1. **[8 marks]** Let **u**, **v**, and **w** be vectors in $\mathbb{R}^3$. State whether the following expressions represent vectors, scalars, or are meaningless.

(a) $5 \times \mathbf{v}$ meaningless; cannot scal x vect

(b) $\mathbf{u} \times (-\mathbf{v})$ vector

(c) $\mathbf{v} + 2$ meaningless; cannot vect + scal

(d) $\mathbf{u} + \mathbf{v} \cdot \mathbf{w}$ meaningless (vect. + scalar)

(e) $(\mathbf{u} \times \mathbf{w}) \times \mathbf{v}$ vector (vec × vec)

(f) $\mathbf{u}\mathbf{v}$ meaningless

(g) $(\mathbf{u} \cdot \mathbf{v}) \times \mathbf{w}$ meaningless; cannot scal x vect.

(h) $(\mathbf{u} \cdot \mathbf{v})(\mathbf{u} \times \mathbf{v})$ vector. (scal, vect)

2. **[5 marks]** Let $\mathbf{a} = \mathbf{i} - 3\mathbf{j} + 3\mathbf{k}$ and $\mathbf{b} = -\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$. Find $\mathbf{a} \cdot \mathbf{b}$ and $|2\mathbf{b} - 3\mathbf{a}|$.

$a \cdot b = 1(-1) - 3(2) + 3(-2)$
$= -1 - 6 - 6 = -13$

~~$2b - 3a = 2(i - 3j + 3k) - 3(-i + 2j - 2k)$~~
~~$= 2i - 6j + 6k + 3i - 6j + 6k$~~
~~$= 5i$~~

$2b - 3a = 2(-i + 2j - 2k) - 3(i - 3j + 3k)$
$= -2i + 4j - 4k - 3i + 9j - 9k$
$= -5i + 13j - 13k$

$\therefore |2b - 3a| = \sqrt{5^2 + 13^2 + (13)^2}$
$= \sqrt{25 + 169 + 169}$
$= \sqrt{363}$

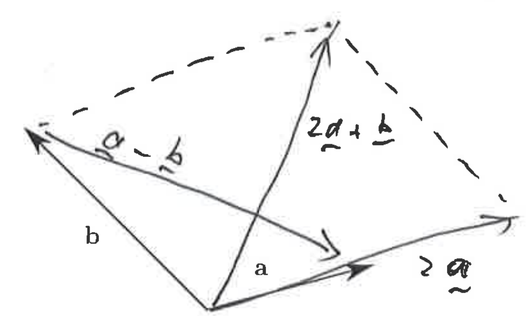
3. In the diagrams below, the vectors **a** and **b** have magnitudes 2 and 3 respectively. The angle between them is 120°.

(a) **[4 marks]** Draw the vectors $\mathbf{a} - \mathbf{b}$ and $2\mathbf{a} + \mathbf{b}$ on the first diagram.

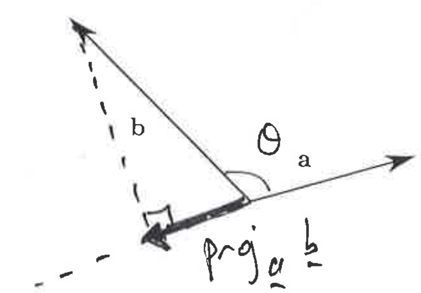
(b) **[4 marks]** What is the magnitude of $\mathbf{a} \times \mathbf{b}$? Does $\mathbf{a} \times \mathbf{b}$ point into or out of the page?

(c) **[4 marks]** Find $\text{comp}_{\mathbf{a}}\, \mathbf{b}$, and draw $\text{proj}_{\mathbf{a}}\, \mathbf{b}$ on the second diagram.

a)



c)



b) $|a \times b| = |a||b| \sin\theta$
$= 2 \cdot 3 \cdot \sin 120°$
$= 6 \cdot \frac{\sqrt{3}}{2} = 3\sqrt{3}$

$a \times b$ point out of the page.

$\text{comp}_a b = \frac{a \cdot b}{|a|}$
$= \frac{|a||b| \cos\theta}{|a|}$
$= |b| \cos\theta$
$= 3 \cos 120°$
$= 3(-\frac{1}{2}) = -\frac{3}{2}$

4. Consider the points $A(1,-1,0)$, $B(3,2,1)$, and $C(4,-2,2)$ in $\mathbb{R}^3$.

(a) **[5 marks]** Calculate $\overrightarrow{AB} \times \overrightarrow{AC}$.

