ISOMETRY GROUPS OF HOMOGENEOUS SPACES WITH POSITIVE SECTIONAL CURVATURE

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Abstract. We calculate the full isometry group in the case G/H admits a homogeneous metric of positive sectional curvature.

Introduction

In this paper, we compute the isometry groups of simply connected, homogeneous, positively curved manifolds. These were classified by Berger, Aloff, Wallach and Bérard Bergery. The primary motivation for computing these isometry groups stemmed from looking for free, isometric actions of finite groups. These were investigated in order to construct counterexamples to an old obstruction proposed by S. S. Chern (also called Chern's conjecture) for fundamental groups of positively curved manifolds. Counterexamples to Chern's conjecture were first constructed in [11] and [6].

In order to compute the full isometry group, G, of a homogeneous space, $M = G_0/H_0$, we first obeserve that the group of components, G/G_0, is finite. We then determine the full isotropy group, H, which helps us get hold of the group of components, $G/G_0 = H/H_0$, by studying the group, $\mathrm{Aut}(G_0, H_0)$. This in essence means that the pair (G, H) may be determined from the pair of Lie algebras, $(\mathfrak{g}, \mathfrak{h})$.

The next step is to study enlargements of transitive actions that are visible at the Lie algebra level. Namely, given a G_0-invariant metric on M, we may have a strict containment, $G_0 \subsetneq \mathbf{I}_0(M)$. Such extensions are handled following the work of A. L. Onishchik (see Section 2). Finally, in Section 4, this program is carried out for the entire list of simply connected, homogeneous spaces with positive sectional curvature. The results are summed up in Table 3 on page 22.

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1. Preliminaries

One systematic way to study positively curved manifolds is to look at the ones with large isometry groups. By large, here, we mean those manifolds on which the isometry group acts transitively. If a compact Lie group, G, acts transitively on M, then M is diffeomorphic to the left coset space G/H. If we put a G-invariant Riemannian metric on M (which always exists if G is compact), then it can be realized by a Riemannian submersion,

$$H \longrightarrow G \xrightarrow{\pi} M = G/H$$

so that the metric on M is submersed from a left invariant metric on G. Here $H \subset G$ is a closed subgroup which is the isotropy group at a point; the metric must necessarily be right invariant with respect to H. It is then possible to compute the sectional curvatures of 2–planes in M using the Gray–O'Neill submersion formulas. Classically, it was known that the spheres, $\mathbb{S}^n$, and the projective spaces, $\mathbb{CP}^m$, $\mathbb{HP}^k$, $\mathrm{Ca}\mathbb{P}^2$ admit positive curvature. They can be described as homogeneous spaces and the well known metrics on them may be obtained by submersing the canonical bi-invariant metric on $\mathrm{SO}(n+1)$, $\mathrm{SU}(m+1)$, $\mathrm{Sp}(k+1)$ and F_4 respectively.

Simply connected homogeneous spaces of positive curvature were classified in [3], [14], [1], [2]. In Table 1 we present the complete classification list.

Even dimensions	**Odd dimensions**
$\mathbb{S}^{2n} = \mathrm{SO}(2n+1)/\mathrm{SO}(2n)$	$\mathbb{S}^{2n+1} = \mathrm{SO}(2n+2)/\mathrm{SO}(2n+1)$
$\mathbb{CP}^m = \mathrm{SU}(m+1)/\mathrm{SU}(m)$	$M^7 = \mathrm{SO}(5)/\mathrm{SO}(3)$
$\mathbb{HP}^k = \mathrm{Sp}(k+1)/(\mathrm{Sp}(k) \times \mathrm{Sp}(1))$	$M^{13} = \mathrm{SU}(5)/(\mathrm{Sp}(2) \times_{\mathbb{Z}_2} S^1)$
$\mathrm{Ca}\mathbb{P}^2 = F_4/\mathrm{Spin}(9)$	$N_{1,1} = \mathrm{SU}(3) \times \mathrm{SO}(3)/\mathrm{U}^*(2)$
$F^6 = \mathrm{SU}(3)/\mathrm{T}^2$	$N_{k,l} = \mathrm{SU}(3)/S^1_{k,l}$, $\gcd(k,l) = 1$,
$F^{12} = \mathrm{Sp}(3)/(\mathrm{Sp}(1) \times \mathrm{Sp}(1) \times \mathrm{Sp}(1))$	$kl(k+l) \neq 0$, $(k,l) \neq (1,1)$
$F^{24} = F_4/\mathrm{Spin}(8)$	

Table 1. Simply connected, homogeneous spaces with positive curvature.

Recently B. Wilking found an error in [3]; he showed that Berger had missed $N_{1,1}$ from the Aloff-Wallach family, $N_{k,l}$, which is normal homogeneous (cf. [17]). Table 1 reflects that correction where the examples above the dotted lines are normal homogeneous. We now recall a simple fact which will be useful to us later.

Lemma 1.1. *Let G act freely and isometrically on a manifold M and let $H \triangleleft G$. Then there is an induced ineffective action of G on M/H with kernel H. In particular, G/H acts freely and isometrically on M/H with quotient M/G.*

From the previous lemma, we see that the natural action of the normalizer, $N(H)$, on G/H induces a free action of $N(H)/H$ on G/H.

$$\begin{aligned} N(H) \times G/H &\longrightarrow G/H \\ (n, gH) &\longmapsto gH \cdot n^{-1} = gn^{-1}H \end{aligned}$$

The group, $N(H)/H$, will show up frequently. We call it the *generalized Weyl group* and denote it by W_H. Note that we have extended the standard action of G on G/H to an action of $G \times W_H$. However, the total action need not be effective. The next proposition is well known.

Proposition 1.2. *Let G be a simple, compact, connected Lie group. The kernel of the extended $G \times W_H$ action on G/H is precisely $\Delta Z(G)$, the diagonal image of the center in $G \times W_H$.*

The isotropy representation.

Consider the identity coset eH, the orbit of the identity element $e \in G$. Then the action of $H \subset G$ fixes eH. This affords a representation of H on the complement $\mathfrak{p}$ which is identified with $T_{eH}G/H$,

$$\begin{aligned} H \times \mathfrak{p} &\longrightarrow \mathfrak{p} \\ (h, X) &\longrightarrow dh(X) \end{aligned}$$

The differential dh acting on $\mathfrak{p}$ is simply the adjoint representation of H, where $\mathfrak{g} = \mathfrak{h} + \mathfrak{p}$ is an $\mathrm{Ad}(H)$-invariant decomposition.

Definition. Let (M, g) be a Riemannian manifold with isometry group $\hat{G}$ and isotropy group $\hat{H}_p$ for $p \in M$. If the isotropy representation of $\hat{H}_p$ on the tangent space T_pM is irreducible for every $p \in M$, then M is said to be *isotropy irreducible.*

By considering the action of the isotropy group $\hat{H}_p$ on a principal orbit, it follows that an isotropy irreducible space must be homogeneous. A homogeneous space G/H is said to be *strongly isotropy irreducible* if the identity component, H_0, acts irreducibly on the complement $\mathfrak{p}$. J. Wolf classified all strongly isotropy irreducible homogeneous spaces of compact groups in [18]. In particular, among our list of homogeneous manifolds with positive curvature, all compact, rank one, symmetric spaces are strongly isotropy irreducible. The other spaces that are isotropy irreducible are the normal homogeneous Berger example, $M^7 = \mathrm{SO}(5)/\mathrm{SO}(3)$, and the spheres, $\mathbb{S}^6 = G_2/\mathrm{SU}(3)$ and $\mathbb{S}^7 = \mathrm{Spin}(7)/G_2$. The next lemma is straightforward.

Lemma 1.3. *If G/H is an isotropy irreducible homogeneous space, then the Weyl group W_H is finite.*

Automorphism groups of compact Lie groups.

Let G be a compact, connected Lie group and let $\sigma : G \to G$ be an automorphism. Then $\sigma(Z) = Z$, where $Z = Z(G)$ is the center. This is because $\sigma(Z) \subseteq Z$ and σ is invertible. This defines a map,

$$\varphi : \mathrm{Aut}(G) \to \mathrm{Aut}(G/Z)$$
$$\sigma \mapsto \bar{\sigma}$$

where $\bar{\sigma}(gZ) = \sigma(g)Z$. If $\alpha \in \ker(\varphi)$, then $\alpha(g) = \mathfrak{z}(g)\cdot g$, where $\mathfrak{z} : G \to Z$ is a homomorphism into the center. $\ker(\varphi)$ is trivial if and only if $\mathfrak{z}$ is the trivial map. When G is a compact semisimple group, the center is discrete, hence finite. So for compact, connected, semisimple groups, $\mathrm{Aut}(G) \cong \mathrm{Aut}(G/Z)$.

Theorem 1.4. *Let G be a compact, connected, semisimple Lie group. Then $\mathrm{Inn}(G) \subset \mathrm{Aut}(G)$ is a normal subgroup of finite index. The groups fit into a (split) exact sequence,*

$$1 \to \mathrm{Inn}(G) \to \mathrm{Aut}(G) \to \mathrm{Out}(G) \to 1$$

where $\mathrm{Out}(G)$, which is called the group of outer automorphisms, is given by symmetries of the Dynkin diagram of G.

