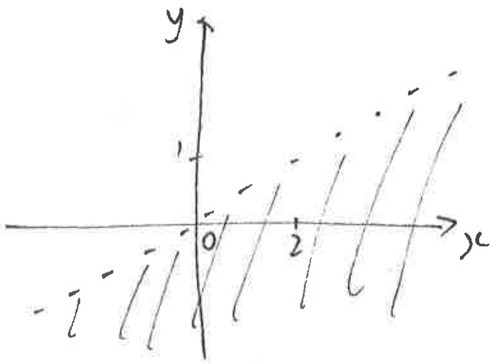
1. Let $f(x, y) = xy + \ln(x - 2y)$.
   (a) **[4 marks]** Find the domain of f and draw a neat sketch of it.
   (b) **[4 marks]** Compute the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.
   (c) **[5 marks]** Find the directional derivative of f at $(1, -2)$ in the direction of $\mathbf{v} = \langle -3, 4\rangle$.

a) Domain of $f = \{(x,y) \in \mathbb{R}^2 \mid x > 2y\}$



b) $\frac{\partial f}{\partial x} = y + \frac{1}{x-2y}$, $\frac{\partial f}{\partial y} = x + \frac{1}{x-2y} \cdot -2$

$= x - \frac{2}{x-2y}$

c) $\nabla f(1,-2) = \left\langle -2 + \frac{1}{1-2\cdot(-2)}, 1 - \frac{2}{1-2(-2)} \right\rangle$

$= \langle -2 + \frac{1}{5}, 1 - \frac{2}{5} \rangle$

$= \langle -\frac{9}{5}, \frac{3}{5} \rangle$

Unit vector in direction of $\underset{\sim}{v} = \langle -3, 4\rangle$ is

$\underset{\sim}{u} = \frac{\underset{\sim}{v}}{|\underset{\sim}{v}|} = \frac{\langle -3, 4\rangle}{\sqrt{3^2+4^2}} = \langle -\frac{3}{5}, \frac{4}{5} \rangle$

Since f is differentiable, we have

$D_{\underset{\sim}{u}} f(1,-2) = \nabla f(1,-2) \cdot \underset{\sim}{u}$

$= \langle -\frac{9}{5}, \frac{3}{5} \rangle \cdot \langle -\frac{3}{5}, \frac{4}{5} \rangle$

$= \frac{27}{25} + \frac{12}{25} = \frac{39}{25}$

2. **[5 marks]** Compute the following limit, or explain why it does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^3+y^3}.$$

Let $f(x,y) = \frac{x^2y}{x^3+y^3}$

Approaching $(0,0)$ along the line $y = mx$

$$f(x,mx) = \frac{x^2mx}{x^3+m^3x^3} = \frac{mx^3}{(1+m^3)x^3} = \frac{m}{1+m^3} \text{ for } x \neq 0$$

$$\to \frac{m}{1+m^3} \text{ as } x \to 0.$$

We obtain different limits for different values of m,

i.e. if $m=0$, the limit is 0 and

if $m=1$, the limit is $\frac{1}{1+1^3} = \frac{1}{2}$.

$\therefore \lim_{(x,y)\to(0,0)} \frac{x^2y}{x^3+y^3}$ does not exist.

3. **[2 marks]** Let $f(x,y,z,w)$ be differentiable, and suppose x, y, z and w are themselves differentiable functions of r, s and t. Write an expression for $\frac{\partial f}{\partial s}$ using the chain rule.

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial s}$$

4. Let S be the ellipsoid in $\mathbb{R}^3$ given by $x^2 + 2y^2 + 4z^2 = 1$. Let $P(x_0, y_0, z_0)$ be a point on S.

(a) **[5 marks]** By first computing an appropriate gradient vector, find the equation of the normal line to S at P in parametric Cartesian form.

(b) **[5 marks]** Show that the equation of the tangent plane to S at P can be written as

$$x_0x + 2y_0y + 4z_0z = 1.$$

a) Let $F(x,y,z) = x^2 + 2y^2 + 4z^2$.
Then the surface S is the level surface
$F(x,y,z) = 1$.

$\nabla F = \langle 2x, 4y, 8z \rangle$

$\therefore \nabla F(x_0, y_0, z_0) = \langle 2x_0, 4y_0, 8z_0 \rangle$
this vector is normal to S at P.

$\therefore$ Vector eq'n of normal line:

$\underset{\sim}{x} - \underset{\sim}{x}_0 = t\,\nabla F(x_0,y_0,z_0)$ where $\underset{\sim}{x}_0 = \langle x_0, y_0, z_0 \rangle$

In parametric Cartesian form:

$x - x_0 = t(2x_0)$, $y - y_0 = t(4y_0)$, $z - z_0 = t(8z_0)$
$= 2tx_0$, $= 4ty_0$, $= 8tz_0$

b) Eq'n of tangent plane at P in vector form:

$\nabla F(x_0,y_0,z_0) \cdot (\underset{\sim}{x} - \underset{\sim}{x}_0) = 0$

~~$\langle 2x_0, 4y_0, 8z_0 \rangle$~~

$\nabla F(x_0,y_0,z_0) \cdot \underset{\sim}{x} - \nabla F(x_0,y_0,z_0) \cdot \underset{\sim}{x}_0 = 0$

$\nabla F(x_0,y_0,z_0) \cdot \underset{\sim}{x} = \nabla F(x_0,y_0,z_0) \cdot \underset{\sim}{x}_0$

$\langle 2x_0, 4y_0, 8z_0 \rangle \cdot \langle x, y, z \rangle = \langle 2x_0, 4y_0, 8z_0 \rangle \cdot \langle x_0, y_0, z_0 \rangle$

$2x_0x + 4y_0y + 8z_0z = 2x_0^2 + 4y_0^2 + 8z_0^2$

$x_0x + 2y_0y + 4z_0z = x_0^2 + 2y_0^2 + 4z_0^2$

$= 1$ (since (x_0, y_0, z_0) lies on S)

5. **[5 marks]** Suppose z is defined implicitly by the equation $xe^{yz} = \sin z$. Compute $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

Let $F(x,y,z) = xe^{yz} - \sin z$.

Then $\frac{\partial z}{\partial x} = \frac{-F_x}{F_z} = -\frac{e^{yz}}{xye^{yz} - \cos z}$

$$\frac{\partial z}{\partial y} = \frac{-F_y}{F_z} = -\frac{xze^{yz}}{xye^{yz} - \cos z}$$

6. **[5 marks]** Let $f(x, y)$ and $g(x, y)$ be differentiable functions. Prove the *product rule* for the gradient operator:

$$\nabla(fg) = f\nabla g + g\nabla f.$$

By definition

$$\nabla(fg) = \left\langle \frac{\partial}{\partial x}(fg), \frac{\partial}{\partial y}(fg) \right\rangle$$

$$= \left\langle f\frac{\partial g}{\partial x} + g\frac{\partial f}{\partial x}, f\frac{\partial g}{\partial y} + g\frac{\partial f}{\partial y} \right\rangle$$ by product rule for functions of one variable

$$= \left\langle f\frac{\partial g}{\partial x}, f\frac{\partial g}{\partial y} \right\rangle + \left\langle g\frac{\partial f}{\partial x}, g\frac{\partial f}{\partial y} \right\rangle$$

$$= f\left\langle \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y} \right\rangle + g\left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$$

$$= f\nabla g + g\nabla f$$