partial\mathbf{H}^2$. From this point of view, the above proposition used the fact that $\sqrt{k}$ is a fixed point of P_k^{-1} (respectively M_k^{-1}) to derive a contradiction. We now pursue this train of thought a bit further. Specifically, the orbit of $1/0$ (the "point at infinity" on $\partial\mathbf{H}^2$) has a very interesting property for certain choices of k. For the definitions of *plus continued fraction* and *minus continued fraction* and notation, see the next section.

Let k be a positive integer, not a perfect square, and let $[d; d_1, d_2, \ldots]$ be the plus continued fraction expansion of $\sqrt{k}$, and $\frac{p_n}{q_n}$ the convergents of that continued fraction. Let $(d+1; e_1, e_2, \ldots)$ be the minus continued fraction expansion of $\sqrt{k}$, and $\frac{r_n}{s_n}$ the convergents of that continued fraction.

Theorem Plus. *The following statements are equivalent for the plus continued fraction expansion of $\sqrt{k}$.*

(a) *For all $n \geq 0$ we have $P_k^n(\frac{1}{0}) = \frac{p_n}{q_n}$. That is, the orbit of $1/0 \in \partial\mathbf{H}^2$ under P_k is precisely the set of convergents of the plus continued fraction expansion of $\sqrt{k}$.*

(b) *The quantity $\frac{2\lfloor\sqrt{k}\rfloor}{k - \lfloor\sqrt{k}\rfloor^2} = -\dfrac{\mathrm{Tr}(P_k)}{\det(P_k)}$ is an integer.*

(c) *The plus continued fraction expansion for $\sqrt{k}$ has period at most 2.*

Theorem Minus. *The following statements are equivalent for the minus continued fraction expansion of $\sqrt{k}$.*

(a) *For all $n \geq 0$ we have $M_k^n\binom{1}{0} = \frac{r_n}{s_n}$. That is, the orbit of $1/0 \in \partial\mathbf{H}^2$ under M_k is precisely the set of convergents of the minus continued fraction expansion of $\sqrt{k}$.*

(b) *The quantity $\frac{2\lceil\sqrt{k}\rceil}{\lceil\sqrt{k}\rceil^2-k} = \dfrac{\mathrm{Tr}(M_k)}{\det(M_k)}$ is an integer.*

(c) *The minus continued fraction expansion for $\sqrt{k}$ has period at most 2.*

3. Continued Fractions Redux

The subject of continued fractions is quite old and well studied. There are several texts and articles about them and as such we refer the interested reader to the excellent books [Ka03] and [RS94] for the proofs of the many well known results stated in this section.

Continued fractions arose from studying expressions of the form,

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}}$$

Given any integer a_0 and a sequence of positive integers $(a_j)_{j\geq 1}$, the above expression defines a sequence of rational numbers $\frac{p_n}{q_n} := [a_0; a_1, a_2, \ldots, a_{n-1}]$. It turns out that $\lim_{n\to\infty} \frac{p_n}{q_n}$ always exists in this case and we say that

$$\alpha := \lim_{n\to\infty} \frac{p_n}{q_n} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{\ddots}}}}$$

is the *plus continued fraction expansion* of α (also called *regular continued fraction expansion*). By simply changing all the $'' + ''$ signs in the definition above to $'' - ''$ signs we obtain similarly the notion of *minus continued fraction expansion*. The next result captures one of the first basic facts about continued fractions.

Theorem 2. *Any irrational number α can be represented as an infinite plus continued fraction and any real numebr β as an infinite minus continued fraction.*

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{\ddots}}}}; \qquad \beta = b_0 - \cfrac{1}{b_1 - \cfrac{1}{b_2 - \cfrac{1}{\ddots - \cfrac{1}{\ddots}}}}$$

where $a_i \geq 1$ and $b_i \geq 2$ for all $i \geq 1$.

