1. **[8 marks]** Let $\mathbf{F}(x, y) = 3x^2\mathbf{i} - 2y\mathbf{j}$. Compute the line integral

$$\int_C \mathbf{F} \cdot dr$$

where C is the straight line segment from $(1, 3)$ to $(-1, 0)$ in $\mathbb{R}^2$.

Sol ①: Parameterise C: $\underset{\sim}{r}(t) = \langle 1, 3\rangle + t\langle -2, -3\rangle, \quad 0 \le t \le 1$

$$= \langle 1-2t, 3-3t\rangle$$

$$\underset{\sim}{r}'(t) = \langle -2, -3\rangle$$

$$\underset{\sim}{F}(x(t), y(t)) = \underset{\sim}{F}(1-2t, 3-3t)$$
$$= 3(1-2t)^2\underset{\sim}{i} - 2(3-3t)\underset{\sim}{j}$$
$$= \langle 3(1-4t+4t^2), -6+6t\rangle$$
$$= \langle 3-12t+12t^2, -6+6t\rangle$$

$$\therefore \int_C \underset{\sim}{F}\cdot d\underset{\sim}{r} = \int_0^1 \underset{\sim}{F}(\underset{\sim}{r}(t))\cdot \underset{\sim}{r}'(t)\,dt$$
$$= \int_0^1 \langle 3-12t+12t^2, -6+6t\rangle \cdot \langle -2, -3\rangle\,dt$$
$$= \int_0^1 [-2(3-12t+12t^2) - 3(-6+6t)]\,dt$$
$$= \int_0^1 -6+24t-24t^2+18-18t \quad dt$$
$$= \int_0^1 12 + 6t - 24t^2\,dt$$
$$= [12t + 3t^2 - 8t^3]_0^1$$
$$= (12+3-8) - 0 \quad = 7$$

Sol ② Observe that $\underset{\sim}{F} = 3x^2\underset{\sim}{i} - 2y\underset{\sim}{j}$ is conservative since $\frac{\partial}{\partial x}(-2y) = 0 = \frac{\partial}{\partial y}(3x^2)$. Find potential function f, i.e. $\nabla f = \underset{\sim}{F}$

$\frac{\partial f}{\partial x} = 3x^2 \quad \therefore f(x,y) = x^3 + g(y)$

$\frac{\partial f}{\partial y} = -2y \qquad \therefore \frac{\partial f}{\partial y} = \frac{dg}{dy}$

$\therefore \frac{dg}{dy} = -2y$

$\therefore g(y) = -y^2 + C$

$\therefore f(x,y) = x^3 - y^2$ is a potential fn.

By FTC for line integrals:

$$\int_C \underset{\sim}{F}\cdot d\underset{\sim}{r} = f(-1,0) - f(1,3)$$
$$= (-1)^3 - (1^3 - 3^2) = 7$$

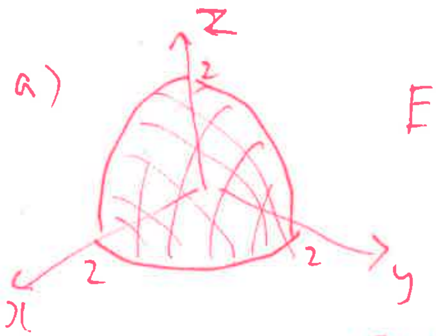
2. Let E be the solid region lying inside the sphere $x^2 + y^2 + z^2 = 4$ and the first octant in $\mathbb{R}^3$.

(a) [**3 marks**] Express the region E using spherical co-ordinates.

(b) [**5 marks**] Evaluate the integral

$$\iiint_E z^2 \, dV.$$

a)



$E = \{(\rho, \theta, \phi) \mid 0 \le \rho \le 2, \ 0 \le \theta \le \frac{\pi}{2}, \ 0 \le \phi \le \frac{\pi}{2}\}$

b) $\iiint_E z^2 dV = \int_0^{\pi/2} \int_0^{\pi/2} \int_0^2 (\rho \cos\phi)^2 \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$

$= \int_0^{\pi/2} d\theta \int_0^{\pi/2} \cos^2\phi \sin\phi \, d\phi \int_0^2 \rho^4 \, d\rho$

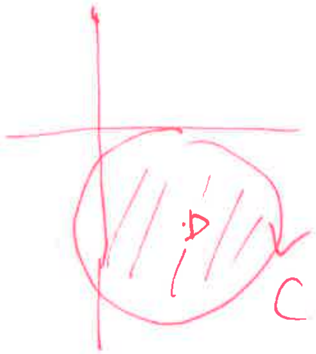
$= \frac{\pi}{2} \cdot \left[-\frac{1}{3}\cos^3\phi\right]_0^{\pi/2} \cdot \left[\frac{1}{5}\rho^5\right]_0^2$