The full automorphism group of a semisimple group can be computed with the help of the previous theorem. The proof for the simple case can be found in [7]. The extension to the semisimple case is easy (see for instance [12]).

The automorphism group of $\mathrm{U}(n)$. For a compact, connected Lie group G, recall the map, $\varphi : \mathrm{Aut}(G) \to \mathrm{Aut}(G/Z)$. If $\sigma \in \ker(\varphi)$, then $\sigma(g) = \mathfrak{z}(g) \cdot g$ for all $g \in G$, where $\mathfrak{z} : G \to Z$ is a homomorphism into the center. A homomorphism of $\mathrm{U}(n)$ into its center, S^1, is a 1-dimensional representation. $\mathrm{U}(n)$ has infinitely many 1-dimensional representations (cf. [4]). All of these factor through the determinant, since the map, $\det : \mathrm{U}(n) \to \mathrm{U}(1)$, can be identified with the quotient of $\mathrm{U}(n)$ by its commutator, $\mathrm{SU}(n)$. It follows that any $\mathbb{C}$-representation of $\mathrm{U}(n)$ looks like $A \mapsto \det(A)^m$ for some $m \in \mathbb{Z}$. Now $\sigma \in \ker(\varphi)$ implies $\sigma^{-1} \in \ker(\varphi)$ and we have,

$$\sigma(A) = \det(A)^m A$$
$$\sigma^{-1}(A) = \det(A)^k A$$

for some fixed $k, m \in \mathbb{Z}$. A simple calculation yields

$$\sigma^{-1} \circ \sigma(A) = \det(A)^{nkm+k+m} \cdot A$$

Since this map must be the identity we must have $nkm + k + m = 0$ which is equivalent to $n + \frac{1}{m} + \frac{1}{k} = 0$ for non-zero k, m. This has a unique solution in integers: $n = -2m = -2k = 2$ which implies $\sigma^2 = \mathrm{id}$. We have,

Theorem 1.5. *The map,* $\varphi : \mathrm{Aut}(\mathrm{U}(n)) \to \mathrm{Aut}(\mathrm{PSU}(n))$*, is an isomorphism for* $n > 2$*.*

A compact, connected, semisimple Lie group G does not admit any non-trivial $\mathbb{C}$-representation (cf. [4]). The above considerations then show

Theorem 1.6. *For any compact, connected, semisimple Lie group* G*,* $\mathrm{Aut}(G \times S^1) \cong \mathrm{Aut}(G) \times \mathrm{Aut}(S^1)$*.*

The group of components.

Given a homogeneous space, $M = G/H$, we now assume that $G = \mathbf{I}(M)$ is the full isometry group for some homogeneous metric on M. Let G_0 and H_0 be the identity components of the isometry group and the isotropy group respectively. Then $G/G_0 \cong H/H_0$ is the *group of components*. Most of the arguments below are along the lines of [16, Section 3].

Definition. The set of automorphisms of a compact Lie group, G, that fix a subgroup, H, will be denoted by $\mathrm{Aut}(G, H)$. The groups $\mathrm{Inn}(G, H)$ and $\mathrm{Out}(G, H)$ are defined analogously.

The adjoint action (conjugation) allows us to define a map,

$$\begin{aligned} \mathrm{Ad} : & H \longrightarrow \mathrm{Aut}(G_0, H_0) \\ & h \longmapsto (\mathrm{Ad}_h : x \mapsto hxh^{-1}) \end{aligned}$$

This is actually a map into $\mathrm{Aut}(G_0)$ whose image lies in $\mathrm{Aut}(G_0, H_0)$. The kernel of this map is,

$$\begin{aligned} \ker(\mathrm{Ad}) &= \{h \in H : hxh^{-1} = x \quad \forall\, x \in G_0\} \\ &= C_G(G_0) \cap H \end{aligned}$$

By assumption, G is the isometry group, so $M = G/H = G_0/H_0$ is an effective coset space. This will allow us to show

Proposition 1.7. *The map,* $\mathrm{Ad} : H \to \mathrm{Aut}(G_0, H_0)$*, is an injection.*

Proof. Suppose $h \in \ker(\mathrm{Ad})$. Then the left action of h fixes the identity coset, eH. Let U be a small neighborhood of the identity, e, in G; then $U \subset G_0$. If $\pi : G \to G/H$, then $\pi(U)$ is an open set containing eH. Then for all $x \in U$, $h \cdot xH = xH$, since $h \in C_G(G_0)$. This implies that h acts ineffectively on $\pi(U)$ and its differential at eH must be the identity and hence $h = e$. ■

This shows that H/H_0 is a subgroup of $\mathrm{Aut}(G_0, H_0)/\mathrm{Ad}(H_0)$. Consider the short exact sequence,

$$1 \longrightarrow \mathrm{Inn}(G_0, H_0) \longrightarrow \mathrm{Aut}(G_0, H_0) \longrightarrow Q \longrightarrow 1$$

where Q is the quotient. Since H_0 is a subgroup of both groups, $\mathrm{Inn}(G_0, H_0)$ and $\mathrm{Aut}(G_0, H_0)$, we may consider the reduced sequence, which leaves the quotient unchanged,

$$1 \longrightarrow \mathrm{Inn}(G_0, H_0)/H_0 \longrightarrow \mathrm{Aut}(G_0, H_0)/H_0 \longrightarrow Q \longrightarrow 1$$

Since the action of G_0 is effective, $H_0 \cap Z(G_0)$ must be trivial. This allows us to identify, $\mathrm{Inn}(G_0, H_0) \cong N(H_0)/Z(G_0)$. We now have a commutative diagram with exact rows; the maps i_1 and i_2 are inclusions.

$$\begin{tikzcd}
1 \arrow[r] & N(H_0)/Z(G_0) \arrow[r] \arrow[d, hook, "i_1"'] & \mathrm{Aut}(G_0, H_0) \arrow[r] \arrow[d, hook, "i_2"'] & Q \arrow[r] \arrow[d, dotted, "\phi"'] & 1 \\
1 \arrow[r] & \mathrm{Inn}(G_0) \arrow[r] & \mathrm{Aut}(G_0) \arrow[r] & \mathrm{Out}(G_0) \arrow[r] & 1
\end{tikzcd}$$

The following lemma will clarify things a bit by helping us compute the induced map ϕ. The proof of the lemma is a simple exercise in 'diagram chasing'.

Lemma 1.8. *Given a commutative diagram of groups with exact rows,*

$$\begin{tikzcd}
1 \arrow[r] & H_1 \arrow[r, "f"] \arrow[d, hook, "\phi_1"'] & H_2 \arrow[r, "g"] \arrow[d, hook, "\phi_2"'] & H_3 \arrow[r] \arrow[d, dotted, "\phi_3"'] & 1 \\
1 \arrow[r] & G_1 \arrow[r, "f'"'] & G_2 \arrow[r, "g'"'] & G_3 \arrow[r] & 1
\end{tikzcd}$$

where the maps ϕ_1 and ϕ_2 are injections and ϕ_3 is the induced map. If $H_1 = G_1 \cap H_2$, then ϕ_3 is a monomorphism also.

This now implies that the quotient $\mathrm{Aut}(G_0, H_0)/\mathrm{Inn}(G_0, H_0) = Q$ is a subgroup of $\mathrm{Out}(G_0)$; the splitting of the short exact sequence for $\mathrm{Aut}(G_0)$ yields the splitting for $\mathrm{Aut}(G_0, H_0)$. So we may regard Q as the subgroup of outer automorphisms that fixes H_0. Note also that the previous lemma allows us to bound the number of components, H/H_0; since the group of components is a subgroup of $\mathrm{Aut}(G_0, H_0)/H_0$, its order is bounded by the orders of $\mathrm{Inn}(G_0, H_0) = N(H_0)/Z(G_0)H_0$ and Q.

To compute the full isometry group, we need two ingredients: the identity component and the group of components. Lemma 1.8 will then allow us to put the two together. All the positively curved homogeneous spaces can be written as quotients of simple groups (see Table 1). We saw above that we can always extend the action by the generalized Weyl group W_H. In the next section we will see that enlargements of transitive actions are restricted to a few types and we will use this to compute the identity component.

2. Enlargements of transitive actions

We briefly describe the theory of enlargements of transitive actions. The theory was developed by A. L. Onishchik (cf. [8]) and the results stated in this section are proved in his original paper. Recall that a Lie algebra, $\mathfrak{g}$, is said to be *compact* if it is the Lie algebra of some compact Lie group.

Definition. Let $(\mathfrak{g}', \mathfrak{g}, \mathfrak{k})$ be a triple of Lie algebras, where $\mathfrak{g}$ and $\mathfrak{k}$ are subalgebras of $\mathfrak{g}'$. The triple is said to be a *decomposition* if $\mathfrak{g}' = \mathfrak{g} + \mathfrak{k}$.

If G' is the simply connected Lie group corresponding to $\mathfrak{g}'$ and G, K are the corresponding connected Lie subgroups of G', then $(\mathfrak{g}', \mathfrak{g}, \mathfrak{k})$ is a decomposition if and only if G acts transitively on G'/K. This is equivalent to saying $G' = G \cdot K$.

