Solutions to Sample Midterm 1

Lecturer: Dr. Robert Tang
Time allowed: 45 minutes

Surname:

Given names:

Student ID:

This exam consists of **4 questions** worth **10 marks** each. Each question is divided into several parts. You may use your answers from previous parts to do later parts.

Please write your solutions in the spaces provided. You may use the blank pages on the back of each page for working out. **Non-programmable** calculators may be used (i.e. no graphing calculators allowed).

No notes or books may be used. Do not take any part of this exam paper out of the room.

To get a good grade on each question, you must show your working out and/or provide correct reasoning. If your working and reasoning is correct but you make minor calculational errors, you will still earn most of the available marks. Conversely, if you only provide an answer with little or no reasoning, you will only get a few marks.

Good luck!

1. Evaluate the following integrals.

(a) [**3 marks**] $\int_0^4 6x^2 - 8x + 3\ dx$

Solution:

$$\begin{aligned}\int_0^4 6x^2 - 8x + 3\ dx &= \left[2x^3 - 4x^2 + 3x\right]_0^4 \\ &= (2 \cdot 4^2 - 4 \cdot 4^2 + 3 \cdot 4) - 0 \\ &= 86\end{aligned}$$

(b) [**3 marks**] $\int \frac{\cos(\sqrt{x}+1)}{\sqrt{x}}\ dx$

Solution: Using a substitution $u = \sqrt{x} + 1 = x^{\frac{1}{2}} + 1$, we have

$$du = \frac{1}{2}x^{-\frac{1}{2}}dx = \frac{dx}{2\sqrt{x}}.$$

Therefore,

$$\begin{aligned}\int \frac{\cos(\sqrt{x}+1)}{\sqrt{x}}\ dx &= 2\int \cos(\sqrt{x}+1) \cdot \frac{dx}{2\sqrt{x}}\ dx \\ &= 2\int \cos u\ du \\ &= 2\sin u + C = 2\sin(\sqrt{x}+1) + C.\end{aligned}$$

(c) [**4 marks**] $\int_{-\frac{\pi}{8}-\frac{1}{2}}^{\frac{\pi}{8}-\frac{1}{2}} \tan^2(2x+1)\ dx$

Solution: Using the substitution $u = 2x + 1$ we get $du = 2dx$. When $x = -\frac{\pi}{8} - \frac{1}{2}$, $u = 2(-\frac{\pi}{8} - \frac{1}{2}) + 1 = \frac{\pi}{4}$; and when $x = \frac{\pi}{8} - \frac{1}{2}$, $u = 2(\frac{\pi}{8} - \frac{1}{2}) + 1 = \frac{\pi}{4}$. Thus,

$$\begin{aligned}\int_{-\frac{\pi}{8}-\frac{1}{2}}^{\frac{\pi}{8}-\frac{1}{2}} \tan^2(2x+1)\ dx &= \frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \tan^2(u)\ du \\ &= \frac{1}{2} \cdot 2\int_0^{\frac{\pi}{4}} \tan^2(u)\ du,\ \text{since}\ \tan^2(u)\ \text{is even.} \\ &= \int_0^{\frac{\pi}{4}} \sec^2(u) - 1\ du \\ &= [\tan u - u]_0^{\frac{\pi}{4}} = \left(\tan\left(\frac{\pi}{4}\right) - \frac{\pi}{4}\right) - 0 = 1 - \frac{\pi}{4}.\end{aligned}$$

2. (a) Let $V(t)$ be the amount of water (in L) in a tank at time t (in seconds).

(i) [**1 mark**] What does the integral $\displaystyle\int_{t_1}^{t_2} V'(t)dt$ represent?

Solution: It represents the net change in the amount of water (in L) from $t = t_1$ to $t = t_2$.

The rate $V'(t)$ of water flowing into the tank is measured over $0 \le t \le 10$, with values given in the table below:

t	0	2	4	6	8	10
$V'(t)$	4	5	7	3	2	9

(ii) [**4 marks**] Estimate $\int_0^{10} V'(t)\,dt$ using right endpoints. Can you conclude whether your answer is an overestimate or an underestimate?

