1. [**6 marks**] Evalutate the integral

$$\int_0^1 \int_0^2 \int_0^3 xyz^2 \, dx \, dy \, dz.$$

$$= \int_0^1 z^2 dz \int_0^2 y \, dy \int_0^3 x \, dx$$

$$= \left[\frac{1}{3} z^3\right]_0^1 \left[\frac{1}{2} y^2\right]_0^2 \left[\frac{1}{2} x^2\right]_0^3$$

$$= \frac{1}{3} \cdot \frac{4}{2} \cdot \frac{9}{2} = 3$$

4 4

2. [**10 marks**] Find and classify all critical points of

$$f(x, y) = x^2 + 3xy - y^3.$$

2

Evaluate the function at each critical point.

$f_x = 2x + 3y \qquad f_y = 3x - 3y^2$

For crit. pts: $f_x = 0$, $f_y = 0$

$\therefore \; 2x + 3y = 0$

$3x - 3y^2 = 0 \quad \therefore \; x = y^2$

$\therefore \; 2y^2 + 3y = 0$

$y(2y + 3) = 0 \quad \therefore \; y = 0 \text{ or } y = -\frac{3}{2}$

$\therefore \; x = 0 \text{ or } x = \left(-\frac{3}{2}\right)^2 = 9/4$

(A)

$\therefore$ Crit. pts are $(0,0)$ & $(9/4, -3/2)$.

$f_{xx} = 2, \quad f_{yy} = -6y, \quad f_{xy} = 3$

$D = f_{xx} f_{yy} - f_{xy}^2 = -12y - 9$

$D(0,0) = -9 < 0 \quad \therefore$ saddle point

$D(9/4, -3/2) = -12\left(-\frac{3}{2}\right) - 9 = 18 - 9 = 9 > 0$

$f_{xx}(9/4, -3/2) = 2 > 0 \quad \therefore$ local minimum

- Saddle pt at $f(0,0) = 0$
- Local min at $f\left(\frac{9}{4}, -\frac{3}{2}\right) = (9/4)^2 + 3\left(\frac{9}{4}\right)\left(-\frac{3}{2}\right) - \left(-\frac{3}{2}\right)^3$

$= \frac{81}{16} - \frac{81}{8} + \frac{27}{8}$

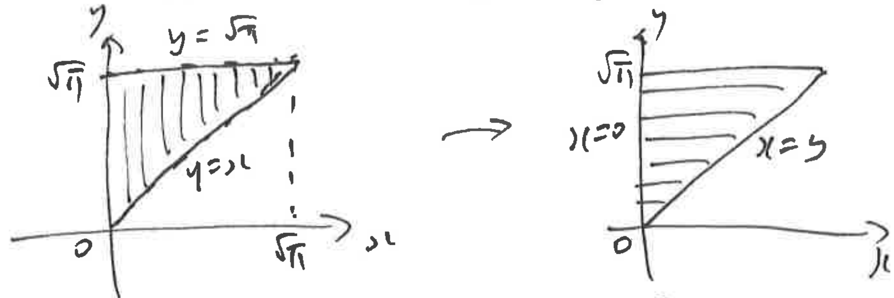
$= \frac{81 - 162 + 54}{16} = \frac{-27}{16}$

3 (4) (2)

3. [**6 marks**] Evaluate the integral

$$\int_0^{\sqrt{\pi}} \int_x^{\sqrt{\pi}} \cos(y^2)\, dy\, dx$$

by first swapping the order of integration. [Hint: Draw the region of integration]



$$\int_0^{\sqrt{\pi}} \int_x^{\sqrt{\pi}} \cos(y^2)\, dy\, dx = \int_0^{\sqrt{\pi}} \int_0^{y} \cos(y^2)\, dx\, dy$$

$$= \int_0^{\sqrt{\pi}} \left[x \cos(y^2)\right]_0^y dy$$

$$= \int_0^{\sqrt{\pi}} y \cos(y^2)\, dy$$

sub $u = y^2$,
$du = 2y\, dy$

$$= \int_0^{\pi} \frac{1}{2} \cos u\, du$$

$$= \frac{1}{2}\left[\sin u\right]_0^{\pi}$$

$$= \frac{1}{2}(\sin \pi - \sin 0)$$

$$= 0$$

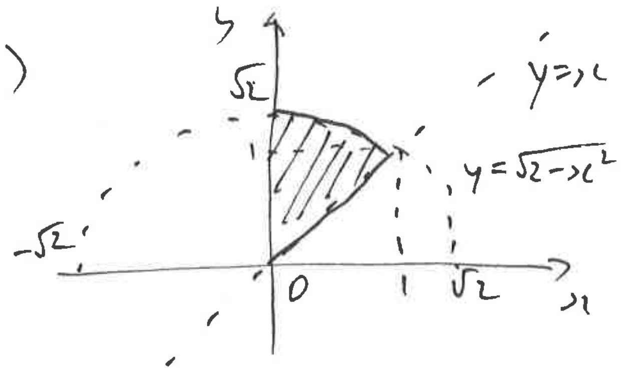
4. Consider the integral

$$\int_0^1 \int_x^{\sqrt{2-x^2}} x^2 + y^2 \; dy \; dx.$$

(a) [**4 marks**] Sketch the region of integration and express it using polar co-ordinates.

