1. The acceleration of an object at time t is given by $\mathbf{a}(t) = 6t\,\mathbf{i} - 4\mathbf{j}$, with initial velocity and position given by $\mathbf{v}(0) = 2\mathbf{j}$ and $\mathbf{r}(0) = \mathbf{0}$ respectively.

(a) **[6 marks]** Express the object's velocity $\mathbf{v}(t)$ and position $\mathbf{r}(t)$ as vector functions of time t. (Hint: Use the fundamental theorem of calculus for vector functions twice.)

(b) **[2 marks]** What is the object's displacement from $t = 1$ to $t = 4$?

(c) **[3 marks]** Write down a formula for the object's speed as a (scalar) function of time t. Use this to express the distance travelled from $t = 0$ to $t = T$ as a definite integral. (Do not evaluate the integral.)

a) $\underset{\sim}{v}(t) = \underset{\sim}{v}(0) + \int_0^t \underset{\sim}{a}(u)\,du$

$= \langle 0,2\rangle + \int_0^t \langle 6u,-4\rangle\,du$

$= \langle 0,2\rangle + [\langle 3u^2,-4u\rangle]_0^t$

$= \langle 0,2\rangle + \langle 3t^2,-4t\rangle = \langle 3t^2, 2-4t\rangle$

$\underset{\sim}{r}(t) = \underset{\sim}{r}(0) + \int_0^t \underset{\sim}{v}(u)\,du$

$= \underset{\sim}{0} + \int_0^t \langle 3u^2, 2-4u\rangle\,du$

$= [\langle u^3, 2u-2u^2\rangle]_0^t = \langle t^3, 2t-2t^2\rangle - \langle 0,0\rangle$

$= \langle t^3, 2t-2t^2\rangle$

b) displacement $= \underset{\sim}{r}(4) - \underset{\sim}{r}(1)$

$= \langle 4^3, 2\cdot4-2\cdot4^2\rangle - \langle 1, 2-2\rangle$

$= \langle 64-1, \underset{-24}{\cancel{}}\rangle = \langle 63, \cancel{}\ -24\rangle$

c) speed at time t $= |\underset{\sim}{v}(t)| = |\langle 3t^2, 2-4t\rangle|$

$= \sqrt{(3t^2)^2 + (2-4t)^2}$

$= \sqrt{9t^4 + 16t^2 - 16t + 4}$

$\therefore$ distance from $t=0$ to $t=T$

$= \int_0^T |\underset{\sim}{v}(t)|\,dt = \int_0^T \sqrt{9t^4+16t^2-16t+4}\,dt$

2. The velocity and acceleration of a car at a point P are respectively $\mathbf{v} = \mathbf{i} + \mathbf{j}$ and $\mathbf{a} = 2\mathbf{i} + 4\mathbf{j}$.

(a) [**6 marks**] Compute the following vectors, and show that they are perpendicular.

(i) $\operatorname{proj}_{\mathbf{v}} \mathbf{a} = \frac{(\mathbf{v} \cdot \mathbf{a})}{|\mathbf{v}|^2} \mathbf{v}$ (ii) $\operatorname{orth}_{\mathbf{v}} \mathbf{a} = \mathbf{a} - \operatorname{proj}_{\mathbf{v}} \mathbf{a}$

(b) [**4 marks**] Find scalars a_T and a_N so that $\mathbf{a} = a_T \mathbf{T} + a_N \mathbf{N}$, where $\mathbf{T}$ and $\mathbf{N}$ are the respective unit tangent and (principal) unit normal vectors to the car's path at P.

(c) [**6 marks**] Use the formula $\mathbf{a} = s''\mathbf{T} + \kappa(s')^2\mathbf{N}$ and part (b) to find the curvature κ of the car's path at P. (The variable s is the arc length parameter, and so s' is the speed.) What is the radius of the osculating circle to the car's path at P?