Solution: We have $\Delta t = 2$, so using right endpoints we get

$$\int_0^{10} V'(t)\,dt \approx (5 + 7 + 3 + 2 + 9) \times 2 = 56.$$

We cannot conclude whether this is an overestimate or an underestimate since $V'(t)$ is neither always increasing or always decreasing.

(b) Find $h'(x)$, where $h(x)$ is given by:

(i) [**2 marks**] $h(x) = \displaystyle\int_0^x \frac{\sqrt{u}}{\sin^2 u + 5}\,du$

Solution: By the Fundamental Theorem of Calculus (part 1), we get

$$h'(x) = \frac{\sqrt{x}}{\sin^2 x + 5}.$$

(ii) [**3 marks**] $h(x) = \displaystyle\int_x^{2x} t^2 + \cos t\,dt$

Solution: Let $g(x) = \displaystyle\int_0^x t^2 + \cos t\,dt$. Then

$$\begin{aligned} h(x) &= \int_x^{2x} t^2 + \cos t\,dt = \int_x^0 t^2 + \cos t\,dt + \int_0^{2x} t^2 + \cos t\,dt \\ &= -\int_0^x t^2 + \cos t\,dt + \int_0^{2x} t^2 \\ &= -g(x) + g(2x). \end{aligned}$$

Therefore,

$$\begin{aligned} h'(x) &= -g'(x) + g'(2x) \cdot 2, \text{ by Chain rule} \\ &= -(x^2 + \cos x) + 2((2x)^2 + \cos(2x)), \text{ by FTC 1} \\ &= 7x^2 + 2\cos(2x) - \cos x. \end{aligned}$$

3. A car is driving along a straight road. Its velocity north at time t (in seconds) is given by $v(t) = t^2 - 3t + 2$ (in m/s).

(a) [**3 marks**] What is the displacement of the car from $t = 0$ to $t = T$?

Solution: Its displacement can be calculated from the integral

$$\begin{aligned}\int_0^T v(t)\ dt &= \int_0^T t^2 - 3t + 2\ dt \\ &= \left[\frac{t^3}{3} - \frac{3t^2}{2} + 2t\right]_0^T \\ &= \frac{T^3}{3} - \frac{3T^2}{2} + 2T.\end{aligned}$$

(b) [**2 marks**] On what time interval is the car travelling south?

Solution: The car is travelling south when $v(t) < 0$. i.e. when

$$\begin{aligned} t^2 - 3t + 2 &< 0 \\ (t-1)(t-2) &< 0. \end{aligned}$$

This holds exactly on the interval $1 < t < 2$.

(c) [**5 marks**] What is the total distance travelled by the car from $t = 0$ to $t = 3$?

Solution: Using part (b), we know $v(t) \geq 0$ when $0 \leq t \leq 1$ or $t \geq 2$; and $v(t) \leq 0$ when $1 \leq t \leq 2$. Thus

$$|v(t)| = \begin{cases} v(t) = t^2 - 3t + 2, & \text{when } 0 \leq t \leq 1 \text{ or } t \geq 2, \\ -v(t) = -(t^2 - 3t + 2), & \text{when } 1 \leq t \leq 2. \end{cases}$$

