Veech surfaces and simple closed curves

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Abstract

We study the $\mathrm{SL}(2,\mathbb{R})$–infimal lengths of simple closed curves on half-translation surfaces. Our main result is a characterization of Veech surfaces in terms of these lengths.

We also revisit the "no small virtual triangles" theorem of Smillie and Weiss and establish the following dichotomy: the virtual triangle area spectrum of a half-translation surface either has a gap above zero or is dense in a neighborhood of zero.

These results make use of the *auxiliary polygon* associated to a curve on a half-translation surface, as introduced by Tang and Webb.

1 Introduction

Let S be a closed surface and let $\mathrm{QD}(S)$ be the space of quadratic differentials on S. Each element $q \in \mathrm{QD}(S)$ naturally endows S with a locally Euclidean metric with isolated conical singularities and linear holonomy restricted to $\{\pm\mathrm{id}\}$. We also refer to elements of $\mathrm{QD}(S)$ as half-translation surfaces. There is a natural action of $\mathrm{SL}(2,\mathbb{R})$ on $\mathrm{QD}(S)$ preserving the signature of the singularities. A half-translation surface q is called a *Veech surface* if its group of (derivatives of) affine self-diffeomorphisms is a lattice in $\mathrm{SL}(2,\mathbb{R})$. Veech surfaces possess remarkable dynamical properties akin to flat tori, and naturally arise in the contexts of rational billiards and Teichmüller curves in moduli space [Vee89].

In this paper, we study $\mathrm{QD}(S)$ from the point of view of simple closed curves on S. On a half-translation surface q, a simple closed curve α either has a unique geodesic representative or there is a maximal flat cylinder on q foliated by closed geodesics in the homotopy class of α. In the former case, the geodesic representative α^q of α is a concatenation of saddle connections. We say that α is a *crooked curve* on q if α^q has at least two saddle connections whose associated holonomy vectors are not parallel. We define the $\mathrm{SL}(2,\mathbb{R})$–*infimal length* of α on q to be

$$l_\alpha^{\mathrm{SL}}(q) = \inf_{q' \in \mathrm{SL}(2,\mathbb{R})\cdot q} l_\alpha(q'),$$

where $l_\alpha(q)$ denotes the geodesic length of α on q. A curve α is crooked on q if and only if $l_\alpha^{\mathrm{SL}}(q)$ is positive (see Proposition 2.1). Our first main result is a characterization of Veech surfaces in terms of their $\mathrm{SL}(2,\mathbb{R})$–infimal length spectra.

Theorem 1.1. *Let q be a half-translation surface. Then q is a Veech surface if and only if it has no short crooked curves: there is an $\epsilon > 0$ such that $l_\alpha^{\mathrm{SL}}(q) \geq \epsilon$ for every crooked curve α.*

This result is reminiscent of the no-small-(virtual)-triangles theorem due to Smillie and Weiss [SW10]. They characterize Veech surfaces as the half-translation surfaces which possess a positive lower bound on the areas of Euclidean triangles on q with edges formed by saddle connections. One advantage of working with simple closed curves is that they are topological objects; they do not depend on the half-translation structure, and therefore we can study them at once over the entire quadratic differential space. In contrast, a saddle connection on q persists only in a (typically proper) open subset of the relevant stratum of $\mathrm{QD}(S)$.

One of the main tools we use in this paper is the *auxiliary polygon* $P_\alpha(q)$ associated to a simple closed curve α on q, as introduced by the second author and Webb in [TW15]. The area of $P_\alpha(q)$ gives an estimate for $l_\alpha^{\mathrm{SL}}(q)^2$ up to bounded multiplicative error (see Proposition 2.1(iii)). It follows that Theorem 1.1 is equivalent to the statement that q is a Veech surface if and only if the *polygonal area spectrum*

$$\mathrm{Poly}(q) = \left\{\mathrm{Area}(P_\alpha(q)) : \ \alpha \text{ is a simple closed curve}\right\}$$

has a gap above zero. In fact, our arguments will prove a slightly stronger statement: either there is a gap (exactly when q is a Veech surface), or this spectrum is dense in a neighborhood $[0,a)$ for some $a > 0$.