One can compute the numbers a_i by setting $a_0 = \lfloor \alpha \rfloor, \alpha_1 = \frac{1}{\alpha - a_0}$, and $a_{i+1} = \lfloor \alpha_{i+1} \rfloor$, where $\alpha_{i+1} = \frac{1}{\alpha_i - a_i}$. Similarly, the b_i's are computed from β by setting $b_0 = \lceil \beta \rceil, \beta_1 = -\frac{1}{\beta - b_0}$, and then $b_{i+1} = \lceil \beta_{i+1} \rceil$, where $\beta_{i+1} = -\frac{1}{\beta_i - b_i}$. The first of these recursion formulas will be used in the proof of Theorem Plus.

We now introduce some notation. From now on, we will denote plus/minus continued fractions as,

$$[a_0; a_1, a_2, \ldots] := a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{\ddots}}}}$$

$$(b_0; b_1, b_2, \ldots) := b_0 - \cfrac{1}{b_1 - \cfrac{1}{b_2 - \cfrac{1}{\ddots - \cfrac{1}{\ddots}}}}$$

If a continued fraction is periodic, for example $\sqrt{7} = [2; 1, 1, 1, 4, 1, 1, 1, 4, \ldots]$, then we will denote it as $\sqrt{7} = [2; \overline{1, 1, 1, 4}]$. Similarly, $\sqrt{29} = [5; \overline{2, 1, 1, 2, 10}]$ and $\sqrt{15} = (4; \overline{8})$. Now we will summarize the properties of plus and minus continued fractions in the form of a table. The statements and their proofs can be found in any standard text on the subject; for instance, see [RS94] for plus continued fractions and [Ka03] for minus continued fractions.

Note that the (plus) continued fraction expansion of α is finite if and only if α is rational. By the same token, the minus continued fraction $\beta = (b_0; b_1, b_2, \ldots)$ is rational if and only if there exists N such that $b_j = 2$ for all $j \geq N$ (since $(2; \overline{2}) = 1$).

In the table that follows, let $a_0, a_1, a_2, \ldots$ and $b_0, b_1, b_2, \ldots$ be infinite sequences of integers such that $a_i \geq 1$ and $b_i \geq 2$ if $i \geq 1$. Then $[a_0; a_1, a_2, \ldots]$ converges to a unique real number α, and similarly $(b_0; b_1, b_2, \ldots)$ converges to a unique real number β.

Define the following recursive sequences:

$$p_{i+1} = a_i p_i + p_{i-1}, \qquad r_{i+1} = b_i r_i - r_{i-1} \quad \text{for } i \geq 0$$

$$q_{i+1} = a_i q_i + q_{i-1}, \qquad s_{i+1} = b_i s_i - s_{i-1} \quad \text{for } i \geq 0$$

with initial values $\begin{matrix} p_{-1} = 0, & p_0 = 1 \\ q_{-1} = 1, & q_0 = 0 \end{matrix}$ and $\begin{matrix} r_{-1} = 0, & r_0 = 1 \\ s_{-1} = -1, & s_0 = 0 \end{matrix}$