Definition. The pair $(\mathfrak{g}', \mathfrak{k})$ is called an *extension* or *enlargement* of the pair, $(\mathfrak{g}, \mathfrak{h})$, if $(\mathfrak{g}', \mathfrak{g}, \mathfrak{k})$ is a decomposition and $\mathfrak{h} = \mathfrak{g} \cap \mathfrak{k}$.

An extension is *effective* if $\mathfrak{g}'$ and $\mathfrak{k}$ have no non-trivial ideal in common. This corresponds to saying that the action of G' on G'/K is almost effective. We will only be concerned with extensions of simple algebras i.e. we will assume that $\mathfrak{g}$ is simple. Although this is not required for the definitions, this is assumed in the results. We identify three types of extensions.

Extensions of Type I. This is essentially an extension by the normalizer. $(\mathfrak{g}', \mathfrak{k})$ is said to be a *Type I extension* of $(\mathfrak{g}, \mathfrak{h})$ if there exists a subalgebra $\mathfrak{a} \subset \mathfrak{g}$ such that, $\mathfrak{h} \oplus \mathfrak{a} \subset \mathfrak{g}$, $\mathfrak{g}' = \mathfrak{g} \oplus \mathfrak{a}$, $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{a}$. The inclusion, $\mathfrak{a} \hookrightarrow \mathfrak{g}'$, is the diagonal embedding. So to have a Type I extension, the normalizer of $\mathfrak{h}$ in $\mathfrak{g}$ must be non-empty.

Example. Consider the Aloff-Wallach space, $N^7_{1,1} = \mathrm{SU}(3)/S^1_{1,1}$. This has positive curvature for a left-invariant, $\mathrm{Ad}(\mathrm{U}(2))$-invariant metric on SU(3). The normalizer of the circle $S^1_{1,1}$ is U(2) and the Weyl group is SO(3). Now we take $\mathfrak{g} = \mathfrak{su}(3)$, $\mathfrak{h} = \mathfrak{s}^1 = i\mathbb{R}$ and $\mathfrak{a} = \mathfrak{so}(3)$ to see that we have a Type I extension.

Extensions of Type II. $(\mathfrak{g}', \mathfrak{g}'')$ is called a *Type II extension* of $(\mathfrak{g}, \mathfrak{h})$ if $\mathfrak{g}'$ is simple. All such extensions are classified in Onishchik's original paper (cf. [8]); we present these in Table 2.

$\mathfrak{g}'$	$\mathfrak{g}$	$\mathfrak{g}''$	$\mathfrak{h} = \mathfrak{g} \cap \mathfrak{g}''$	
$\mathfrak{su}(2n)$,	$\mathfrak{sp}(n)$	$\mathfrak{su}(2n-1)$	$\mathfrak{sp}(n-1)$	◀
$n > 1$		$\mathfrak{su}(2n-1) \oplus \mathfrak{u}(1)$	$\mathfrak{sp}(n-1) \oplus \mathfrak{u}(1)$	◀
$\mathfrak{so}(7)$	$\mathfrak{g}_2$	$\mathfrak{so}(5)$	$\mathfrak{su}(2)$	
		$\mathfrak{so}(5) \oplus \mathfrak{u}(1)$	$\mathfrak{su}(2) \oplus \mathfrak{u}(1)$	
		$\mathfrak{so}(6)$	$\mathfrak{su}(3)$	◀
$\mathfrak{so}(2n+2)$	$\mathfrak{su}(n+1)$	$\mathfrak{so}(2n+1)$	$\mathfrak{su}(n)$	◀
$n > 2$	$\mathfrak{su}(n+1) \oplus \mathfrak{u}(1)$		$\mathfrak{su}(n) \oplus \mathfrak{u}(1)$	◀
	$\mathfrak{sp}(n)$		$\mathfrak{sp}(n-1)$	◀
$\mathfrak{so}(4n)$	$\mathfrak{sp}(n) \oplus \mathfrak{u}(1)$	$\mathfrak{so}(4n-1)$	$\mathfrak{sp}(n-1) \oplus \mathfrak{u}(1)$	◀
$n > 1$	$\mathfrak{sp}(n) \oplus \mathfrak{sp}(1)$		$\mathfrak{sp}(n-1) \oplus \mathfrak{sp}(1)$	◀
$\mathfrak{so}(16)$	$\mathfrak{spin}(9)$	$\mathfrak{so}(15)$	$\mathfrak{spin}(7)$	◀
$\mathfrak{so}(8)$	$\mathfrak{sp}(2)$	$\mathfrak{so}(7)$	$\mathfrak{sp}(1)$	◀
	$\mathfrak{sp}(2) \oplus \mathfrak{u}(1)$		$\mathfrak{sp}(1) \oplus \mathfrak{u}(1)$	◀
	$\mathfrak{sp}(2) \oplus \mathfrak{sp}(1)$		$\mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$	◀
	$\mathfrak{su}(4)$		$\mathfrak{su}(3)$	◀
	$\mathfrak{su}(4) \oplus \mathfrak{u}(1)$		$\mathfrak{su}(3) \oplus \mathfrak{u}(1)$	◀
	$\mathfrak{spin}(7)$		$\mathfrak{g}_2$	◀

TABLE 2. **Type II extensions.**

We will be concerned with the pairs that actually yield homogeneous manifolds with positive curvature. In Table 2, these are marked by the "◀" symbol. Note that most exceptional examples are spheres with the exception of $\mathbb{CP}^n$ appearing the pair $(\mathfrak{sp}(n), \mathfrak{sp}(n-1) \oplus \mathfrak{u}(1))$. The simply connected, exceptional, homogeneous spaces with positive curvature can be read off from the marked pairs $(\mathfrak{g}, \mathfrak{h})$.

Extensions of Type III. These are described as follows: let $(\mathfrak{m}, \mathfrak{m}', \mathfrak{m}'')$ be a decomposition with $\mathfrak{m}$ simple. Let $\mathfrak{a}$ be a subalgebra such that $\mathfrak{m}'' \subsetneq \mathfrak{a} \subset \mathfrak{m}$. Now we set $\mathfrak{g}' = \mathfrak{m} + \mathfrak{a}$, $\mathfrak{k} = \mathfrak{m}' + \mathfrak{m}''$, $\mathfrak{g} = \Delta\mathfrak{a}$ and $\mathfrak{h} = \mathfrak{k} \cap \Delta\mathfrak{a}$. In this case $(\mathfrak{g}', \mathfrak{k})$ is said to be a *Type III extension* of $(\mathfrak{g}, \mathfrak{h})$. Note that the homogeneous space G'/K is diffeomorphic to a product manifold, $M/M' \times A/M''$.

Example. There exists an embedding $G_2 \hookrightarrow \mathrm{Spin}(8)$ such that the quotient is, $\mathrm{Spin}(8)/G_2 = \mathbb{S}^7 \times \mathbb{S}^7$. This admits a Type III extension to $\mathrm{Spin}(8) \times \mathrm{Spin}(8)$.

Type III extensions were also classified by Onishchik in his paper [8]; for a simpler exposition, see [15]. It is clear from the classification that none of the examples of positive curvature admit Type III extensions which implies that none of our examples are diffeomorphic to products. In fact, all known examples of positively curved manifolds are not even homotopy equivalent to products (cf. [12]). So we may safely ignore this type of extension for homogeneous positively curved manifolds. If a homogeneous space admits a Type II or Type III extension, we will refer to it as an *exceptional* space.

Theorem 2.1 (Onishchik)**.** *Let* $(\mathfrak{g}, \mathfrak{h})$ *be an effective pair of compact Lie algebras, with* $\mathfrak{g}$ *simple. Then any effective, compact extension of* $(\mathfrak{g}, \mathfrak{h})$ *is either a Type I extension or a Type I extension of an extension of Type II or Type III.*

One has to be careful applying the theorem; it outlines topological extensions only. For instance, we have the inclusions, $\mathrm{Sp}(n) \subset \mathrm{SU}(2n) \subset \mathrm{SO}(4n)$, all of which act transitively on the sphere, $\mathbb{S}^{4n-1}$. However, for the metric submersed from the bi-invariant metric, the quotient spaces are pairwise not isometric. The various pinchings are computed in [20]. So $\mathbb{S}^{4n-1}$ only admits a Type I extension *metrically* for the normal homogeneous metric. In the case of a non-exceptional space, we need to perform a Type I extension only once; this follows easily from the theorem.

3. The generalized Weyl group

We now exhibit the Weyl group, W_H, for our examples; we drop the term, 'generalized'. Before we plunge into computations, we state some useful lemmas.

Lemma 3.1 (Berger)**.** *Any circle acting isometrically on an even dimensional manifold with positive curvature has a fixed point.*

The proof involves showing the vanishing of Killing fields on such manifolds (cf. [14]). The analogous statement in odd dimensions is well known; see for instance [13].

Lemma 3.2. *If a torus* $\mathrm{T}^2 = S^1 \times S^1$ *acts isometrically on a manifold* M^{2n+1} *with positive curvature, then there exists* $x \in M$ *for which the isotropy group,* T^2_x*, contains a circle i.e. not all isotropy groups can be finite.*

It is evident from the previous lemma that a Lie group G acting freely and isometrically on a positively curved manifold must have $\mathrm{rank}(G) \leq 1$; this is also well known (see for instance [14]). This restricts the Weyl group W_H in some odd dimensional cases as we shall see. Another way to estimate the Weyl group is to consider the fixed point set, $(G/H)^H$, of the action of H on G/H since in fact $(G/H)^H \approx N(H)/H$.