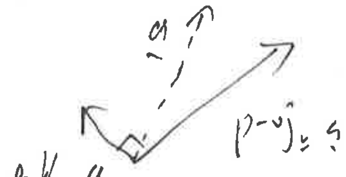
a) $|\underset{\sim}{v}|^2 = 1^2 + 1^2 = 2$, $\underset{\sim}{v} \cdot \underset{\sim}{a} = \langle 1,1\rangle \cdot \langle 2,4\rangle = 6$

$\therefore \operatorname{proj}_{\underset{\sim}{v}} \underset{\sim}{a} = \dfrac{(\underset{\sim}{v} \cdot \underset{\sim}{a})}{|\underset{\sim}{v}|^2} \underset{\sim}{v} = \dfrac{6}{2}\langle 1,1\rangle = \langle 3,3\rangle$

$\operatorname{orth}_{\underset{\sim}{v}} \underset{\sim}{a} = \underset{\sim}{a} - \operatorname{proj}_{\underset{\sim}{v}}(\underset{\sim}{a}) = \langle 2,4\rangle - \langle 3,3\rangle = \langle -1,1\rangle$

$\operatorname{proj}_{\underset{\sim}{v}} \underset{\sim}{a} \cdot \operatorname{orth}_{\underset{\sim}{v}} \underset{\sim}{a} = \langle 3,3\rangle \cdot \langle -1,1\rangle = -3+3 = 0$

$\therefore$ they are perpendicular.



b) Note: $\underset{\sim}{a} = \operatorname{proj}_{\underset{\sim}{v}} \underset{\sim}{a} + \operatorname{orth}_{\underset{\sim}{v}} \underset{\sim}{a}$

and $\operatorname{proj}_{\underset{\sim}{v}} \underset{\sim}{a}$ is parallel to $\underset{\sim}{T}$, $\operatorname{orth}_{\underset{\sim}{v}} \underset{\sim}{a}$ is normal to $\underset{\sim}{T}$,

$\therefore$ parallel to $\underset{\sim}{N}$.

$\therefore a_T \underset{\sim}{T} = \operatorname{proj}_{\underset{\sim}{v}} \underset{\sim}{a}$ & $a_N \underset{\sim}{N} = \operatorname{orth}_{\underset{\sim}{v}} \underset{\sim}{a}$

So, $a_T = |a_T \underset{\sim}{T}| = |\operatorname{proj}_{\underset{\sim}{v}} \underset{\sim}{a}| = |\langle 3,3\rangle| = 3\sqrt{2}$ (using the fact that

$a_N = |a_N \underset{\sim}{N}| = |\operatorname{orth}_{\underset{\sim}{v}} \underset{\sim}{a}| = |\langle -1,1\rangle| = \sqrt{2}$ $\underset{\sim}{T}, \underset{\sim}{N}$ are unit vectors)

c) Since $\underset{\sim}{a} = a_T \underset{\sim}{T} + a_N \underset{\sim}{N} = s''\underset{\sim}{T} + \kappa(s')^2 \underset{\sim}{N}$,

we have $a_N = \kappa(s')^2$. Note: $(s')^2 = |\underset{\sim}{v}|^2 = 2$

$\therefore \kappa = \dfrac{a_N}{(s')^2} = \dfrac{\sqrt{2}}{2} = \dfrac{1}{\sqrt{2}}$. speed

$\therefore$ Radius of osculating circle $= \dfrac{1}{\kappa} = \sqrt{2}$.

3. Let C be the curve given by $r = \cos 3\theta$ in polar co-ordinates.

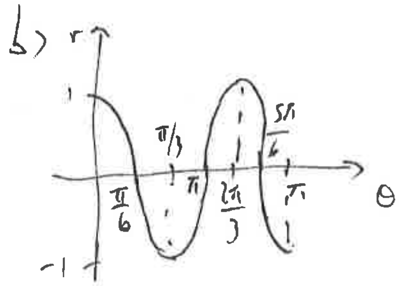
(a) **[3 marks]** Find the point on C corresponding to $\theta = 0$ in Cartesian co-ordinates. What is the smallest value of $\theta > 0$ for which the curve returns to this point?