<table>
<tr><th>Plus</th><th>Minus</th></tr>
<tr><td>$\alpha = [a_0; a_1, a_2, \ldots]$</td><td>$\beta = (b_0; b_1, b_2, \ldots)$</td></tr>
<tr><td>If $A_n := \frac{p_n}{q_n}$, then $A_n = [a_0; a_1, \ldots, a_{n-1}]$
and $\lim_{n\to\infty} A_n = [a_0; a_1, \ldots] = \alpha$.
$p_{i-1}q_i - p_i q_{i-1} = (-1)^{i-1}$ for all $i \geq 0$</td><td>If $B_n := \frac{r_n}{s_n}$, then $B_n = (b_0; b_1, \ldots, b_{n-1})$
and $\lim_{n\to\infty} B_n = (b_0; b_1, \ldots) = \beta$.
$r_{i-1}s_i - r_i s_{i-1} = 1$ for all $i \geq 0$</td></tr>
<tr><th colspan="2">Periodic continued fractions</th></tr>
<tr><td>$\alpha = [a_0; a_1, a_2, \ldots]$ is periodic i.e.,
$\alpha = [a_0; a_1, a_2, \ldots, \overline{a_l, a_{l+1}, \ldots, a_m}]$
$\iff$ α is a quadratic irrational.</td><td>$(b_0; b_1, b_2, \ldots)$ is periodic i.e.,
$\beta = (b_0; b_1, b_2, \ldots, \overline{b_l, b_{l+1}, \ldots, b_m})^*$
$\iff$ β is a quadratic irrational.
$*$ except $(b_0; b_1, \ldots, b_{l-1}, \overline{2})$ which is rational.</td></tr>
<tr><th colspan="2">$k \in \mathbf{Z}$ such that $k > 0$ and $d = \lfloor \sqrt{k} \rfloor < \sqrt{k} < \lceil \sqrt{k} \rceil = d + 1$</th></tr>
<tr><td>$\sqrt{k} = \left[d; \overline{a_1, a_2, \ldots, a_2, a_1, 2d}\right]$
For example,
$\sqrt{14} = [3; \overline{1, 2, 1, 6}]$
$\sqrt{73} = [8; \overline{1, 1, 5, 5, 1, 1, 16}]$</td><td>$\sqrt{k} = (d + 1; \overline{b_1, b_2, \ldots, b_2, b_1, 2(d+1)})$
For example,
$\sqrt{7} = (3; \overline{3, 6})$
$\sqrt{13} = (4; \overline{3, 3, 2, 2, 2, 2, 3, 3, 8})$</td></tr>
</table>

The following formula allows one to freely travel between the worlds of plus and minus continued fractions. The proof is left as an exercise for the interested reader (as in [Ka03]; see also [Zag81]).

$$[a_0; a_1, a_2, a_3, \ldots] = (a_0 + 1; \underbrace{2, 2, \ldots, 2}_{a_1 - 1}, a_2 + 2, \underbrace{2, 2, \ldots, 2}_{a_3 - 1}, a_4 + 2, \ldots)$$

For example, $\sqrt{13} = \left[3; \overline{1, 1, 1, 6}\right] = (4; \overline{3, 3, 2, 2, 2, 2, 3, 3, 8})$. We are now ready to prove the main theorems, plus and minus.

4. Proofs of Theorems Plus and Minus

Proof of Theorem Plus.

Recall that we have $\sqrt{k} = [d; d_1, d_2, \ldots]$.

(a) $\Rightarrow$ (b)

The equation $P_k^2(\frac{1}{0}) = \frac{p_2}{q_2}$ implies

$$\frac{d^2 + k}{2d} = \frac{dd_1 + 1}{d_1},$$

which in turn yields $d_1 = \frac{2d}{k - d^2}$. Then, since d_1 is an integer and $d := \lfloor \sqrt{k} \rfloor$, the result follows.

(b) $\Rightarrow$ (c)

The inequality $d < \sqrt{k} < d + 1$ gives us

$$\frac{2d}{k - d^2} < \frac{\sqrt{k} + d}{k - d^2} = \frac{1}{\sqrt{k} - d} < \frac{2d + 1}{k - d^2}.$$

Then, since $k - d^2$ is an integer greater than 0, there can be no integer between the two quantities on the far left and right sides of the above inequality. Thus, $d_1 := \left\lfloor \frac{1}{\sqrt{k} - d} \right\rfloor = \left\lfloor \frac{2d}{k - d^2} \right\rfloor = \frac{2d}{k - d^2}$. From this, we compute

$$d_2 := \left\lfloor \frac{1}{\frac{1}{\sqrt{k} - d} - d_1} \right\rfloor = \left\lfloor \frac{k - d^2}{\sqrt{k} - d} \right\rfloor = 2d$$

Continuing on, we see that $d_3 := \left\lfloor \frac{1}{(\sqrt{k}+d)-2d} \right\rfloor = \left\lfloor \frac{1}{\sqrt{k}-d} \right\rfloor = d_1$, and similarly $d_4 = 2d = d_2$. The continued fraction for $\sqrt{k}$ becomes $\left[d; \overline{\frac{2d}{k-d^2}, 2d}\right]$, which proves (c).