$= \frac{-\pi}{30} (0 - 1^3) \cdot (2^5 - 0)$

$= \frac{32}{30}\pi = \frac{16\pi}{15}$

3. **[8 marks]** Let C be the circle in $\mathbb{R}^2$ given by the equation $(x-3)^2 + (y+4)^2 = 16$ with the clockwise orientation. Use Green's Theorem to evaluate the integral

$$\int_C \underbrace{(x + 3x^2y - 2y)}_{P}\, dx + \underbrace{(x^3 + 2x - y)}_{Q}\, dy.$$



C is negatively oriented

Let $P(x,y) = x + 3x^2y - 2y$

$Q(x,y) = x^3 + 2x - y$

By Green's Thm

$$\int_C P\,dx + Q\,dy = -\int_{-C} P\,dx + Q\,dy$$

where D is region enclosed by C.

$$= -\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$= -\iint_D \left[(3x^2+2) - (3x^2-2)\right] dA$$

$$= -\iint_D 4\, dA$$

$$= -4 \iint_D dA$$

$$= -4\, \text{area}(D)$$

$$= -4 \cdot \pi \cdot 4^2 = -64\pi.$$

4. Let $\mathbf{F}(x,y,z) = ye^z\mathbf{i} + xe^z\mathbf{j} + xye^z\mathbf{k}$, and let C be the curve in $\mathbb{R}^3$ with parameterisation $\mathbf{r}(t) = \sin t\,\mathbf{i} + \cos^2 t\,\mathbf{j} + t(t-\pi)\,\mathbf{k}$ for $0 \le t \le \pi$.

(a) [**4 marks**] Compute $\operatorname{curl}\mathbf{F}$.

(b) [**4 marks**] Find $\int_C \mathbf{F}\cdot dr$. You must state any theorems you use. [Hint: What does your answer in part (a) say about $\mathbf{F}$? What is special about the curve C?]

a) $$\operatorname{curl} \underset{\sim}{F} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ye^z & xe^z & xye^z \end{vmatrix}$$

$$= (xe^z - xe^z)\underset{\sim}{i} + (ye^z - ye^z)\,\cancel{\underset{\sim}{k}}\,\underset{\sim}{j} + (e^z - e^z)\underset{\sim}{k}$$

$$= \underset{\sim}{0}$$

b) Since $\underset{\sim}{F}$ is defined on all of $\mathbb{R}^3$, ~~and~~ has continuous partial derivatives and $\operatorname{curl} \underset{\sim}{F} = 0$, it follows that $\underset{\sim}{F}$ is conservative.

Also, $\underset{\sim}{r}(0) = 0\underset{\sim}{i} + \underset{\sim}{j} + 0\underset{\sim}{k} = \underset{\sim}{j}$

$\underset{\sim}{r}(\pi) = 0\underset{\sim}{i} + (-1)^2\underset{\sim}{j} + 0\underset{\sim}{k} = \underset{\sim}{j}$

$\therefore$ C is a closed curve since $\underset{\sim}{r}(0) = \underset{\sim}{r}(\pi)$.

$\therefore \int_C \underset{\sim}{F}\cdot d\underset{\sim}{r} = 0$, since line integrals of conservative fields along closed curves $= 0$.

5. [**8 marks**] Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ be a vector field and $g(x, y, z)$ a scalar function on $\mathbb{R}^3$. Prove the identity

$$\operatorname{div}(g\mathbf{F}) = g \operatorname{div} \mathbf{F} + \nabla g \cdot \mathbf{F}.$$

Assume all appropriate partial derivatives exist and are continuous. [Hint: Find the component functions of the vector field $g\mathbf{F}$, then apply the definition of div.]

$$g\underset{\sim}{F} = \langle gP, gQ, gR \rangle$$

$$\operatorname{div}(g\underset{\sim}{F}) = \frac{\partial}{\partial x}(gP) + \frac{\partial}{\partial y}(gQ) + \frac{\partial}{\partial z}(gR)$$

by product rule

$$= \left\{\frac{\partial g}{\partial x}\right\} P + g\frac{\partial P}{\partial x} + \frac{\partial g}{\partial y} Q + g\frac{\partial Q}{\partial y} + \frac{\partial g}{\partial z} R + g\frac{\partial R}{\partial z}$$

$$= g\left\{\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right\} + \frac{\partial g}{\partial x} P + \frac{\partial g}{\partial y} Q + \frac{\partial g}{\partial z} R$$

$$= g \operatorname{div} \underset{\sim}{F} + \left\langle \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z} \right\rangle \cdot \langle P, Q, R \rangle$$

$$= g \operatorname{div} \underset{\sim}{F} + \nabla g \cdot \underset{\sim}{F}$$