The Weyl group in even dimensions.

From Lemma 3.1 we know that the Weyl group is finite. Synge's theorem implies that the largest subgroup of W_H acting isometrically is $\mathbb{Z}_2$. In the case of the compact, rank one, symmetric spaces, the Weyl group is well known; for explicit computations, see [12]. We state them for completeness:

$\mathbb{S}^{2n} = \mathrm{SO}(2n+1)/\mathrm{SO}(2n)$. The Weyl group, $W_{\mathrm{SO}(2n)}$ is isomorphic to $\mathbb{Z}_2$.

$\mathbb{CP}^n = \mathrm{SU}(n+1)/\mathrm{U}(n)$. The Weyl group is trivial unless $n = 1$ in which case it is $\mathbb{Z}_2$.

$\mathbb{HP}^k = \mathrm{Sp}(k+1)/\mathrm{Sp}(k) \times \mathrm{Sp}(1)$. The Weyl group is trivial unless $k = 1$ in which case it is $\mathbb{Z}_2$.

$\mathrm{Ca}\mathbb{P}^2 = F_4/\mathrm{Spin}(9)$. The Weyl group is trivial.

$\mathbb{CP}^{2m+1} = \mathrm{Sp}(m+1)/\mathrm{Sp}(m) \times \mathrm{U}(1)$. The Weyl group is isomorphic to $\mathbb{Z}_2$.

The flag manifolds, F^6 and F^{12}. There are two of the three non-symmetric examples in even dimensions. They are described as the spaces of complete flags over $\mathbb{CP}^2$ and $\mathbb{HP}^2$ respectively.

$F^6 = \mathrm{SU}(3)/\mathrm{T}^2$, so the Weyl group, $W_{\mathrm{T}^2} = N(\mathrm{T})/\mathrm{T}$ is simply the usual Weyl group of SU(3). Hence $W_{\mathrm{T}^2} = \mathrm{S}_3$, the symmetric group on 3 letters.

$F^{12} = \mathrm{Sp}(3)/(\mathrm{Sp}(1) \times \mathrm{Sp}(1) \times \mathrm{Sp}(1))$. Let $\{e_1, e_2, e_3\}$ be the standard basis in $\mathbb{H}^3$. Consider the following transformations (where the subscripts $i = 1, 2, 3$ are to be understood cyclically),

$$\sigma_i(e_i) = e_{i+1}, \quad \sigma_i(e_{i+1}) = e_i, \quad \sigma_i(e_{i-1}) = -e_{i-1}$$

The adjoint action of these involutions normalizes $\mathrm{Sp}(1) \times \mathrm{Sp}(1) \times \mathrm{Sp}(1)$ by permuting the diagonal entries. They generate S_3, the symmetric group. The cohomology ring of F^{12} yields $\chi(F^{12}) = 6$ (see for instance [5]) which bounds the order of the Weyl group. It follows that the Weyl group is isomorphic to S_3.

The Cayley flag, $F^{24} = F_4/\mathrm{Spin}(8)$. This is the space of flags over $\mathrm{Ca}\mathbb{P}^2$, the Cayley plane. The isotropy representation of Spin(8) on F_4 splits into three equivalent subrepresentations V_1, V_2, V_3 (cf. [14]). In [16, Theorem B] the authors show the following for even dimensional homogeneous spaces: if $H_0 \neq 1$ and G/H is isotropy irreducible, then there exists a finite group of automorphisms of $\mathfrak{h}$ which permutes transitively the dominant weights of the isotropy representation of H_0.

Let $\hat{G} = F_4 \times W_{\mathrm{Spin}(8)}$ acting on F^{24} with isotropy group $\hat{H}$. In [16] the authors show that $\hat{G}/\hat{H}$ is isotropy irreducible. By their theorem mentioned above, it follows the Weyl group must be the symmetric group, S_3, acting via the permutation representation on $\mathfrak{p}$.

Remark. The three flag manifolds all have Weyl group isomorphic to S_3. However, by Synge's theorem S_3 cannot act isometrically. For a homogeneous metric of positive curvature, it is evident that at most $\mathbb{Z}_2 \subset \mathrm{S}_3$ acts freely and isometrically.

The Weyl group in odd dimensions.

The only positively curved symmetric space in odd dimensions is the sphere. It is well known that its Weyl group is $\mathbb{Z}_2$, generated by the antipodal map.

The Berger space $\mathrm{SO}(5)/\mathrm{SO}(3)$. This space is rather curious; the embedding of $\mathrm{SO}(3)$ is nonstandard. Consider the space of 3×3 symmetric, trace zero matrices. This is isomorphic to the the vector space $\mathbb{R}^5$ and $\mathrm{SO}(3)$ acts on it by conjugation. This gives a (maximal) representation, $\rho : \mathrm{SO}(3) \to \mathrm{SO}(5)$. This space is isotropy irreducible (cf. [18]) which shows

Lemma 3.3. *The embedding,* $\rho : \mathrm{SO}(3) \to \mathrm{SO}(5)$*, is maximal as an embedding of connected groups.*

Proposition 3.4. *The Weyl group* $W_{\rho(\mathrm{SO}(3))}$ *is trivial.*

Proof. Let $n \in N(\mathrm{SO}(3))$ be an element of the normalizer. Then $\mathrm{Ad}(n)$ is an automorphism of $\mathrm{SO}(3)$. Since $\mathrm{SO}(3)$ has no outer automorphisms, the restriction of $\mathrm{Ad}(n)$ to $\mathrm{SO}(3)$ must be inner. So we may modify it by an element $x \in \mathrm{SO}(3)$ so that $\mathrm{Ad}(xn)$ is trivial on $\mathrm{SO}(3)$. Hence, $xn \in C(\mathrm{SO}(3))$, the centralizer of $\mathrm{SO}(3)$ in $\mathrm{SO}(5)$.

Let $g \in C(\mathrm{SO}(3))$. Then $C(g)$ is a connected subgroup of maximal rank, since g lies in some maximal torus. Also, $\rho(\mathrm{SO}(3)) \subsetneq C(g) \subset \mathrm{SO}(5)$. The first containment is strict since $C(g)$ has full rank. By the previous lemma, $C(g)$ must be all of $\mathrm{SO}(5)$ which implies that $g \in Z(\mathrm{SO}(5))$, the center. But $\mathrm{SO}(5)$ has trivial center, so $g = xn = \mathrm{id}$. Hence $n \in \mathrm{SO}(3)$. ∎

The Berger space $\mathrm{SU}(5)/\mathrm{Sp}(2) \times_{\mathbb{Z}_2} S^1$. We give a brief description of this space. $\mathrm{Sp}(2)$ is embedded into $\mathrm{SU}(4)$ in the standard way, which is embedded in $\mathrm{SU}(5)$ canonically. We have,

$$\begin{aligned} \mathrm{Sp}(2) \times S^1 &\longrightarrow \mathrm{SU}(5) \\ (A, z) &\longmapsto \begin{pmatrix} \mathrm{im}(A) \cdot z & 0 \\ 0 & \bar{z}^4 \end{pmatrix} \end{aligned}$$

Note however, that the map has a kernel given by $\{(\mathrm{id}, 1), (-\mathrm{id}, -1)\} \cong \mathbb{Z}_2$. The quotient, $\mathrm{Sp}(2) \times_{\mathbb{Z}_2} S^1$, then embeds into $\mathrm{SU}(5)$ and the corresponding normal homogeneous space has positive curvature.

Proposition 3.5. *The Weyl group $W_{\mathrm{Sp}(2)\times_{\mathbb{Z}_2} S^1}$ is trivial.*

Proof. We have the following containments, $\mathrm{Sp}(2) \times_{\mathbb{Z}_2} S^1 \subset \mathrm{U}(4) \subset \mathrm{SU}(5)$. Let $H = \mathrm{Sp}(2) \times_{\mathbb{Z}_2} S^1$ and $K = \mathrm{U}(4)$. The isotropy representation of H splits into two irreducible summands: a 5-dimensional real representation and a 4-dimensional complex representation. This shows that W_H is finite.