(c) $\Rightarrow$ (a)

Suppose $\sqrt{k}$ has a period 2 continued fraction expansion. From the well known theorems on the continued fraction expansions of quadratic surds (see Table), we know that it must be of the form $\sqrt{k} = [d; \overline{d_1, 2d}]$. Upon examining this expression, one can see that

$$\frac{1}{\sqrt{k}-d} = d_1 + \cfrac{1}{2d + \cfrac{1}{\cfrac{1}{\sqrt{k}-d}}}$$

Simplifying this equation gives $d_1 = \frac{2d}{k-d^2}$. Now, we use this information to get a different way to express p_n and q_n, the numerators and denominators of our convergents.

Define two sequences, $\{a_n\}$ and $\{b_n\}$, recursively using the following matrix notation. Define

$$\begin{pmatrix} a_1 & a_0 \\ b_1 & b_0 \end{pmatrix} := \begin{pmatrix} d & 1 \\ 1 & 0 \end{pmatrix},$$

and

$$\begin{pmatrix} a_n & a_{n-1} \\ b_n & b_{n-1} \end{pmatrix} = \begin{pmatrix} a_{n-1} & a_{n-2} \\ b_{n-1} & b_{n-2} \end{pmatrix} \begin{pmatrix} 2d & 1 \\ k-d^2 & 0 \end{pmatrix},$$

for all $n \geq 2$. We claim that

$$p_n = \frac{a_n}{(k-d^2)^{\lfloor n/2 \rfloor}} \text{ and } q_n = \frac{b_n}{(k-d^2)^{\lfloor n/2 \rfloor}}$$

for all $n \geq 0$. The reader can easily verify that the assertion holds for $n = 0, 1$. Then, for $n \geq 2$, we have two cases. First, if n is even, $\lfloor \frac{n-1}{2} \rfloor = n/2 - 1 = \lfloor \frac{n-2}{2} \rfloor$ and the recursion formula for p_n yields $p_n = \frac{2d}{k-d^2} p_{n-1} + p_{n-2}$ which, by induction is

$$\frac{2d}{(k-d^2)^{1+\lfloor (n-1)/2 \rfloor}} a_{n-1} + \frac{1}{(k-d^2)^{\lfloor (n-2)/2 \rfloor}} a_{n-2}$$

Using the remarks above and the formula for a_n verifies our claim for p_n. A similar argument proves the statement about q_n. Now, we have $p_n/q_n = a_n/b_n$, so to finish the proof, we will show that $P_k^n(\frac{1}{0}) = a_n/b_n$. To this end, define

$$\begin{pmatrix} s_0 \\ t_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \text{ and } \begin{pmatrix} s_n \\ t_n \end{pmatrix} = \begin{pmatrix} d & k \\ 1 & d \end{pmatrix} \begin{pmatrix} s_{n-1} \\ t_{n-1} \end{pmatrix}$$

It is clear that $P_k^n(\frac{1}{0}) = \frac{s_n}{t_n}$, so if we show that $s_n = a_n$ and $t_n = b_n$ we are done. The formula for s_n and t_n can be written as

$$\begin{pmatrix} s_n \\ t_n \end{pmatrix} = \begin{pmatrix} d & k \\ 1 & d \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

In addition, the formula for a_n and b_n can be written as

$$\begin{pmatrix} a_n \\ b_n \end{pmatrix} = \begin{pmatrix} d & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2d & 1 \\ k-d^2 & 0 \end{pmatrix}^{n-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Write the right hand side of the former equation as $P_k^n\binom{1}{0}$, and the right hand side of the latter equation as $AB^{n-1}\binom{1}{0}$. Then, observe that $AB = P_kA$, so $(a_n, b_n) = (s_n, t_n)$ follows by induction. □

Proof of Theorem Minus.

