# Solutions to Sample Midterm 3

Lecturer: Dr. Robert Tang
Time allowed: 45 minutes

Surname:

Given names:

Student ID:

This exam consists of **4 questions** worth **10 marks** each. Each question is divided into several parts. You may use your answers from previous parts to do later parts.

Please write your solutions in the spaces provided. You may use the blank pages on the back of each page for working out. **Non-programmable** calculators may be used (i.e. no graphing calculators allowed).

No notes or books may be used. Do not take any part of this exam paper out of the room.

To get a good grade on each question, you must show your working out and/or provide correct reasoning. If your working and reasoning is correct but you make minor calculational errors, you will still earn most of the available marks. Conversely, if you only provide an answer with little or no reasoning, you will only get a few marks.

Good luck!

**1.** Evaluate the following integrals.

(a) [**3 marks**] $\displaystyle\int \frac{dx}{\sqrt{x^2-9}}$ [Hint: Use a substitution $x = 3\cosh t$]

(b) [**3 marks**] $\displaystyle\int x^2 \ln x \; dx$

(c) [**4 marks**] $\displaystyle\int \cos 3\theta \sin 5\theta \; d\theta$

**2.** (a) Evaluate the following limits.

(i) [**3 marks**] $\lim_{x\to 1} \frac{\ln x}{x-1}$.

(ii) [**4 marks**] $\lim_{x\to\infty} \sqrt{x+1} - \sqrt{x}$.

(b) [**3 marks**] Simplify $\sec(\tan^{-1} x)$. [You may use a diagram]

**3.** (a) (i) [**4 marks**] Find constants $A$, $B$ and $C$ so that

$$\frac{x^2+x+3}{(x+1)^2(x-2)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-2}.$$

(ii) [**3 marks**] Evaluate $\displaystyle\int \frac{x^2+x+3}{(x+1)^2(x-2)}\,dx$.

(b) [**3 marks**] Evaluate $\displaystyle\int \frac{dx}{x^2-4x+8}$.

**4.** (a) [**3 marks**] Prove that $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$.

***Solution:***

$$\begin{aligned} RHS &= \cosh x \cosh y + \sinh x \sinh y \\ &= \left(\frac{e^x + e^{-x}}{2}\right)\left(\frac{e^y + e^{-y}}{2}\right) + \left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^y - e^{-y}}{2}\right) \\ &= \frac{e^{x+y} + e^{x-y} + e^{-x+y} + e^{-x-y}}{4} + \frac{e^{x+y} - e^{x-y} - e^{-x+y} + e^{-x-y}}{4} \\ &= \frac{2e^{x+y} + 2e^{-x-y}}{4} = \frac{e^{x+y} + e^{-x-y}}{2} = \cosh(x+y) = LHS. \end{aligned}$$

(b) Let $I_n = \int x^n e^x \, dx$ for $n \geq 0$.

(i) [**3 marks**] Use integration by parts to show that $I_n = x^n e^x - nI_{n-1}$ for $n \geq 1$. [There original question erroneously asked you to prove $I_n = x^n e^x - nI_n$]

***Solution:*** Setting $u = x^n$ and $dv = e^x \, dx$, we get $du = nx^{n-1} \, dx$ and $v = e^x$. Using integration by parts, we obtain

$$I_n = \int x^n e^x \, dx = x^n e^x - \int nx^{n-1} e^x \, dx = x^n e^x - n \int x^{n-1} e^x = x^n e^x - nI_{n-1}.$$

(ii) [**4 marks**] Compute $\int x^3 e^x \, dx$.

***Solution:*** First note that $I_0 = \int e^x \, dx = e^x + C$. Applying part (i), we get

$$\begin{aligned} \int x^3 e^x \, dx &= I_3 \\ &= x^3 e^x - 3I_2 \\ &= x^3 e^x - 3(x^2 e^x - 2I_1) \\ &= x^3 e^x - 3(x^2 e^x - 2(xe^x - I_0)) \\ &= x^3 e^x - 3x^2 e^x + 6xe^x - 6I_0 \\ &= x^3 e^x - 3x^2 e^x + 6xe^x - 6e^x - 6C \\ &= (x^3 - 3x^2 + 6x - 6)e^x + C'. \end{aligned}$$