Let $\mathfrak{su}(5) = \mathfrak{h} + \mathfrak{p} + \mathfrak{m}$, where $\mathfrak{u}(4) = \mathfrak{h} + \mathfrak{p}$. Let $V = \mathfrak{p} + \mathfrak{m}$. The decomposition $\mathfrak{su}(5) = \mathfrak{h} + V$ is $\mathrm{Ad}(H)$ invariant. Since $N(H)$ preserves $\mathfrak{h}$, it must preserve V as well. If V were irreducible for the action of $N(H)$, then $\mathrm{SU}(5)/N(H)$ would be an isotropy irreducible space whose universal cover is our space, M^{13}. This contradicts the classification of isotropy irreducible spaces in [16]. So let $U \subset V$ be an invariant subspace for the isotropy representation of $N(H)$. Then U is preserved by H also and hence $U = \mathfrak{p}$ or $U = \mathfrak{m}$. It follows that $\mathfrak{p}$ and $\mathfrak{m}$ are invariant subspaces for the action of $N(H)$ also. In particular $N(H)$ normalizes $\mathrm{U}(4)$. Now $N(\mathrm{U}(4)) = \mathrm{U}(4)$ in $\mathrm{SU}(5)$ which implies $N(H) \subset \mathrm{U}(4)$, and we have $N_{\mathrm{SU}(5)}(H) = N_{\mathrm{U}(4)}(H)$. But $K/H = \mathrm{SO}(6)/\mathrm{O}(5) = \mathbb{RP}^5$ is an isotropy irreducible symmetric space with trivial Weyl group. It follows that $N_{\mathrm{SU}(5)}(H) = H$. ∎

The Aloff-Wallach space $N_{1,1} = (\mathrm{SU}(3) \times \mathrm{SO}(3))/\mathrm{U}^*(2)$. Following Wilking (cf. [17]), the subgroup $\mathrm{U}^*(2)$ is the image under the embedding $(i, \pi) : \mathrm{U}(2) \to \mathrm{SU}(3) \times \mathrm{SO}(3)$. For $A \in \mathrm{U}(2)$,

$$i(A) = \begin{pmatrix} A & 0 \\ 0 & \det(A)^{-1} \end{pmatrix}$$

and $\pi : \mathrm{U}(2) \to \mathrm{SO}(3) \cong \mathrm{U}(2)/Z$ is the canonical projection.

Proposition 3.6. *The Weyl group $W_{\mathrm{U}^*(2)}$ is trivial.*

Proof. Let $(A, B) \in N(\mathrm{U}^*(2))$ be an element of the normalizer. Since π is surjective, B is an element of $\mathrm{SO}(3)$. Now $\mathrm{SU}(3)/i(\mathrm{U}(2)) = \mathbb{CP}^2$ which has the fixed point property. This implies that $N_{\mathrm{SU}(3)}(\mathrm{U}(2)) = \mathrm{U}(2)$ which forces $(A, B) \in \mathrm{U}^*(2)$. ∎

The Aloff-Wallach spaces, $N_{k,l} = \mathrm{SU}(3)/S^1_{k,l}$. These spaces are described as circle quotients of $\mathrm{SU}(3)$ by the subgroups,

$$S^1_{k,l} = \{\mathrm{diag}(z^k, z^l, \bar{z}^{k+l}) : \gcd(k,l) = 1, z \in S^1\}$$

When $(k, l) = (1, 1)$, we get the normal homogeneous space described above. Note that the pairs (k, l), $(-k, -l)$, $(k, -k - l)$ etc. give the same space. So we may assume, without loss of generality, that $k, l > 0$. Since we already considered $(k, l) = (1, 1)$, we will also assume that at least one of k or l is bigger than 1. The following is straightforward.

Lemma 3.7. *Let k, l be positive integers with $gcd(k, l) = 1$ and $(k, l) \neq (1, 1)$. Then k, l and $-k - l$ are pairwise distinct.*

Let $C_{k,l}$ denote the centralizer of $S^1_{k,l}$. The centralizer of any toral subgroup S in a compact, connected Lie group is the union of all maximal tori containing S. So $C_{k,l}$ is a connected subgroup of maximal rank. Let,

$$\mathrm{T} = \{\mathrm{diag}(z, w, \bar{z}\bar{w}) : z, w \in S^1\}$$

be the usual maximal torus of SU(3). Then $S^1_{k,l} \subset \mathrm{T} \subset C_{k,l}$. Now the Lie algebra of SU(3) splits as,

$$\mathfrak{su}(3) = \mathfrak{t} + V_1 + V_2 + V_3$$

where V_i are the root spaces for the maximal torus. They are generated by matrices of the form,

$$\begin{pmatrix} 0 & z & 0 \\ -\bar{z} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & w \\ 0 & 0 & 0 \\ -\bar{w} & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & x \\ 0 & -\bar{x} & 0 \end{pmatrix}$$

respectively, where $z, w, x \in \mathbb{C}$. The Lie algebra of $S^1_{k,l}$ is generated by the element $\mathrm{diag}(k, l, -k - l)$. It is routine to check that the Lie bracket of $\mathrm{diag}(k, l, -k - l)$ with any element in the root spaces is non-trivial by Lemma 3.7. This implies that the Lie algebra of the centralizer is the same as $\mathfrak{t}$. We have shown

Proposition 3.8. *The centralizer $C_{k,l}$ is the maximal torus* T *containing $S^1_{k,l}$.*

The normalizer, $N(S^1_{k,l})$, acts on $S^1_{k,l}$ via the adjoint action with kernel $C_{k,l}$. By Lemma 1.1, we have a non-trivial action of $N/C_{k,l}$ on the circle. Since the circle has only one non-trivial automorphism (complex conjugation), it follows that the normalizer is at most a $\mathbb{Z}_2$ extension of $C_{k,l}$.

Proposition 3.9. *The Weyl group $W_{S^1_{k,l}}$ is isomorphic to S^1.*

Proof. The centralizer is the identity component of the normalizer. So the adjoint action of $N(S^1_{k,l})$ preserves $C_{k,l}$ as well. But $C_{k,l}$ is a maximal torus which implies that $N(S^1_{k,l})/C_{k,l} \subset S_3$, the Weyl group of SU(3). The Weyl group, S_3, acts by permuting the eigenvalues of T. By Lemma 3.7, we see that no non-trivial permutation preserves the subgroup $S^1_{k,l}$. Hence, $N(S^1_{k,l})/C_{k,l}$ is trivial and the result follows. ■

We now deal with other representations of spheres most of whose Weyl groups are also well known.

$\mathbb{S}^{2n+1} = \mathrm{SU}(n+1)/\mathrm{SU}(n)$, $n > 1$. $W_{\mathrm{SU}(n)}$ is isomorphic to S^1.

$\mathbb{S}^{4n+3} = \mathrm{Sp}(n+1)/\mathrm{Sp}(n)$. The Weyl group $W_{\mathrm{Sp}(n)}$ is isomorphic to $\mathrm{Sp}(1)$.

$\mathbb{S}^{15} = \mathrm{Spin}(9)/\mathrm{Spin}(7)$. The embeddings, $\mathrm{Spin}(7) \subset \mathrm{Spin}(8) \subset \mathrm{Spin}(9)$, are standard. Since, $\mathrm{Spin}(9)/\mathrm{Spin}(8) = \mathbb{S}^8$ and $\mathrm{Spin}(8)/\mathrm{Spin}(7) = \mathbb{S}^7$, we have a fibration

$$\mathbb{S}^7 \longrightarrow \mathbb{S}^{15} \longrightarrow \mathbb{S}^8$$

The isotropy representation of $\mathrm{Spin}(7)$ splits into two irreducible pieces which correspond to the tangent spaces of the fiber and base. The normalizer, $N(\mathrm{Spin}(7))$, must preserve this decomposition and hence it normalizes $\mathrm{Spin}(8)$ as well. This implies that $N(\mathrm{Spin}(7)) \subset N(\mathrm{Spin}(8))$ which is a $\mathbb{Z}_2$ extension of $\mathrm{Spin}(8)$.

Proposition 3.10. *The normalizer,* $N(\mathrm{Spin}(7))$*, is contained in* $\mathrm{Spin}(8)$*.*

Proof. We want to show that any non-trivial element that normalizes $\mathrm{Spin}(8)$ cannot normalize $\mathrm{Spin}(7)$. Suppose $x \in N(\mathrm{Spin}(8))$ normalizes $\mathrm{Spin}(7)$ such that $x \notin \mathrm{Spin}(8)$. Then $x^2 \in \mathrm{Spin}(8)$. There are two possibilities: either $x^2 \in \mathrm{Spin}(8) - \mathrm{Spin}(7)$ or $x^2 \in \mathrm{Spin}(7)$. If x^2 were in $\mathrm{Spin}(7)$, then $W_{\mathrm{Spin}(7)}$ would contain two elements of order two, namely, x^2 and the non-trivial element in $N_{\mathrm{Spin}(8)}(\mathrm{Spin}(7))/\mathrm{Spin}(7)$. This would violate Milnor's condition which says that any finite group acting freely on a sphere has at most one element of order two.

So we must have $x^2 \in \mathrm{Spin}(8)\backslash\mathrm{Spin}(7)$. Since x^2 also normalizes $\mathrm{Spin}(7)$, x^2 must generate $N_{\mathrm{Spin}(8)}(\mathrm{Spin}(7))$. It follows that $W_{\mathrm{Spin}(7)} = \mathbb{Z}_4$ which acts freely and isometrically for the bi-invariant metric. In particular, $\mathbb{S}^{15}/\mathbb{Z}_4$, has positive curvature. Since $N(\mathrm{Spin}(7))$ is contained in $N(\mathrm{Spin}(8))$, we have an induced (ineffective) action on the base, $\mathbb{S}^8$, with quotient $\mathbb{RP}^8$. The kernel of the action is $\mathbb{Z}_2$ which then acts on the fiber and the fibration now looks like,

$$\mathbb{RP}^7 \longrightarrow \mathbb{S}^{15}/\mathbb{Z}_4 \longrightarrow \mathbb{RP}^8$$

The base, $\mathbb{RP}^8$, is non-orientable which implies that the total space is also non-orientable. This contradicts Synge's theorem. ∎

The proposition tells us that the Weyl group is isomorphic to $\mathbb{Z}_2$. Note that the $\mathrm{Spin}(9)$ action on $\mathbb{S}^{15}$ is effective, so $\mathrm{Spin}(7)$ does not intersect $Z(\mathrm{Spin}(9))$. Since the center normalizes every subgroup, we must have, $Z(\mathrm{Spin}(9)) \subset N(\mathrm{Spin}(7))$. It follows that the Weyl group comes from the center, $Z(\mathrm{Spin}(9))$.