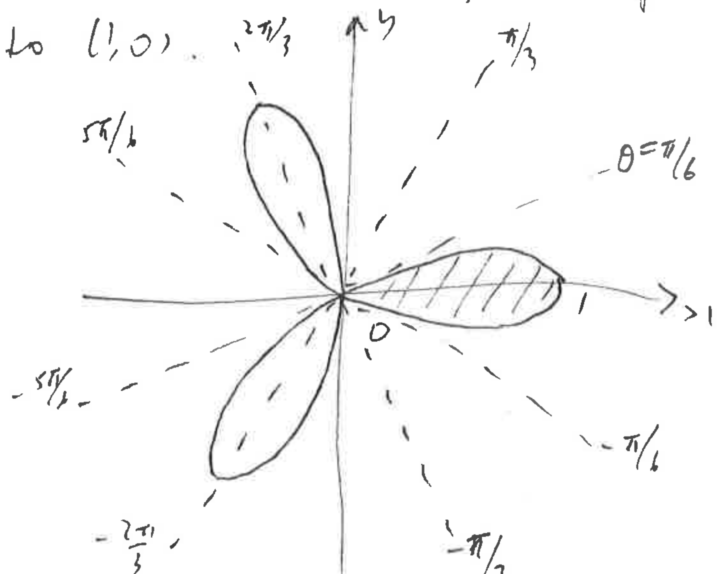
(b) **[5 marks]** Draw a neat diagram of C, clearly indicating the angles at which it passes through the origin. (It should be a rose with some number of petals.)

(c) **[5 marks]** How many petals does C have? Find the area contained in one petal of C. (You may use the formulae $\sin 2X = 2\sin X \cos X$ and $\cos 2X = \cos^2 X - \sin^2 X$.)

a) when $\theta = 0$, $r = \cos 0 = 1$ $\rightsquigarrow$ gives point $(1,0)$ in Cartesian co-ords.
If (r,θ) also represent this pt., then θ must be an integer multiple of π. Try $\theta = \pi$, $r = \cos 3\pi = -1$
$\rightsquigarrow$ $(-1, \pi)$ in polar coords $\rightsquigarrow$ $(-1\cos\pi, -1\sin\pi) = (1,0)$
in Cartesian co-ords. $\therefore$ $\theta = \pi$ is the first value of θ where the curve returns to $(1,0)$.

b)





c) 3 petals

$$\text{Area of one petal} = \int_{-\pi/6}^{\pi/6} \frac{1}{2} r^2 \, d\theta$$

$$= \int_{-\pi/6}^{\pi/6} \frac{1}{2}\cos^2 3\theta \, d\theta$$

$$= \int_{0}^{\pi/6} \cos^2 3\theta \, d\theta \quad \text{(integrand is even)}$$

$$= \int_{0}^{\pi/6} \frac{\cancel{\cos 2\theta}\ \cos 6\theta + 1}{2} \, d\theta$$

$$= \left[\frac{\sin 6\theta}{12} + \frac{\theta}{2} \right]_0^{\pi/6}$$

$$= \frac{\sin \pi}{12} + \frac{\pi}{12} - \frac{\sin 0}{12} - 0$$

$$= \frac{\pi}{12}.$$

4. **[6 marks]** Let $\mathbf{u}(t) = \langle u_1(t), u_2(t), u_3(t)\rangle$ and $\mathbf{v}(t) = \langle v_1(t), v_2(t), v_3(t)\rangle$ be differentiable vector functions. Prove that

$$\frac{d}{dt}(\mathbf{u}(t)\cdot\mathbf{v}(t)) = \mathbf{u}'(t)\cdot\mathbf{v}(t) + \mathbf{u}(t)\cdot\mathbf{v}'(t).$$

$$\underset{\sim}{u}(t)\cdot\underset{\sim}{v}(t) = u_1(t)v_1(t) + u_2(t)v_2(t) + u_3(t)v_3(t)$$

$$\frac{d}{dt}(\underset{\sim}{u}(t)\cdot\underset{\sim}{v}(t)) = u_1'(t)v_1(t) + u_1(t)v_1'(t) + u_2'(t)v_2(t) + u_2(t)v_2'(t) + u_3'(t)v_3(t) + u_3(t)v_3'(t)$$ (By product rule)