(b) [**4 marks**] Evaluate the integral.

a)



$R = \{(r,\theta) \mid \frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}, \; 0 \leq r \leq \sqrt{2}\}$

b) $\displaystyle \int_0^1 \int_x^{\sqrt{2-x^2}} x^2 + y^2 \, dy \, dx = \int_{\pi/4}^{\pi/2} \int_0^{\sqrt{2}} r^2 \cdot r \, dr \, d\theta$

$\displaystyle = \int_{\pi/4}^{\pi/2} d\theta \int_0^{\sqrt{2}} r^3 \, dr$

$\displaystyle = \left(\frac{\pi}{2} - \frac{\pi}{4}\right) \left[\frac{1}{4} r^4\right]_0^{\sqrt{2}}$

$\displaystyle = \frac{\pi}{4} \cdot \frac{1}{4} \sqrt{2}^4$

$\displaystyle = \frac{\pi}{4}$

5. **[10 marks]** Use the method of Lagrange multipliers to find the extreme values of

$$f(x, y, z) = xy + xz$$

subject to both constraints $xy = 1$ and $x^2 + z^2 = 1$. [There should be four critical points.]

Let $g(x,y,z) = xy$, $h(x,y,z) = x^2 + z^2$

Need to solve for x, y, z & λ, μ

$$\nabla f(x,y,z) = \lambda \nabla g(x,y,z) + \mu \nabla h(x,y,z)$$

& $\cancel{g(x,y,z)=}$ $xy = 1$, $x^2 + z^2 = 1$

$$\Rightarrow \langle y+z, x, x \rangle = \lambda \langle y, x, 0 \rangle + \mu \cancel{\langle 2x, 2y, 0 \rangle} \langle 2x, 0, 2z \rangle$$

$\therefore$ $y + z = \lambda y + 2\mu x$ ①

$x = \lambda x$ $\therefore \lambda = 1$ ②

$x = 2\mu z$ ③

$xy = 1$ $\therefore x \neq 0, y \neq 0.$ ④

$x^2 + z^2 = 1$ ⑤

$\lambda = 1 \Rightarrow$ ① $y + z = y + 2\mu x$ $\therefore z = 2\mu x$ ⑥

$\Rightarrow$ ③ $x = 2\mu(2\mu x) = 4\mu^2 x$ $\therefore 4\mu^2 = 1$ since $x \neq 0$

$\therefore \mu = \pm \frac{1}{2}$

$\Rightarrow$ ⑥ $\therefore z = 2 \cdot \pm\frac{1}{2} x = \pm x$

$\Rightarrow$ ⑤ $x^2 + x^2 = 1$ $\therefore x^2 = \frac{1}{2}$ $\therefore x = \pm \frac{1}{\sqrt{2}}$

$\Rightarrow$ ④ if $x = \frac{1}{\sqrt{2}}$, $y = \frac{1}{x} = \sqrt{2}$, if $x = -\frac{1}{\sqrt{2}}$, $y = \frac{1}{x} = -\sqrt{2}$

$\therefore$ crit pts

$$\left.\begin{array}{l} \left(\frac{1}{\sqrt{2}}, \sqrt{2}, \frac{1}{\sqrt{2}}\right) \\ \left(\frac{1}{\sqrt{2}}, \sqrt{2}, -\frac{1}{\sqrt{2}}\right) \\ \left(-\frac{1}{\sqrt{2}}, \sqrt{2}, \frac{1}{\sqrt{2}}\right) \\ \left(-\frac{1}{\sqrt{2}}, -\sqrt{2}, \frac{-1}{\sqrt{2}}\right) \end{array}\right\} \xmapsto{\text{apply } f} \left\{\begin{array}{l} 1 + \frac{1}{2} = 3/2 \\ 1 - \frac{1}{2} = 1/2 \\ 1 - \frac{1}{2} = 1/2 \\ 1 + \frac{1}{2} = 3/2 \end{array}\right.$$

$\therefore$ extreme values are $\frac{1}{2}$ & $3/2$.