$N_{1,1} = \mathrm{SU}(3)/S^1_{1,1}$. The space $N^7_{1,1}$ can be written in two different ways; we already saw its normal homogeneous incarnation which was described by Wilking. For a certain choice of left invariant metric on SU(3), the two representations of this space are isometric (cf. [17]). As a quotient of SU(3), it is easy to see that the normalizer of $S^1_{1,1}$ is U(2) and the Weyl group, $W_{S^1_{1,1}} = \mathrm{U}(2)/S^1_{1,1} = \mathrm{SO}(3)$. Hence this space admits a free, isometric SO(3) action.

Remark (*i*)**.** The spheres, $\mathbb{S}^{4k+3} = \mathrm{Sp}(k+1)/\mathrm{Sp}(k)$ admit a free, isometric action of Sp(1). This gives rise to the principal bundles,

$$\begin{array}{ccccc} S^1 & \longrightarrow & \mathbb{S}^{4k+3} & \longrightarrow & \mathbb{CP}^{2k+1} \\ \mathrm{Sp}(1) & \longrightarrow & \mathbb{S}^{4k+3} & \longrightarrow & \mathbb{HP}^k \end{array}$$

This also shows that $\mathbb{CP}^{2k+1}$ is an $\mathbb{S}^2$ bundle over $\mathbb{HP}^k$. In the above action of Sp(1) on $\mathbb{S}^{4k+3}$, consider the action of the subgroup $\mathrm{Pin}(2) \subset \mathrm{Sp}(1) \cong \mathrm{SU}(2)$. The resulting space is a $\mathbb{Z}_2$ isometric quotient of $\mathbb{CP}^{2k+1}$. However, the metric on $\mathbb{CP}^{2k+1}$ submersed from the bi-invariant metric on $\mathrm{Sp}(k+1)$ is not the standard Fubini-Study metric for $k > 0$; it has pinching $\frac{1}{4(k+1)}$ (cf. [20]). Regarding the quotient, $\mathbb{CP}^{2k+1}/\mathbb{Z}_2$, we see that it is a $\mathbb{RP}^2$ bundle over $\mathbb{HP}^k$.

Remark (*ii*)**.** In the case of the standard $\frac{1}{4}$-pinched metric, we saw that the Weyl group of $\mathbb{CP}^n$ is trivial for all n. Yet, we remarked the the odd complex projective spaces, $\mathbb{CP}^{2k+1}$, admit free, isometric involutions which come from the outer automorphism of complex conjugation on $\mathbb{C}^{n+1}$. If $n+1 = 2k+2$ is even, then we can define a map on $\mathbb{C}^{2k+2}$,

$$(z_1, z_2, ..., z_{2k}, z_{2k+1}, z_{2k+2}) \mapsto (-\bar{z}_{k+2}, -\bar{z}_{k+3}, ..., -\bar{z}_{2k+2}, \bar{z}_1, ..., \bar{z}_{k+1})$$

This is an involution on $\mathbb{C}^{2k+2}$ without fixed points that preserves (complex) lines and it induces a fixed point free involution on $\mathbb{CP}^{2k+1}$. Interestingly enough, this can also be realized as an action of Pin(2) on the sphere, $\mathbb{S}^{4k+3}$. The Pin(2) embedding in the full isometry group of $\mathbb{S}^{4k+3}$ is quite strange and yields $\mathbb{CP}^{2k+1}/\mathbb{Z}_2$ as an isometric quotient of the standard $\mathbb{CP}^{2k+1}$.

Remark (*iii*)**.** The spheres, $\mathbb{S}^6 = G_2/\mathrm{SU}(3)$ and $\mathbb{S}^7 = \mathrm{Spin}(7)/G_2$, always admit Type II extensions, so we do not compute their Weyl groups.

4. The isometry group

It is appropriate to mention at this time that Onishchik himself has used his theory of enlargements to indicate how the isometry group of a homogeneous space may be computed. His solution is in general terms and for the bi-invariant metric (cf. [10], [9]). In particular, there are no curvature assumptions in his work. Unless otherwise mentioned, we will compute the isometry group for the normal homogeneous metric and indicate what the isometry groups are for other homogeneous metrics of positive curvature on the same space. As usual, $\mathbf{I}(M)$ denotes the isometry group of the Riemannian manifold M.

Proposition 4.1. *Let G/H be an even dimensional, homogeneous, positively curved manifold. Then $\mathbf{I}_0(G/H)$ is simple and centerless.*

Proof. From [14] we know that for even dimensional G/H of positive curvature, we can assume that G is simple. By Lemma 3.1 and the fact that they all have positive Euler number, Type I extensions are ruled out. Since they do not admit Type III extensions, either $\mathbf{I}_0(G/H) = G$ (almost effectively) or $\mathbf{I}_0$ is a Type II extension of G, which is also simple.

Now suppose we have G/H even dimensional with positive curvature. Since $\mathrm{rank}(H) = \mathrm{rank}(G)$, G and H share a maximal torus. Since the center lies in every maximal torus, the action is effective if and only if G is centerless. ■

In odd dimensions we see from our Weyl group computations, that a nontrivial Type I extension is by S^1, SO(3) or SU(2). Next we state a key reduction theorem which will yield the isometry group for any homogeneous metric of positive curvature.

Reduction Theorem. *Let $M = G'/H'$ be a simply connected, homogeneous, positively curved manifold. Then G' contains a closed subgroup, G, which is simple, compact and acts transitively on M. In particular, G' is a Type I or Type II extension of G.*

Proof. The even dimensional case is an immediate consequence of [14, Corollary 4.2] which proves the following: if $M = G/H$ is an even dimensional homogeneous with positive curvature, then G is simple and G and H have the same rank. Combining this with the previous proposition yields the result.

In odd dimensions we appeal to [2] where the author classifies odd dimensional homogeneous, positively curved manifolds. Theorems 1 and 2 in that paper deal with the case of $M = G'/H'$ of positive curvature when G' is not simple. The theorems prove: if G' is not simple, then $(\mathfrak{g}', \mathfrak{h}')$ is one of the pairs $(\mathfrak{su}(n+1) \oplus \mathbb{R}, \mathfrak{su}(n) \oplus \mathbb{R})$, $(\mathfrak{sp}(n) \oplus \mathbb{R}, \mathfrak{sp}(n-1) \oplus \mathbb{R})$, $(\mathfrak{sp}(n) \oplus \mathfrak{sp}(1), \mathfrak{sp}(n-1) \oplus \mathfrak{sp}(1))$, $(\mathfrak{su}(3) \oplus \mathbb{R}, \mathbb{R}^2)$ or $(\mathfrak{su}(3) \oplus \mathfrak{su}(2), \mathfrak{su}(2) \oplus \mathbb{R})$. A quick glance reveals that all of these are Type I extensions of simple groups; the first three are different representations of odd spheres while the fourth represents the Aloff-Wallach spaces. The last one is the normal homogeneous representation of the Aloff-Wallach space $N_{1,1}$. ■

Remark. The normal homogeneous space, $N_{1,1}$, was ruled out by Bérard Bergery. Since this space is also isometric to a circle quotient of SU(3) with respect to a left invariant metric, we may apply our methods with impunity.

For a positively curved homogeneous space we can find a minimal simple group that acts transitively namely, a transitive simple group such that no subgroup acts transitively. This follows from the Reduction Theorem and the classification theorems in [14] and [2]. The homogeneous spaces in Table 1 are written as quotients of these minimally transitive simple groups with a few exceptions like $\mathbb{S}^3 = \mathrm{SU}(2)$, $\mathbb{S}^6 = G_2/\mathrm{SU}(3)$ and $\mathbb{S}^7 = \mathrm{Spin}(7)/G_2$. These are dealt with on a case by case basis. In any event, we have

Proposition 4.2. *Let M be a homogeneous space of positive curvature. Then there exists a minimally transitive simple group, G, such that for any homogeneous metric, g, of positive curvature, $G \subset \mathbf{I}_0((M,g))$.*

Recall that the group of components depends on the groups $\mathrm{Out}(G_0, H_0)$ and $N(H_0)/H_0Z(G_0)$. The former will be the Weyl group in most cases. From our Weyl group computations, we see that the odd dimensional spheres and the Aloff-Wallach spaces are the only spaces that admit honest Type I extensions. A word of caution is in order: the group of components may be a proper subgroup of $\mathrm{Aut}(G_0, H_0)$ as is evident in the case of the flag manifolds.