The forward implication of Theorem 1.1 will follow from a relatively straightforward application of the no-small-virtual-triangles theorem.

The reverse implication of Theorem 1.1 will make use of the auxiliary polygon mentioned above, as well as two other ingredients: the orbit closure theorem of Eskin, Mirzakhani, and Mohammadi [EMM15] and a rigidity statement for $\mathrm{SL}(2,\mathbb{R})$–orbits of quadratic differentials due to Duchin, Leininger, and Rafi [DLR10]. We remark that the orbit closure theorem is used only to deduce local path connectedness of $\mathrm{SL}(2,\mathbb{R})$–orbit closures in strata of half-translation surfaces.

To prove the reverse implication we use an elementary continuity argument; see Proposition 4.2. An essential step is establishing that the auxiliary polygon $P_\alpha(q)$ is continuous in q, with respect to the Hausdorff topology. This is achieved in several steps. By continuity of the intersection pairing between measured foliations on S, we obtain a continuous function $\mathrm{QD}^1(S) \to \mathscr{C}(\mathbb{RP}^1,\mathbb{R})$ of the form $q \mapsto i\big(v_q^{\frac{\pi}{2}+\theta}, \alpha\big)$ for any fixed curve α on S. The value $i\big(v_q^{\frac{\pi}{2}+\theta}, \alpha\big)$ coincides with the *width* of the auxiliary polygon in direction θ, and so the *width function* $w_{P_\alpha(q)} \in \mathscr{C}(\mathbb{RP}^1,\mathbb{R})$ of the polygon is continuous in q. Finally, standard results from convex geometry on centrally symmetric sets yield continuity of $P_\alpha(q)$ itself.

Next consider the *virtual triangle area spectrum*, defined as follows:

$$\mathrm{VT}(q) = \left\{|u \wedge v| \ : \ u, v \in \mathrm{hol}(q)\right\}$$

where $\mathrm{hol}(q)$ is the set of holonomy vectors of saddle connections in q. The no-small-virtual-triangles theorem of [SW10] states that $\mathrm{VT}(q)$ has a gap above zero if and only if q is a Veech surface. In our second theorem we show further that $\mathrm{VT}(q)$ resembles the polygonal area spectrum as discussed above:

Theorem 1.2. *Let q be a half-translation surface. Then $VT(q)$ either has a gap above zero (exactly when q is a Veech surface) or is dense in a neighborhood $[0,a)$ for some $a > 0$.*

In fact, our argument provides a new proof of the "gap implies Veech" direction of the no-small-virtual-triangles theorem. Moreover, the virtual triangles yielding the dense subset of $[0,a)$ in the non-Veech case are *based* virtual triangles; see Section 5 and Proposition 5.1. The non-Veech case of Theorem 1.2 is proved very similarly to Theorem 1.1.

Finally we have some additional results on polygonal area when q is a Veech surface. The first of these will be derived from the analogous result for $\mathrm{VT}(q)$ in [SW10].

Theorem 1.3. *If q is a Veech surface then* $\mathrm{Poly}(q)$ *is a discrete subset of* $\mathbb{R}$.

Next, for $a > 0$ define the set

$$\mathrm{PA}(a) = \left\{ q \in \mathrm{QD}^1(S) : \ \mathrm{Area}(\mathrm{P}_\alpha(q)) \geq a \text{ for every crooked curve } \alpha \text{ on } q \right\}.$$

We say that two half-translation surfaces are *affinely equivalent* if they are related by the actions of the mapping class group and $\mathrm{SL}(2,\mathbb{R})$.