The total distance travelled is given by the integral

$$\begin{aligned}\int_0^3 |v(t)|\ dt &= \int_0^1 v(t)\ dt + \int_1^2 -v(t)\ dt + \int_2^3 v(t)\ dt \\ &= \int_0^1 t^2 - 3t + 2\ dt - \int_1^2 t^2 - 3t + 2\ dt + \int_2^3 t^2 - 3t + 2\ dt \\ &= \left[\frac{t^3}{3} - \frac{3t^2}{2} + 2t\right]_0^1 - \left[\frac{t^3}{3} - \frac{3t^2}{2} + 2t\right]_1^2 + \left[\frac{t^3}{3} - \frac{3t^2}{2} + 2t\right]_2^3 \\ &= \left(\frac{1}{3} - \frac{3}{2} + 2\right) - \left\{\left(\frac{2^3}{3} - \frac{3 \cdot 2^2}{2} + 2 \cdot 2\right) - \left(\frac{1}{3} - \frac{3}{2} + 2\right)\right\} \\ &\quad + \left\{\left(\frac{3^3}{3} - \frac{3 \cdot 3^2}{2} + 2 \cdot 3\right) - \left(\frac{2^3}{3} - \frac{3 \cdot 2^2}{2} + 2 \cdot 2\right)\right\} \\ &= \frac{5}{6} - \frac{2}{3} + \frac{5}{6} + \frac{3}{2} - \frac{2}{3} = \frac{5 - 4 + 5 + 9 - 4}{6} = \frac{11}{6}\end{aligned}$$

Thus, the total distance travelled by the car is $\frac{11}{6}$ m.

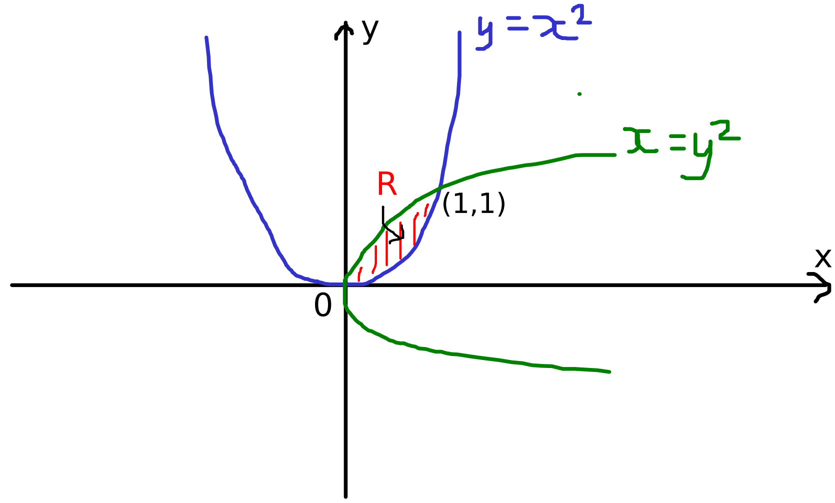
4. Let R be the region in the plane bounded by the curves $y = x^2$ and $x = y^2$.

(a) [**2 marks**] Find all intersection points of the two curves.

Solution: We substitute $y = x^2$ into $x = y^2$ to get $x = y^2 = (x^2)^2 = x^4$. This yields $0 = x^4 - x = x(x^3 - 1)$ which means $x = 0$ or $x^3 - 1 = 0$, the latter equation giving $x^3 = 1$ and hence $x = 1$. Substituting the values $x = 0, 1$ into $y = x^2$ gives $(0, 0)$ and $(1, 1)$ as the intersection points.

(b) [**3 marks**] Draw a neat sketch of the curves, labelling any intercepts. Shade the region R on your diagram.

Solution:



(c) [**2 marks**] Express the area of R as an integral.

Solution: We need to decide which curve is on top. According to our graph, the upper branch of the curve $x = y^2$ is the top curve, and $y = x^2$ is the bottom curve. Since we need to look at the difference in the y–coordinates between the two curves, we should rewrite the first equation as $y = \sqrt{x}$. Thus, the area of R is

$$\int_0^1 \sqrt{x} - x^2 \, dx.$$

(d) [**3 marks**] Calculate the area of R.

Solution:

$$\begin{aligned}
\text{Area of } R &= \int_0^1 \sqrt{x} - x^2 \, dx \\
&= \int_0^1 x^{\frac{1}{2}} - x^2 \, dx \\
&= \left[\frac{x^{\frac{3}{2}}}{3/2} - \frac{x^3}{3} \right]_0^1 = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}.
\end{aligned}$$