(b) **[2 marks]** Explain why A, B, and C are non-collinear.

(c) **[3 marks]** Find an equation of the plane passing through A, B, and C.

(d) **[3 marks]** What is the area of the triangle $\triangle ABC$?

(a) $\overrightarrow{AB} = \langle 3,2,1\rangle - \langle 1,-1,0\rangle = \langle 2,3,1\rangle$

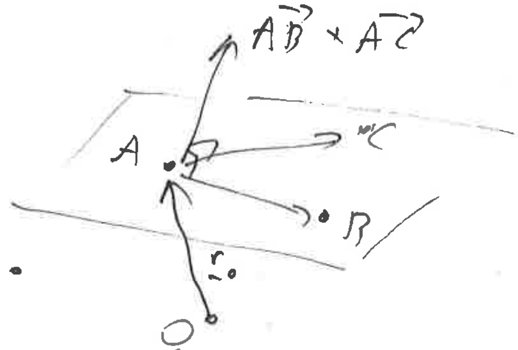
$\overrightarrow{AC} = \langle 4,-2,2\rangle - \langle 1,-1,0\rangle = \langle 3,-1,2\rangle$

$$\begin{array}{ccc|cc} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} & \underset{\sim}{i} & \underset{\sim}{j} \\ 2 & 3 & 1 & 2 & 3 \\ 3 & -1 & 2 & 3 & -1 \end{array}$$

$\overrightarrow{AB} \times \overrightarrow{AC} = (6\underset{\sim}{i} + 3\underset{\sim}{j} - 2\underset{\sim}{k}) - (-\underset{\sim}{i} + 4\underset{\sim}{j} + 9\underset{\sim}{k})$
$= 7\underset{\sim}{i} - \underset{\sim}{j} - 11\underset{\sim}{k}$

b) Since $\overrightarrow{AB} \times \overrightarrow{AC} \neq \underset{\sim}{0}$, $\overrightarrow{AB}$ & $\overrightarrow{AC}$ are <u>not</u> parallel,
$\therefore$ A, B, C are non-collinear. [or just check $\overrightarrow{AB}$, $\overrightarrow{AC}$ are not scalar multiples of each other.]

c) Take $\underset{\sim}{n} = \overrightarrow{AB} \times \overrightarrow{AC} = 7\underset{\sim}{i} - \underset{\sim}{j} - 11\underset{\sim}{k}$
as normal vector to the plane.
Take A as a point on the plane →
it has position vector $\overrightarrow{OA} = \langle 1,-1,0\rangle = \underset{\sim}{r}_0$



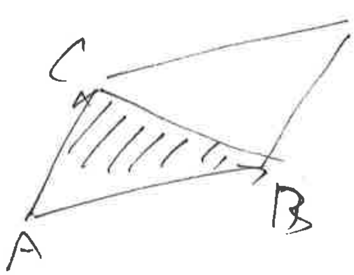
Eq'n of plane: $\underset{\sim}{n} \cdot (\underset{\sim}{r} - \underset{\sim}{r}_0) = 0$

$\langle 7,-1,-11\rangle \cdot \langle x-1, y+1, z\rangle = 0$

$7(x-1) - (y+1) - 11z = 0$

$7x - y - 11z = 8$

d) Area of triangle $= \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}|$
$= \frac{1}{2}|7\underset{\sim}{i} - \underset{\sim}{j} - 11\underset{\sim}{k}|$
$= \frac{1}{2}\sqrt{7^2 + (-1)^2 + (-11)^2}$
$= \frac{1}{2}\sqrt{49 + 1 + 121} = \frac{1}{2}\sqrt{171}$



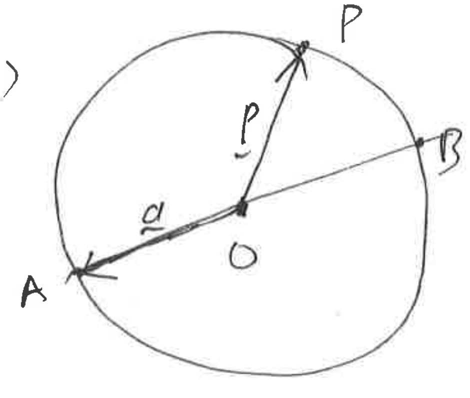
5. Let C be a circle centred at O, and let AB be a diameter of C (i.e. a straight line segment through O with endpoints on C). Suppose P is any point on C which is not A or B.