Let k be a non-square positive integer, and let $(d+1; e_1, e_2, \ldots)$ be the minus continued fraction for $\sqrt{k}$.

(a) ⇒ (b)

The equation $M_k^2(\frac{1}{0}) = \frac{r_2}{s_2}$ implies

$$\frac{(d+1)^2 + k}{2(d+1)} = \frac{(d+1)e_1 - 1}{e_1}$$

from which it follows that $e_1 = \frac{2(d+1)}{(d+1)^2 - k}$. This, along with $d+1 = \lceil\sqrt{k}\rceil$, and the fact that e_1 is an integer, gives the desired result.

(b) ⇒ (c)

If $\frac{2\lceil\sqrt{k}\rceil}{\lceil\sqrt{k}\rceil^2 - k}$ is an integer, then let $d+1 = \lceil\sqrt{k}\rceil, e_1 = \frac{2\lceil\sqrt{k}\rceil}{\lceil\sqrt{k}\rceil^2 - k}$ and $e_2 = 2(d+1)$. Let $x = (d+1; \overline{e_1, e_2})$. It can be seen that x satisfies

$$x = d + 1 - \cfrac{1}{e_1 - \frac{1}{e_2 + (x - (d+1))}}$$

This yields a quadratic equation whose only positive root is $\sqrt{k}$. Therefore, $\sqrt{k} = (d+1; \overline{e_1, e_2})$, and the minus continued fraction for $\sqrt{k}$ has period at most 2.

(c) $\Rightarrow$ (a)

Suppose the minus continued fraction for $\sqrt{k}$ has period at most 2. That is, $\sqrt{k} = (d+1; \overline{e_1, e_2})$, for some $d+1, e_1, e_2$. From this, we see

$$\sqrt{k} = d + 1 - \cfrac{1}{e_1 - \frac{1}{e_2 - ((d+1) - \sqrt{k})}}.$$

Rearranging, this gives

$$e_1(2(d+1) - e_2)\sqrt{k} + ((d+1)e_1e_2 - (d+1)^2e_1 - ke_1 - e_2) = 0$$

Because $\sqrt{k}$ is irrational, we must have

$$\begin{aligned} e_1(2(d+1) - e_2) =& \ 0 \\ (d+1)e_1e_2 - (d+1)^2e_1 - ke_1 - e_2 =& \ 0. \end{aligned}$$

In the first equation, $e_1 \neq 0$, so $e_2 = 2(d+1)$. Substituting for e_2 in the second equation gives $((d+1)^2 - k)e_1 - 2(d+1) = 0$, which in turn gives $e_1 = \frac{2(d+1)}{(d+1)^2 - k}$. Note also that we have $d + 1 = \lceil \sqrt{k} \rceil$.

Next, we define the following sequence. For $n \geq 1$, define

$$\begin{pmatrix} a_n \\ b_n \end{pmatrix} = \begin{pmatrix} d+1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2(d+1) & -1 \\ (d+1)^2 - k & 0 \end{pmatrix}^{n-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

¿From this definition, it can be shown that a_n satisfies the recurrence relation $a_n = 2(d+1)a_{n-1} - ((d+1)^2 - k)a_{n-2}$, with a similar recurrence for b_n. We then claim that

$$r_n = \frac{a_n}{((d+1)^2 - k)^{\lfloor n/2 \rfloor}}$$

and

$$s_n = \frac{b_n}{((d+1)^2 - k)^{\lfloor n/2 \rfloor}}$$

which is true by induction. For $n = 1, 2$, this is easily verified. For $n \geq 3$, we have two cases. When n is even, we have, via the recursion formula for r_n:

$$\begin{aligned} r_n &= e_1 r_{n-1} - r_{n-2} \\ &= \frac{2(d+1)a_{n-1}}{((d+1)^2-k)^{n/2}} - \frac{((d+1)^2-k)a_{n-2}}{((d+1)^2-k)^{n/2}} \\ &= \frac{a_n}{((d+1)^2-k)^{n/2}} \end{aligned}$$

Similar arguments hold when n is odd, and for s_n.

