Homework 5

Due: Friday, 4 March, 2016.
Practice: Questions marked with a $\sqrt{}$ in Sections Three.II.1–2 from the textbook.

1. [**12 marks**] Determine whether the following are linear transformations.

(a) A *translation* $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T(\mathbf{v}) = \mathbf{v} + \mathbf{a}$, where $\mathbf{a} \in \mathbb{R}^2$ is a non-zero vector.

(b) The map $T : \mathcal{P}_2 \to \mathcal{P}_3$ given by $T(a + bx + cx^2) = ax + \frac{b}{2}x^2 + \frac{c}{3}x^3$.

2. [**16 marks**] For each of the following linear maps $T : \mathbb{R}^2 \to \mathbb{R}^2$, find $T(\mathbf{e}_1)$ and $T(\mathbf{e}_2)$ where $\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Write down the matrix representing T (with respect to the standard bases). Draw a neat diagram of the image of the unit square $\{\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \mid 0 \leq x, y \leq 1\}$ under the given transformation, with $T(\mathbf{e}_1)$ and $T(\mathbf{e}_2)$ clearly labelled.

(a) T_1 is a reflection in the line $y = x$.

(b) T_2 is a reflection in the y–axis.

(c) T_3 is an anticlockwise rotation about the origin through an angle of $\frac{\pi}{2}$.

(d) $T_4 = T_2 \circ T_1$, the map defined by applying T_1 followed by T_2.

What can you say about T_3 and T_4? Would $T_1 \circ T_2$ define the same map as $T_2 \circ T_1$?

3. [**12 marks**] Let $T : V \to W$ be a linear transformation. Recall that the *kernel* of T is $\ker T = \{\mathbf{v} \in V \mid T(\mathbf{v}) = \mathbf{0}_W\}$.

(a) Given $\mathbf{v}_1, \mathbf{v}_2 \in V$, prove that $T(\mathbf{v}_1) = T(\mathbf{v}_2)$ if and only if $\mathbf{v}_1 - \mathbf{v}_2 \in \ker T$.

(b) Let $\mathbf{w} \in W$. Show that the set of all $\mathbf{v} \in V$ satisfying the equation $T(\mathbf{v}) = \mathbf{w}$ is either empty, or is of the form

$$\{\mathbf{p} + \mathbf{h} \mid \mathbf{h} \in \ker T\}$$

where $\mathbf{p} \in V$ is any vector satisfying $T(\mathbf{p}) = \mathbf{w}$. (In fact, the set is non-empty if and only if $\mathbf{w} \in \operatorname{im} T$.)