$$= (u_1'(t)v_1(t) + u_2'(t)v_2(t) + u_3'(t)v_3(t)) + (u_1(t)v_1'(t) + u_2(t)v_2'(t) + u_3(t)v_3'(t))$$

$$= \langle u_1'(t), u_2'(t), u_3'(t)\rangle \cdot \langle v_1(t), v_2(t), v_3(t)\rangle + \langle u_1(t), u_2(t), u_3(t)\rangle \cdot \langle v_1'(t), v_2'(t), v_3'(t)\rangle$$

$$= \underset{\sim}{u}'(t)\cdot\underset{\sim}{v}(t) + \underset{\sim}{u}(t)\cdot\underset{\sim}{v}'(t).$$

5. Let E be an ellipse with equation $\frac{x^2}{9} + \frac{y^2}{4} = 1$, and H a hyperbola with equation $x^2 - \frac{y^2}{4} = 1$.

(a) **[3 marks]** Show that E and H have the same foci.

(b) **[6 marks]** Let $P(x_0, y_0)$ be an intersection point of E and H. Let L_1 and L_2 be lines through P tangent to E and H respectively. Show that L_1 and L_2 have gradients $m_1 = -\frac{4x_0}{9y_0}$ and $m_2 = \frac{4x_0}{y_0}$ respectively. (Hint: Use implicit differentiation.)

(c) **[5 marks]** Prove that E and H intersect at right angles.

a) ellipse given by $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (w/ $a^2 \geq b^2$) has foci at $(\pm c, 0)$, where $c^2 = a^2 - b^2$. For E, $a^2 = 9$, $b^2 = 4$ $\therefore c^2 = 9 - 4 = 5$ $\therefore$ foci are at $(\pm\sqrt{5}, 0)$.

Hyperbola w/ eq'n $\frac{x^2}{A^2} - \frac{y^2}{B^2} = 1$ has foci at $(\pm C, 0)$, where $C^2 = A^2 + B^2$. For H, $A^2 = 1$, $B^2 = 4$, $\therefore C^2 = 1 + 4 = 5$ $\therefore$ foci are at $(\pm\sqrt{5}, 0)$

same

b) E: $\frac{x^2}{9} + \frac{y^2}{4} = 1$

$$\frac{2x}{9} + \frac{2yy'}{4} = 0$$

$$\frac{2yy'}{4} = -\frac{2x}{9}$$

$$y' = -\frac{4x}{9y}$$

at $P(x_0, y_0)$, $m_1 = -\frac{4x_0}{9y_0}$

H: $x^2 - y^2/4 = 1$

$$2x - \frac{2yy'}{4} = 0$$

$$x = \frac{yy'}{4}$$

$$\therefore y' = \frac{4x}{y}$$

at $P(x_0, y_0)$, $m_1 = \frac{4x_0}{y_0}$

c) Need to show $m_1 m_2 = -1$, i.e. $m_1 m_2 = -\frac{4x_0}{9y_0} \cdot \frac{4x_0}{y_0} = -\frac{16x_0^2}{9y_0^2} = -1$

Since $P(x_0, y_0)$ lies on both E & H, we have

$$\frac{x_0^2}{9} + \frac{y_0^2}{4} = 1 \quad \& \quad x_0^2 - \frac{y_0^2}{4} = 1$$

$$\therefore \frac{x_0^2}{9} + \frac{y_0^2}{4} = x_0^2 - \frac{y_0^2}{4}$$

$$\frac{2y_0^2}{4} = x_0^2 - \frac{x_0^2}{9}$$

$$\frac{y_0^2}{2} = \frac{8}{9}x_0^2$$

$$\therefore \frac{16x_0^2}{9y_0^2} = 1$$

$$\therefore m_1 m_2 = \frac{-16x_0^2}{9y_0^2} = -1$$

$\therefore L_1, L_2$ are $\perp$.

$\therefore E$ & H intersect at right angles.