Theorem 1.4. *For any $a > 0$, the set $PA(a)$ contains only finitely many affine equivalence classes of half-translation surfaces.*

This result is an application of Theorem 2.2 of [EMM15], which classifies the closed $\mathrm{SL}(2,\mathbb{R})$–invariant subsets of strata of half-translation surfaces: namely, any such subset is a finite union of orbit closures.

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2 Background

2.1 Quadratic differentials and half-translation surfaces

We begin by recalling relevant background regarding quadratic differentials and half-translation surfaces. For further details, consult [Str84].

Let S be a closed surface of genus $g \geq 2$. A *half-translation structure* on S consists of a finite set ς of singular points on S, together with an atlas of charts to $\mathbb{C}$ defined away from ς, where the transition maps are of the form $z \mapsto \pm z + c$ for some $c \in \mathbb{C}$. One can pull back the standard Euclidean metric on $\mathbb{C}$ to obtain a local Euclidean metric defined on S away from ς. Taking the metric completion of this metric on $S - \varsigma$ endows S with a singular Euclidean structure, where each point in ς has cone angle of the form $k\pi$ for some $k \geq 3$. The atlas determines a preferred vertical direction on S, and we consider this to be part of the data in the half-translation structure.

One can construct a half-translation surface by taking a finite collection of disjoint Euclidean polygons in $\mathbb{C}$, with pairs of edges identified by gluing maps of the form $z \mapsto \pm z + c$.

By a *quadratic differential* on S, we mean a complex structure on S equipped with an integrable holomorphic quadratic differential. A quadratic differential q induces a half-translation structure on S as follows: If $z_0 \in S$ is not a zero of q, then the map $z \mapsto \int_{z_0}^{z} \sqrt{q(w)}\,dw$ defines a local coordinate chart to $\mathbb{C}$. A zero of order k then gives rise to a cone point of angle $(k+2)\pi$. We shall use q to denote a quadratic differential on S, as well as S equipped with the corresponding half-translation structure. The integrability assumption on q ensures that the area of the half-translation surface, equal to $\int_S |q|\,dw^2$, is finite.

Let $\mathrm{QD}(S)$ be the space of quadratic differentials on S. This can be identified with the cotangent bundle to Teichmüller space $\mathcal{T}(S)$ via the natural projection $\mathrm{QD}(S) \to \mathcal{T}(S)$ obtained by taking the underlying complex structures. The space $\mathrm{QD}^1(S)$ of *unit area* quadratic differentials on S can be identified with the unit cotangent bundle to $\mathcal{T}(S)$.

There is a natural $\mathrm{SL}(2,\mathbb{R})$–action on $\mathrm{QD}(S)$ defined by post-composing the coordinate charts to $\mathbb{C}$ by an $\mathbb{R}$–linear transformation. One can view this action by applying an element $A \in \mathrm{SL}(2,\mathbb{R})$ to a defining set of polygons for a half-translation surface q to obtain a new half-translation structure $A \cdot q$, noting that parallel edges of the same length remain so under $\mathrm{SL}(2,\mathbb{R})$–deformations. Note that $\mathrm{SL}(2,\mathbb{R})$–deformations preserve area, and so $\mathrm{SL}(2,\mathbb{R})$ also acts naturally on $\mathrm{QD}^1(S)$.

We will write $e^{i\theta} \cdot q$ as shorthand for $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \cdot q$.

2.2 Geodesic representatives and measured foliations

Let $\mathcal{S}$ be the set of (homotopy classes of) simple closed curves on S. For any $\alpha \in \mathcal{S}$, either α has a unique geodesic representative on q, or there is a unique maximal flat cylinder on q foliated by the closed geodesics in the homotopy class of α. In the former case, the geodesic representative of α is a concatenation of *saddle connections* – embedded geodesic arcs or loops with endpoints at singularities with no singularities on their interior. The angle between consecutive saddle connections is at least π on both sides. We shall use α^q to denote any geodesic representative of α on q.