The isometry group of a symmetric space is well known and has been computed (see for instance [19]), so we won't repeat this here. The results are summarized in Table 3. We would like to mention, however, that the compact, rank one, symmetric spaces are isotropy irreducible, so any homogeneous metric on them will be a multiple of the usual normal homogeneous metric. Hence, by Proposition 4.2, the isometry group for any homogeneous metric of positive curvature will be the same as that for the normal homogeneous metric. A few other cases such as other representations of the spheres and projective spaces are also well known (cf. [19]). We will use the Reduction Theorem and Proposition 4.2 to indicate the connected isometry groups for all possible homogeneous metrics of positive curvature. The methods developed here may then be applied in each case to compute the group of components. Note that the short exact sequence,

$$1 \longrightarrow \mathbf{I}_0 \longrightarrow \mathbf{I} \longrightarrow \mathbf{I}/\mathbf{I}_0 \longrightarrow 1$$

is always split for homogeneous metrics, as a consequence of Lemma 1.8.

$\mathbb{S}^6 = G_2/\mathrm{SU}(3)$.

The submersed bi-invariant metric has constant curvature. Since this space is isotropy irreducible, any homogeneous, G_2-invariant metric will also have constant curvature. Hence we have a Type II extension to SO(7) and the isometry group is O(8).

$\mathbb{CP}^{2m+1} = \mathrm{Sp}(m+1)/(\mathrm{Sp}(m)\times \mathrm{U}(1))$.
$\mathbf{I}_0 = \mathrm{Sp}(m+1)/\mathbb{Z}_2$ which is centerless and the Weyl group, $W_{\mathrm{Sp}(m)\times\mathrm{U}(1)}$, is isomorphic to $\mathbb{Z}_2$. Also $\mathrm{Out}(\mathrm{Sp}(m+1))$ is trivial and we have,

$$\mathbf{I}(\mathbb{CP}^{2m+1}) = (\mathrm{Sp}(m+1)/\mathbb{Z}_2)\times\mathbb{Z}_2$$

for the normal homogeneous metric induced by $\mathrm{Sp}(m+1)$. This space is isotropy reducible. Scaling the irreducible factors differently, one obtains a left invariant metric on $\mathrm{Sp}(m+1)$ which admits a Type II extension to $\mathrm{PSU}(2m+2)$. This is the Fubini-Study metric and its isometry group is well known. The isometry group for any other homogeneous metric of positive curvature is a finite extension of $\mathrm{Sp}(m+1)/\mathbb{Z}_2$ or $\mathrm{PSU}(2m+2)$.

$F^6 = \mathrm{SU}(3)/\mathrm{T}^2$.
$\mathbf{I}_0 = \mathrm{PSU}(3)$ and the Weyl group, W_{T^2} is the full symmetric group, S_3. Also, $\mathrm{Out}(\mathrm{PSU}(3)) = \mathbb{Z}_2$ generated by complex conjugation on SU(3). This map acts isometrically on the flag manifold and $\mathrm{Aut}(\mathrm{SU}(3),\mathrm{T}^2) = \mathrm{S}_3\times\mathbb{Z}_2$. Since the Weyl group acts freely and F^6 is even dimensional, we know that at most a $\mathbb{Z}_2$ inside it can act freely and isometrically. Hence, $\mathbf{I}/\mathbf{I}_0$ is generated by two involutions. Putting this together, we have,

$$\mathbf{I}(F^6) = \mathrm{PSU}(3)\rtimes(\mathbb{Z}_2\oplus\mathbb{Z}_2) = (\mathrm{PSU}(3)\rtimes\mathbb{Z}_2)\times\mathbb{Z}_2$$

for the metric submersed from a left-invariant, $\mathrm{Ad}(\mathrm{U}(2))$-invariant metric on SU(3). Since this space does not admit any extensions, any homogeneous metric of positive curvature will have PSU(3) as the identity component of its isometry group with at most four components.

$F^{12} = \mathrm{Sp}(3)/(\mathrm{Sp}(1)\times\mathrm{Sp}(1)\times\mathrm{Sp}(1))$.
$\mathbf{I}_0 = \mathrm{Sp}(3)/\mathbb{Z}_2$ which is centerless and the Weyl group is S_3. Since $\mathrm{Out}(\mathrm{Sp}(3))$ is trivial, the group of components is a subgroup of the Weyl group. As above, we have,

$$\mathbf{I}(F^{12}) = (\mathrm{Sp}(3)/\mathbb{Z}_2)\times\mathbb{Z}_2$$

for the metric submersed from a left invariant, $\mathrm{Ad}(\mathrm{Sp}(2)\times\mathrm{Sp}(1))$-invariant metric on Sp(3). Again as above, any homogeneous metric of positive curvature will have Sp(3) as identity component of its isometry group with at most four components.

$F^{24} = F_4/\mathrm{Spin}(8)$.

The group F_4 is centerless and has no outer automorphisms. Hence, the computation here is the same as the previous case and we have,

$$\mathbf{I}(F^{24}) = F_4 \times \mathbb{Z}_2$$

for the metric submersed from a left invariant, Ad(Spin(9))-invariant metric on F_4. Any homogeneous metric of positive curvature on this space will have isometry group F_4 or $F_4 \times \mathbb{Z}_2$.

$M^7 = \mathrm{SO}(5)/\mathrm{SO}(3)$.

This is the curious Berger space that stands out as an errant son. $\mathbf{I}_0 = \mathrm{SO}(5)$ is centerless and has no outer automorphisms. We also showed that the Weyl group is trivial. The gods smile and we have,

$$\mathbf{I}(M^7) = \mathrm{SO}(5)$$

for the normal homogeneous metric induced by SO(5). This space being isotropy irreducible, any homogeneous metric of positive curvature on this space will have the same isometry group.

$M^{13} = \mathrm{SU}(5)/(\mathrm{Sp}(2) \times_{\mathbb{Z}_2} S^1)$.

$\mathbf{I}_0 = \mathrm{PSU}(5)$ is centerless and has an outer automorphism – complex conjugation. We showed that the Weyl group is trivial, so $\mathbf{I}/\mathbf{I}_0$ is either trivial or $\mathbb{Z}_2$. Complex conjugation is an isometry which does not commute with the action of SU(5) and we have,

$$\mathbf{I}(M^{13}) = \mathrm{PSU}(5) \rtimes \mathbb{Z}_2$$

for the normal homogeneous metric induced by SU(5). From Proposition 4.2, the isometry group of any homogeneous metric of positive curvature on this space would be PSU(5) or $\mathrm{PSU}(5) \rtimes \mathbb{Z}_2$.

$N_{1,1} = (\mathrm{SU}(3) \times \mathrm{SO}(3))/\mathrm{U}^*(2)$.

$\mathbf{I}_0 = \mathrm{PSU}(3) \times \mathrm{SO}(3)$ which is centerless. The Weyl group, $W_{\mathrm{U}^*(2)}$, is trivial. $\mathrm{Out}(\mathrm{SU}(3) \times \mathrm{SO}(3)) = \mathbb{Z}_2$, the complex conjugation map on SU(3) which is trivial on the SO(3) factor. Complex conjugation is an isometry for the bi-invariant metric and we have,

$$\mathbf{I}(N_{1,1}) = (\mathrm{PSU}(3) \times \mathrm{SO}(3)) \rtimes \mathbb{Z}_2 = (\mathrm{PSU}(3) \rtimes \mathbb{Z}_2) \times \mathrm{SO}(3)$$

for the normal homogeneous metric induced by $\mathrm{SU}(3) \times \mathrm{SO}(3)$. Considering this space as a circle quotient of SU(3), we see that the isometry group of any homogeneous metric of positive curvature will be a finite extension of PSU(3) or $\mathrm{PSU}(3) \times \mathrm{SO}(3)$.

$\boldsymbol{N_{k,l} = \mathbf{SU(3)}/S^1_{k,l}}$.

$3 \mid (k^2+l^2+kl)$, $(k,l) \neq (1,1)$. We know that $\mathbf{I}_0 = \mathrm{PSU}(3) \times S^1$, with center $\{\mathrm{id}\} \times \mathrm{S}^1$ with isotropy group T^2, the diagonal image of the normalizer $N(S^1_{k,l})$. By Theorem 2.1, the reduced Weyl group, $N(H_0)/H_0 Z(G_0)$, must be finite and is in fact trivial. We saw earlier that $\mathrm{Out}(\mathrm{PSU}(3) \times S^1) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$, where the involutions are complex conjugation on each factor. Both involutions are isometries and we have,

$$\begin{aligned} \mathbf{I}(N_{k,l}) &= (\mathrm{PSU}(3) \times S^1) \rtimes (\mathbb{Z}_2 \oplus \mathbb{Z}_2) \\ &= (\mathrm{PSU}(3) \rtimes \mathbb{Z}_2) \times (S^1 \rtimes \mathbb{Z}_2) \end{aligned} \qquad \text{for } 3 \mid (k^2+l^2+kl)$$

for the metric submersed from a left invariant, Ad(U(2))-invariant metric on SU(3). By Proposition 4.2, the isometry group of any homogeneous metric of positive curvature would be a finite extension of PSU(3) or $\mathrm{PSU}(3) \times S^1$.

$3 \nmid (k^2+l^2+kl)$. In this case, $\mathbf{I}_0 = \mathrm{U}(3)$ whose center is the set of diagonal matrices, $z \cdot I_n$, where z is a unit complex number. The reduced Weyl group, $N(H_0)/H_0 Z(G_0)$, is again trivial and $\mathrm{Out}(\mathrm{U}(n)) = \mathbb{Z}_2$ is generated by complex conjugation. The latter is an isometry and we have,

$$\mathbf{I}(N_{k,l}) = \mathrm{U}(3) \rtimes \mathbb{Z}_2 \quad \text{for } 3 \nmid (k^2+l^2+kl)$$

for the metric submersed from a left invariant, Ad(U(2))-invariant metric on SU(3). As above, the isometry group of any homogeneous metric of positive curvature would be a finite extension of SU(3) or U(3).

