Solutions to Sample Midterm 2

Lecturer: Dr. Robert Tang
Time allowed: 45 minutes

Surname:

Given names:

Student ID:

This exam consists of **4 questions** worth **10 marks** each. Each question is divided into several parts. You may use your answers from previous parts to do later parts.

Please write your solutions in the spaces provided. You may use the blank pages on the back of each page for working out. **Non-programmable** calculators may be used (i.e. no graphing calculators allowed).

No notes or books may be used. Do not take any part of this exam paper out of the room.

To get a good grade on each question, you must show your working out and/or provide correct reasoning. If your working and reasoning is correct but you make minor calculational errors, you will still earn most of the available marks. Conversely, if you only provide an answer with little or no reasoning, you will only get a few marks.

Good luck!

1. (a) Evaluate the following integrals.

(i) [**2 marks**] $\displaystyle\int \frac{\cos x}{\sin x + 2}\, dx$

(ii) [**3 marks**] $\displaystyle\int_0^3 2^{3x}\, dx$

(b) Differentiate the following.

(i) [**2 marks**] $y = \ln(\cos x + x)$

(ii) [**3 marks**] $y = (\tan x)^x$

2. Consider the function $f(x) = \ln(x + 1)$.

(a) [**2 marks**] What is the domain and range of f? Explain why f is one-to-one.

(b) [**3 marks**] Find an expression for $f^{-1}(x)$. What is the domain and range of f^{-1}?

(c) [**2 marks**] Draw a neat sketch of $y = f(x)$ and $y = f^{-1}(x)$ on the same axes.

(d) [**3 marks**] Find an expression for $(f^{-1})'(x)$.

3. (a) (i) [**2 marks**] What is the average value of $f(x) = e^{-x}$ over the interval $0 \leq x \leq 10$?

Solution:

$$f_{ave} = \frac{1}{10}\int_0^{10} e^{-x}\,dx = \frac{1}{10}\left[-e^{-x}\right]_0^{10} = \frac{1-e^{-10}}{10}.$$

(ii) [**2 marks**] Find a number $0 \leq c \leq 10$ so that $f(c) = f_{ave}$.

Solution: Solving for c:

$$\begin{aligned} f(c) &= f_{ave} \\ e^{-c} &= \frac{1-e^{-10}}{10} \\ -c &= \ln\left(\frac{1-e^{-10}}{10}\right) \\ c &= -\ln\left(\frac{1-e^{-10}}{10}\right) = \ln\left(\frac{10}{1-e^{-10}}\right). \end{aligned}$$

(b) The region between the curve $x = y^2$ and the y–axis from $y = 0$ to $y = 2$ is rotated about the line $x = -1$ to form a solid S.

(i) [**2 marks**] Draw a diagram of a typical cross-sectional slice obtained by slicing perpendicular to the y–axis. Label the radius and thickness.

(ii) [**1 marks**] Express the volume of S as an integral.

(iii) [**3 marks**] Calculate the volume of S.

4. (a) Consider the region R between the curve $y = \sin(x^2)$ and the x–axis from $x = 0$ to $x = \sqrt{\pi}$. The region R is rotated around the y–axis to form a solid S.

(i) [**3 marks**] Write down an expression for the volume of S obtained using the cylindrical shells method. Draw a typical shell labelling its height, thickness and radius.

Solution:

$$\text{Volume}(S) = \int_0^{\sqrt{\pi}} 2\pi x \sin(x^2)\ dx$$

[Picture omitted, but it should be a cylindrical shell of height $\sin(x^2)$, thickness Δx and radius x.]

(ii) [**3 marks**] Compute the volume of S.

Solution: Using the substitution $u = x^2$, we get $du = 2xdx$. When $x = 0$, $u = 0$ and when $x = \sqrt{\pi}$, $u = \pi$. So

$$\begin{aligned}
\text{Volume}(S) &= \int_0^{\sqrt{\pi}} 2\pi x \sin(x^2)\ dx \\
&= \int_0^{\pi} \pi \sin u\ du \\
&= \pi\left[-\cos u\right]_0^{\pi} \\
&= -\pi(\cos\pi - \cos 0) \\
&= \pi(-1-1) = 2\pi.
\end{aligned}$$

(b) [**4 marks**] A spring has a natural length of 60 cm. The work done in stretching the spring from 70 cm to 90 cm is 20 J. How much force is required to hold the spring at 40 cm?