Homework 4

Due: Friday, 26 February, 2016.
Practice: Questions marked with a $\surd$ in Sections Two.II.1, Two.III.1–3 from the textbook.

1. [**6 marks**] Let A be a 3×7 matrix. What are the possible values for rank(A) and nullity(A)? What does this say about the number of solutions to $A\mathbf{x} = \mathbf{0}$? Is it possible for the columns of A to be linearly independent?

2. [**12 marks**] Let $A = \begin{pmatrix} 1 & 2 & 0 & 3 & -3 & 10 \\ 3 & 6 & 2 & 3 & 3 & 8 \\ 2 & 4 & 1 & 1 & 4 & 3 \\ 4 & 8 & 1 & 6 & 0 & 20 \end{pmatrix}$. The RREF of A is $\begin{pmatrix} 1 & 2 & 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$.
Let S be the set of columns of A.
   (a) What is rank(A) and nullity(A)?
   (b) Choose a linearly independent subset $S' \subseteq S$ such that Span(S') = Span(S). Write each vector of S not in S' as a linear combination of vectors in S'.
   (c) Does S' form a basis for $\mathbb{R}^3$? Give reasons.

3. [**16 marks**]
   (a) Show that $\mathcal{B} = \{\begin{pmatrix} 3 \\ 7 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \end{pmatrix}\}$ forms a basis for $\mathbb{R}^2$. Find the column vector (i.e. the co-ordinates) representing $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$ with respect to the (ordered) basis $\mathcal{B}$.
   (b) Does $\mathcal{D} = \{2x^2 + 1,\ x - 3,\ x^2 + 2x + 1\}$ form a basis for $\mathcal{P}_2$? (Hint: represent each element of $\mathcal{D}$ as a column vector with respect to the basis $\mathcal{B} = \{1,\ x,\ x^2\}$ for $\mathcal{P}_2$.) If so, find $[x + 1]_{\mathcal{D}}$, the column vector representing the polynomial $x + 1$ with respect to the basis $\mathcal{D}$.

4. [**6 marks**] Let $\mathcal{P}$ be the vector space of real polynomials under the usual operations, and suppose $S \subset \mathcal{P}$ is a finite set. Explain why S cannot span $\mathcal{P}$. (Hint: consider the degree of each polynomial in S.) This shows that $\mathcal{P}$ does not have a finite basis, and therefore must be infinite dimensional.