# Homework 6

**Due:** Friday, 11 March, 2016.
**Practice:** Questions marked with a $\surd$ in Sections Three.III.1–2 and Three.IV.2 from the textbook.

**1.** [**16 marks**] Consider the matrices

$$A = \begin{pmatrix} 2 & 1 & -1 \\ 3 & 0 & 2 \end{pmatrix}, B = \begin{pmatrix} 1 & 3 \\ 2 & 2 \\ 1 & 4 \end{pmatrix}, C = \begin{pmatrix} 2 & 3 \\ 1 & 5 \end{pmatrix}.$$

Compute the following matrix products, or state that they are not defined:

$$AB,\ BA,\ AC,\ CA,\ BC,\ CB,\ A^2,\ B^2,\ C^2.$$

**2.** [**8 marks**] Let $\mathcal{B} = \{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}\}$ and $\mathcal{D} = \{\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 7 \\ 3 \end{pmatrix}\}$. You may assume $\mathcal{B}$ and $\mathcal{D}$ are bases for $\mathbb{R}^3$ and $\mathbb{R}^2$ respectively. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear map defined by

$$T \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \qquad T \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad T \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

and extending linearly. Compute $[T]_{\mathcal{B}}^{\mathcal{D}}$, the matrix representing $T$ with respect ot $\mathcal{B}$ and $\mathcal{D}$.

**3.** [**16 marks**] Let $V$ be the vector space spanned by the functions

$$\mathcal{B} = \{e^x, xe^x, \frac{x^2e^x}{2}, \frac{x^3e^x}{6}\}.$$

(You may assume $\mathcal{B}$ forms a basis for $V$).

(a) Find the derivative of each function $f \in \mathcal{B}$. Show that the matrix $[\frac{d}{dx}]_{\mathcal{B}}^{\mathcal{B}}$, representing the differentiation operator $\frac{d}{dx} : V \to V$ with respect to $\mathcal{B}$ and $\mathcal{B}$, is given by

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(Hint: for each $f \in \mathcal{B}$, write $f'$ as a linear combination of the elements of $\mathcal{B}$, then use this to write down the column vector $[f']_{\mathcal{B}}$ representing $f'$ with respect to $\mathcal{B}$.)

(b) Compute $A^3$, then use this to find the third derivative of $x^3e^x$. (Do not differentiate this directly.)