Next, we claim that if we view the map M_k as a matrix, then

$$M_k^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a_n \\ b_n \end{pmatrix}$$

To see this, let $A = \begin{pmatrix} d+1 & -1 \\ 1 & 0 \end{pmatrix}, B = \begin{pmatrix} 2(d+1) & -1 \\ (d+1)^2-k & 0 \end{pmatrix}$. Then, the equation above reduces to showing that $AB^{n-1}\binom{1}{0} = M_k^n\binom{1}{0}$. If we note that $AB = M_k A$ and that $\binom{1}{0}$ is a fixed point of BA^{-1}, we get our result:

$$M_k^n\tbinom{1}{0} = (ABA^{-1})^n\tbinom{1}{0} = AB^nA^{-1}\tbinom{1}{0} = AB^{n-1}(BA^{-1})\tbinom{1}{0} = AB^{n-1}\tbinom{1}{0}$$

which implies that

$$\frac{r_n}{s_n} = \frac{\frac{a_n}{((d+1)^2-k)^{\lfloor n/2 \rfloor}}}{\frac{b_n}{((d+1)^2-k)^{\lfloor n/2 \rfloor}}} = \frac{a_n}{b_n} = M_k^n\tbinom{1}{0}$$

which completes the proof. □

5. Closing Remarks

5.1. How many k such that $\sqrt{k}$ has period 2 continued fraction?

Given consecutive squares, $d^2, (d+1)^2$, there are $2d$ numbers between them. So if $d^2 < k < (d+1)^2$, then the plus continued fraction of $\sqrt{k}$ has period at most 2 if and only if $k - d^2$ is a divisor of $2d$, where $1 \leq k - d^2 \leq 2d$. Thus there are precisely $\sigma_0(2d)$ numbers between the consecutive squares $d^2, (d+1)^2$ that have period at most 2, where $\sigma_0(j)$ is the number of divisors of j. Similarly, there are $\sigma_0(2d+2) - 1$ numbers with minus continued fraction of period at most 2 (we subtract 1, since the divisor $2d+2$ cannot be realized).

In general, $\sigma_0(2d) \geq 4$ (counting the obvious divisors $1, 2, d, 2d$), and the lower bound is sharp for d prime. Moreover, it is not hard to see that for $d > 3$, there are exactly two numbers between d^2 and $(d+1)^2$ which have period at most 2 simultaneously for their plus and minus continued fractions; they are $d^2 + d$ and $d^2 + 2d = (d+1)^2 - 1$. Along these lines, observe that $d^2 + d + 1 = d(d+1) + 1$ always has period larger than 2 for either continued fraction.

5.2. **Matrix convergents vs. continued fraction convergents.**

When k does not satisfy the conditions of either theorem, then the matrix yields a sequence of rational numbers that converge quite fast to $\sqrt{k}$, but have no apparent relation to the continued fraction convergents. The fact that P_k and M_k applied to $\binom{1}{0}$ converge to $\sqrt{k}$ is a fairly straightforward exercise in the linear algebra of irreducible Perron–Frobenius matrices.