(a) **[3 marks]** Draw a neat diagram of C and the diameter AB. Label all given points.

(b) **[4 marks]** Let $\mathbf{a} = \overrightarrow{OA}$ and $\mathbf{p} = \overrightarrow{OP}$. Write $\overrightarrow{AP}$ and $\overrightarrow{PB}$ in terms of $\mathbf{a}$ and $\mathbf{p}$.

(c) **[5 marks]** Use vectors to prove that $\angle APB = 90°$. Give full reasoning.

a)



b) $\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA}$

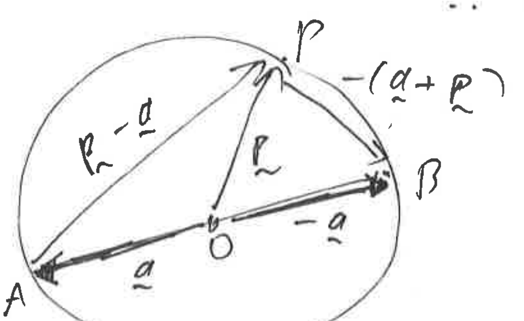
$= \underset{\sim}{p} - \underset{\sim}{a}$

Since AB is a diameter, the centre O must be the midpoint of AB.

$\therefore \overrightarrow{OB} = -\overrightarrow{BO} = -\overrightarrow{OA} = -\underset{\sim}{a}$

$\therefore \overrightarrow{PB} = \overrightarrow{OB} - \overrightarrow{OP} = -\underset{\sim}{a} - \underset{\sim}{p}$

$= -(\underset{\sim}{a} + \underset{\sim}{p})$



c) $\overrightarrow{AP} \cdot \overrightarrow{PB} = (\underset{\sim}{p} - \underset{\sim}{a}) \cdot (-\underset{\sim}{p} - \underset{\sim}{a})$

$= -\underset{\sim}{p} \cdot \underset{\sim}{p} - \underset{\sim}{p} \cdot \underset{\sim}{a} + \underset{\sim}{a} \cdot \underset{\sim}{p} + \underset{\sim}{a} \cdot \underset{\sim}{a}$

$= -|\underset{\sim}{p}|^2 + |\underset{\sim}{a}|^2$

Now, OA and OP are both radii of the circle C

$\therefore |\overrightarrow{OA}| = |\overrightarrow{OP}|$, i.e. $|\underset{\sim}{a}| = |\underset{\sim}{p}|$

$\therefore \overrightarrow{AP} \cdot \overrightarrow{PB} = 0$

$\therefore \overrightarrow{AP}$ is $\perp$ to $\overrightarrow{PB}$

and so $\angle APB = 90°$.

6. Consider the vector function $\mathbf{r}(t) = \langle e^{-2t}, -e^{-t} \rangle$.

(a) **[4 marks]** What is the natural domain of $\mathbf{r}(t)$? Evaluate $\mathbf{r}(-1)$, $\mathbf{r}(0)$, and $\mathbf{r}(1)$.

(b) **[2 marks]** Find $\lim_{t\to\infty} \mathbf{r}(t)$.

(c) **[3 marks]** Eliminate t from the component functions to obtain an equation involving only x and y.

(d) **[4 marks]** Draw a neat diagram of the path paramaterised by $\mathbf{r}(t)$. Mark the points on the curve corresponding to $t = -1, 0, 1$. Indicate the direction of increasing t.

a) Natural domain $t \in \mathbb{R}$. (all real numbers)

$\underline{r}(-1) = \langle e^2, -e \rangle$, $\underline{r}(0) = \langle 1, -1 \rangle$, $\underline{r}(1) = \langle \frac{1}{e^2}, -\frac{1}{e} \rangle$.

b) $\lim_{t\to\infty} \underline{r}(t) = \lim_{t\to\infty} \langle e^{-2t}, -e^{-t} \rangle$