$\boldsymbol{\mathbb{S}^{2m+1} = \mathbf{SU}(m+1)/\mathbf{SU}(m)}$.

We assume $m > 1$. $\mathbf{I}_0 = \mathrm{SU}(m+1) \times_{\mathbb{Z}_{m+1}} \mathrm{U}(1) = \mathrm{U}(m+1)$ for the normal homogeneous metric even though this example admits a Type II extension. $\mathrm{Out}(\mathbf{I}_0) = \mathbb{Z}_2$ which acts isometrically while the reduced Weyl group, $N(H_0)/H_0 Z(G_0)$, is trivial. We have,

$$\mathbf{I}(\mathbb{S}^{2m+1}) = \mathrm{U}(m+1) \rtimes \mathbb{Z}_2$$

for the normal homogeneous metric. The isotropy representation of this space splits into two irreducible summands. Scaling the metric appropriately on the two summands yields a space isometric to the round sphere (cf. [20]). For this metric, the space admits a Type II extension and its isometry group is the full orthogonal group. The isometry group of any homogeneous metric of positive curvature would be a finite extension of $\mathrm{U}(m+1)$ or $\mathrm{SO}(2m+2)$ by Proposition 4.2.

$\mathbb{S}^{4m+3} = \mathrm{Sp}(m+1)/\mathrm{Sp}(m)$.

$\mathbf{I}_0 = \mathrm{Sp}(m+1) \times_{\mathbb{Z}_2} \mathrm{Sp}(1)$. As before, the reduced Weyl group is trivial, and $\mathrm{Out}(\mathrm{Sp}(m+1))$ is trivial. We have,

$$\mathbf{I}(\mathbb{S}^{4m+3}) = \mathrm{Sp}(m+1) \times_{\mathbb{Z}_2} \mathrm{Sp}(1)$$

for the normal homogeneous metric induced by $\mathrm{Sp}(m+1)$. The normal homogeneous metric does not have constant curvature, although it is possible to scale the irreducible factors of the isotropy representation to obtain other homogeneous metrics of positive curvature. In particular, it is possible to get the metric in the previous case or the round sphere metric. The isometry group for any homogeneous metric of positive curvature would be a finite extension of $\mathrm{Sp}(m+1) \times_{\mathbb{Z}_2} \mathrm{Sp}(1)$, $\mathrm{U}(2m+2)$ or $\mathrm{SO}(4m+4)$.

$\mathbb{S}^3 = \mathrm{SU}(2)$.

The normal homogeneous metric has constant curvature so it will have the orthogonal group as isometry group. However, it is possible to find other left invariant metrics of positive curvature whose isometry groups would be finite extensions of $\mathrm{SU}(2)$, $\mathrm{U}(2)$ or $\mathrm{SO}(4)$ depending on the right invariance of the metric. Determing the moduli space of *all* homogeneous metrics of positive curvature is a hard problem, in general.

$\mathbb{S}^7 = \mathrm{Spin}(7)/G_2$.

The normal homogeneous metric is the metric of constant curvature. Hence, this admits a legitimate Type II extension to $\mathrm{SO}(8)$ and we have,

$$\mathbf{I}(\mathbb{S}^7) = \mathrm{SO}(8) \rtimes \mathbb{Z}_2 = \mathrm{O}(8)$$

The isotropy representation of G_2 on the quotient is irreducible. Hence, any $\mathrm{Spin}(7)$-invariant metric has constant curvature and any homogeneous metric of positive curvature will have the orthogonal group as isometry group.

$\mathbb{S}^{15} = \mathrm{Spin}(9)/\mathrm{Spin}(7)$.

The normal homogeneous metric does not have constant curvature, its pinching is $\frac{9}{121}$ (cf. [20]). The Weyl group, $W_{\mathrm{Spin}(7)}$, is $\mathbb{Z}_2$ and $\mathrm{Spin}(9)$ has no outer automorphisms. However, since the Weyl group is precisely the center, the reduced Weyl group, $N(H_0)/H_0Z(G_0)$, is trivial. Hence, the group of components is trivial and we have,

$$\mathbf{I}(\mathbb{S}^{15}) = \mathrm{Spin}(9)$$

for the normal homogeneous metric induced by $\mathrm{Spin}(9)$. The isotropy representation is reducible and it is possible to scale the invariant factors differently to yield other left-invariant metrics on $\mathrm{Spin}(9)$. For some choice of constants, it is possible to recover the metric of constant curvature on $\mathbb{S}^{15}$

$M = G/H$	$\mathbf{I}(M)$	Other $\mathbf{I}_0(M)$
$\mathbb{S}^n = \mathrm{SO}(n+1)/\mathrm{SO}(n)$	$\mathrm{O}(n+1)$	—
$\mathbb{CP}^m = \mathrm{SU}(m+1)/\mathrm{SU}(m)$	$\mathrm{PSU}(m+1) \rtimes \mathbb{Z}_2$	—
$\mathbb{HP}^k = \mathrm{Sp}(k+1)/(\mathrm{Sp}(k) \times \mathrm{Sp}(1))$, $k > 1$	$\mathrm{Sp}(k+1)/\mathbb{Z}_2$	—
$\mathrm{Ca}\mathbb{P}^2 = F_4/\mathrm{Spin}(9)$	F_4	—
$\mathbb{S}^6 = G_2/\mathrm{SU}(3)$	$\mathrm{O}(7)$	—
$\mathbb{CP}^{2m+1} = \mathrm{Sp}(m+1)/(\mathrm{Sp}(m) \times \mathrm{U}(1))$	$(\mathrm{Sp}(m+1)/\mathbb{Z}_2) \times \mathbb{Z}_2$	$\mathrm{PSU}(2m+2)$
$F^6 = \mathrm{SU}(3)/\mathrm{T}^2$	$(\mathrm{PSU}(3) \rtimes \mathbb{Z}_2) \times \mathbb{Z}_2$	—
$F^{12} = \mathrm{Sp}(3)/(\mathrm{Sp}(1) \times \mathrm{Sp}(1) \times \mathrm{Sp}(1))$	$(\mathrm{Sp}(3)/\mathbb{Z}_2) \times \mathbb{Z}_2$	—
$F^{24} = F_4/\mathrm{Spin}(8)$	F_4	—
$M^7 = \mathrm{SO}(5)/\mathrm{SO}(3)$	$\mathrm{SO}(5)$	—
$M^{13} = \mathrm{SU}(5)/(\mathrm{Sp}(2) \times_{\mathbb{Z}_2} S^1)$	$\mathrm{PSU}(5) \rtimes \mathbb{Z}_2$	—
$N_{1,1} = \mathrm{SU}(3) \times \mathrm{SO}(3)/\mathrm{U}^*(2)$	$(\mathrm{PSU}(3) \rtimes \mathbb{Z}_2) \times \mathrm{SO}(3)$	$\mathrm{PSU}(3)$
$N_{k,l} = \mathrm{SU}(3)/S^1_{k,l}$ $(k,l) \neq (1,1)$, $3 \mid (k^2+l^2+kl)$	$(\mathrm{PSU}(3) \rtimes \mathbb{Z}_2) \times (S^1 \rtimes \mathbb{Z}_2)$	$\mathrm{PSU}(3)$
$N_{k,l} = \mathrm{SU}(3)/S^1_{k,l}$ $3 \nmid (k^2+l^2+kl)$	$\mathrm{U}(3) \rtimes \mathbb{Z}_2$	$\mathrm{SU}(3)$
$\mathbb{S}^{2m+1} = \mathrm{SU}(m+1)/\mathrm{SU}(m)$	$\mathrm{U}(m+1) \rtimes \mathbb{Z}_2$	$\mathrm{SO}(2m+2)$
$\mathbb{S}^{4m+3} = \mathrm{Sp}(m+1)/\mathrm{Sp}(m)$	$\mathrm{Sp}(m+1) \times_{\mathbb{Z}_2} \mathrm{Sp}(1)$	$\mathrm{U}(2m+2)$, $\mathrm{SO}(4m+4)$
$\mathbb{S}^3 = \mathrm{SU}(2)$	$\mathrm{O}(4)$	$\mathrm{SU}(2)$, $\mathrm{U}(2)$
$\mathbb{S}^7 = \mathrm{Spin}(7)/G_2$	$\mathrm{O}(8)$	—
$\mathbb{S}^{15} = \mathrm{Spin}(9)/\mathrm{Spin}(7)$	$\mathrm{Spin}(9)$	$\mathrm{SO}(16)$

TABLE 3. Isometry groups of 1-connected, homogeneous, positively curved manifolds.

(cf. [20]). Hence, the isometry group of any homogeneous metric of positive curvature on this space is either Spin(9) or the orthogonal group.

The results are summarized in Table 3. The last column of the table answers the following question: Given G/H of positive cuvature, what are the possible connected isometry groups for homogeneous metrics of positive curvature on G/H for metrics submersed from a left-invariant metric on G. This is answered using the Reduction Theorem and Proposition 4.2.

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