Homework 8

Due: Friday, 8 April, 2016.
Practice: Questions marked with a $\surd$ in Sections Three.V.1–2 and Three.VI.1–2 from the textbook.

1. [**15 marks**] Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y+z \\ x+z \\ x+y \end{pmatrix}$. Let $\mathcal{B} = \{\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}\}$ be a basis for $\mathbb{R}^3$.

(a) Write down the matrix $[T]_{\mathcal{E}}^{\mathcal{E}}$ representing T with respect to the standard basis $\mathcal{E}$ for $\mathbb{R}^3$, and the change of basis matrices $[id]_{\mathcal{B}}^{\mathcal{E}}$ and $[id]_{\mathcal{E}}^{\mathcal{B}}$.

(b) Use part (a) to find the matrix $[T]_{\mathcal{B}}^{\mathcal{B}}$ respresenting T with respect to $\mathcal{B}$. It should be a diagonal matrix. (Hint: $[T]_{\mathcal{B}}^{\mathcal{B}} = [id]_{\mathcal{E}}^{\mathcal{B}}[T]_{\mathcal{E}}^{\mathcal{E}}[id]_{\mathcal{B}}^{\mathcal{E}}$.)

(c) Use part (b) to find $[T^3]_{\mathcal{B}}^{\mathcal{B}}$, then use the change of basis matrices to find $[T^3]_{\mathcal{E}}^{\mathcal{E}}$.

2. [**10 marks**] Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation which preserves lengths of vectors, i.e. $|T(\mathbf{v})| = |\mathbf{v}|$ for all $\mathbf{v} \in \mathbb{R}^n$. Show that T must preserve dot products, i.e.

$$T(\mathbf{u}) \cdot T(\mathbf{v}) = \mathbf{u} \cdot \mathbf{v}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$. (Hint: expand the equation $|T(\mathbf{u}-\mathbf{v})|^2 = |\mathbf{u}-\mathbf{v}|^2$ using the fact that $|\mathbf{v}|^2 = \mathbf{v} \cdot \mathbf{v}$ for all $\mathbf{v} \in \mathbb{R}^n$.)

3. [**15 marks**] Determine whether the vectors

$$\boldsymbol{\beta}_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \boldsymbol{\beta}_2 = \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix}, \boldsymbol{\beta}_3 = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$$

are mutually orthogonal. Apply the Gram–Schmidt orthogonalisation process to the vectors $\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \boldsymbol{\beta}_3$ to obtain an orthonormal basis for $\mathbb{R}^3$. Check that your answer does indeed give an orthonormal basis.