However, there is one strange phenomenon for which we do not have a complete proof (and leave as a conjecture for the interested reader). When the quantity $\frac{2d}{k-d^2}$ is a half-integer, then the matrix convergents form a proper subset of the (plus) continued fraction convergents in a precise way. For example, consider $\sqrt{13}$ and $\sqrt{44}$.

$$\sqrt{\mathbf{13}}; \quad d = 3, k = 13, \quad \frac{2d}{k-d^2} = \tfrac{3}{2}, \quad \sqrt{13} = [3; \overline{1, 1, 1, 1, 6}], \quad P_{13} = \begin{pmatrix} 3 & 13 \\ 1 & 3 \end{pmatrix}$$

Continued fraction convergents:

$$\left\{3; 4, \tfrac{7}{2}, \frac{11}{3}, \frac{18}{5}, \frac{119}{33}, \tfrac{137}{38}, \tfrac{256}{71}, \frac{393}{109}, \frac{649}{180}, \frac{4287}{1189}, \tfrac{4936}{1369}, \tfrac{9223}{2558}, \frac{14159}{3927}, \frac{23382}{6485}, \ldots\right\}$$

Matrix convergents; $P_{13}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$\left\{3; \frac{11}{3}, \frac{18}{5}, \frac{119}{33}, \frac{393}{109}, \frac{649}{180}, \frac{4287}{1189}, \frac{14159}{3927}, \frac{23382}{6485}, \frac{154451}{42837}, \ldots\right\}$$

The matrix convergents are picked from the continued fraction convergents in the following pattern: **skip two, pick three** (and also note that the continued fraction of $\sqrt{13}$ has period $5 = 2 + 3$). Now observe a similar phenomenon for $\sqrt{44}$.

$\sqrt{\mathbf{44}}$;

$$d = 6, k = 44, \quad \frac{2d}{k-d^2} = \tfrac{3}{2}, \quad \sqrt{44} = [6; \overline{1,1,1,2,1,1,1,12}], P_{44} = \begin{pmatrix} 6 & 44 \\ 1 & 6 \end{pmatrix}$$

Continued fraction convergents:

$$\left\{6; 7, \tfrac{13}{2}, \frac{20}{3}, \tfrac{53}{8}, \tfrac{73}{11}, \frac{126}{19}, \frac{199}{30}, \frac{2514}{379}, \tfrac{2713}{409}, \tfrac{5227}{788}, \frac{7940}{1197}, \tfrac{21107}{3182}, \tfrac{29047}{4379}, \frac{50154}{7561}, \ldots\right\}$$

Matrix convergents; $P_{44}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$\left\{6; \frac{20}{3}, \frac{126}{19}, \frac{199}{30}, \frac{2514}{379}, \frac{7940}{1197}, \frac{50154}{7561}, \frac{79201}{11940}, \frac{1000566}{150841}, \ldots\right\}$$

The matrix convergents are picked from the continued fraction convergents in the following pattern: **skip two, pick one, skip two, pick three** (and note that the continued fraction of $\sqrt{44}$ has period $8 = 2+1+2+3$).

Elementary arithmetic shows that such k do not exist if d is a power of 2. In general, the number of such $k \in (d^2, (d+1)^2)$ for which $\frac{2d}{k-d^2}$ is a half integer depends on the number of odd factors of d: if $d = 2^j m$, where m is odd, then the number of such k is $\sigma_0(m) - 1$. When such k exist, we formulate the following conjecture.

Conjecture 1. *Let k be a positive integer, not a perfect square, and let $d = \lfloor \sqrt{k} \rfloor$. If $\frac{2d}{k-d^2}$ is a half-integer, then the matrix convergents of $\sqrt{k}$ are a proper subset of the continued fraction convergents in the following pattern: "skip two, pick three", if d is odd and "skip two, pick one, skip two, pick three" if d is even.*

In addition, it seems that the period of the continued fraction divides 10 if d is odd, and the period divides 8 if d is even. In the case of d odd, the period is always 5 if $k = d^2 + 4$, and the period is 10 in other cases. Moreover, there are possibly more conjectures for patterns of the continued fraction when $\frac{2d}{k-d^2}$ is in $\mathbf{Z}_{(2)} = \{\frac{m}{2^n}$, where $n \in \mathbf{Z}\}$, but we don't have precise statements, so we leave that to an enterprising and interested reader.

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