$= \langle \lim_{t\to\infty} e^{-2t}, \lim_{t\to\infty} -e^{-t} \rangle$

$= \langle 0, 0 \rangle$

c) $x = e^{-2t}$, $y = -e^{-t}$

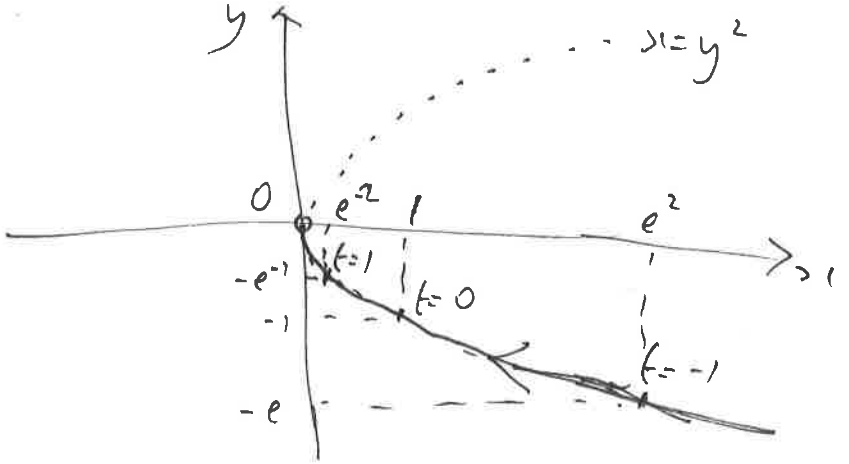
$y^2 = (-e^{-t})^2 = e^{-2t} = x$

$\therefore$ $x = y^2$ is eq'n involving only x & y.

d) Path must lie on parabola given by $x = y^2$.

Note: $x = e^{-2t} > 0$ & $y = -e^{-t} < 0$ for all t.

$\therefore$ only lies in 4th quadrant.



from b) path converges to (0, 0) as $t \to \infty$.

7. Let C be the curve in $\mathbb{R}^3$ given by the vector function $\mathbf{r}(t) = \langle \sin t, t, \cos t \rangle$, $t \in \mathbb{R}$.

(a) **[3 marks]** Calculate the derivative $\mathbf{r}'(t)$.

(b) **[5 marks]** Find an equation of the tangent line to C at the point corresponding to $t = \pi$.

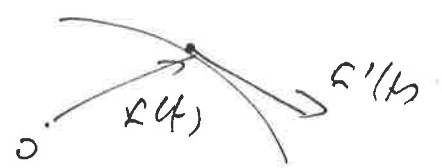
(c) **[4 marks]** What is the angle of intersection between this tangent line and the plane given by $2x - 3y + z = 4$?

a) $\underset{\sim}{r}'(t) = \langle \cos t, 1, -\sin t \rangle$

b) position: $\underset{\sim}{r}(\pi) = \langle \sin \pi, \pi, \cos \pi \rangle$
$= \langle 0, \pi, -1 \rangle$

tangent vector: $\underset{\sim}{r}'(\pi) = \langle \cos \pi, 1, -\sin \pi \rangle$
$= \langle -1, 1, 0 \rangle$

Eqn of tangent line:

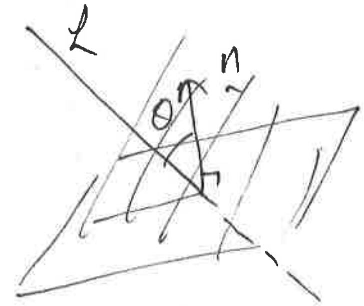


$\langle x, y, z \rangle = \langle 0, \pi, -1 \rangle + t\langle -1, 1, 0 \rangle$
$= \langle -t, \pi + t, -1 \rangle$, $t \in \mathbb{R}$ (parametric vector form)

$\therefore$ $x = -t$, $y = \pi + t$, $z = -1$ (parametric Cartesian form)

or $y = \pi - x$, $z = -1$ (symmetric form)

c) Normal vector to plane



$\underset{\sim}{n} = \langle 2, -3, 1 \rangle$

Angle between tangent vector $\underset{\sim}{v} = \underset{\sim}{r}'(\pi) = \langle -1, 1, 0 \rangle$ and $\underset{\sim}{n}$:

$$\cos \theta = \frac{\underset{\sim}{n} \cdot \underset{\sim}{v}}{|\underset{\sim}{n}||\underset{\sim}{v}|} = \frac{-2 - 3 + 0}{\sqrt{2^2 + (-3)^2 + 1^2}\sqrt{(-1)^2 + 1^2}} = \frac{-5}{2\sqrt{7}}$$

$\therefore \theta = \cos^{-1}\left(\frac{-5}{2\sqrt{7}}\right) \approx 160.89°$

$\therefore$ angle between line & plane is
$\approx 160.89° - 90° = 70.89°$