If α^q is a core curve of a flat cylinder, then we call α a *cylinder curve* on q. Let $\mathsf{cyl}(q)$ denote the set of cylinder curves on q, and $\widehat{\mathsf{cyl}}(q)$ the set of curves whose geodesic representatives have constant direction on q. Any curve in $\mathcal{S} - \widehat{\mathsf{cyl}}(q)$ is called a *crooked curve* on q.

We can consider several notions of length of a curve α on q. Let $l_\alpha(q)$ denote the Euclidean length of α^q. Integrating α^q with respect to $|dx|$ and $|dy|$ (in local coordinates) gives the *horizontal* and *vertical* lengths $l^H_\alpha(q)$ and $l^V_\alpha(q)$ respectively. Finally, define the $\mathrm{SL}(2,\mathbb{R})$–*infimal length* of α with respect to q to be

$$l^{\mathrm{SL}}_\alpha(q) = \inf_{q' \in \mathrm{SL}(2,\mathbb{R}) \cdot q} l_\alpha(q').$$

This should be viewed as a measure of length of α with respect to the $\mathrm{SL}(2,\mathbb{R})$–orbit of q, rather than with respect to q itself.

One can pull back the foliation in the direction $\theta \in \mathbb{RP}^1$ on $\mathbb{C}$ to obtain a measured foliation ν^θ_q on q, where the transverse measure is the Euclidean distance between leaves. In particular, taking the horizontal and vertical directions respectively give rise to the *horizontal* and *vertical* foliations $\nu^H_q = \nu^0_q$ and $\nu^V_q = \nu^{\pi/2}_q$. The map $\mathrm{QD}(S) \times \mathbb{RP}^1 \to \mathcal{MF}(S)$ defined by $(q,\theta) \mapsto \nu^\theta_q$ is continuous, where $\mathcal{MF}(S)$ is the space of measured foliations on S. Let $\mathcal{MF}(q) = \left\{ t \cdot \nu^\theta_q \,:\, \theta \in \mathbb{RP}^1, t \in \mathbb{R}_+ \right\}$.

Let $\mathscr{PMF}(S)$ and $\mathscr{PMF}(q)$ be the projectivizations of $\mathscr{MF}(S)$ and $\mathscr{MF}(q)$, respectively. Note that these sets are invariant under $\mathrm{SL}(2,\mathbb{R})$–deformations.

The *geometric intersection number* $i\colon \mathscr{S}\times\mathscr{S}\to\mathbb{R}$ extends continuously to $\mathscr{MF}(S)\times\mathscr{MF}(S)$. For any curve α, we have $i(v_q^H,\alpha)=l_\alpha^V(q)$ and $i(v_q^V,\alpha)=l_\alpha^H(q)$. See [FLP79] for additional background on measured foliations.

2.3 Auxiliary polygons

We now recall the construction from [TW15] of the *auxiliary polygon* $P_\alpha(q)$ associated to a curve α and a quadratic differential $q\in\mathrm{QD}(S)$.

To a saddle connection e on q, we assign a *holonomy vector* $v_e\in\mathbb{R}^2$ which is parallel to it and is of the same length. This is unique up to scaling by ±1. Consider a geodesic representative α^q on q. If α is a cylinder curve, we may choose α^q to be a boundary component of the maximal flat cylinder with core curve α. Let m_e be the number of times α^q runs over a saddle connection e. Define

$$P_\alpha(q)=\left\{\sum_e t_e v_e\ :\ -\frac{m_e}{2}\le t_e\le\frac{m_e}{2}\right\}\subset\mathbb{R}^2,$$

where the sum is taken over all saddle connections used by α^q. Note that this definition does not depend on the choices of direction for each v_e, nor on the choice of boundary component in the case of cylinder curves. However, to simplify the exposition, we will assume that parallel saddle connections are associated holonomy vectors with the same orientation. The set $P_\alpha(q)$ is a convex Euclidean polygon unless α^q has constant direction, in which case $P_\alpha(q)$ degenerates to a line segment parallel to α^q of length $l_\alpha(q)$. Moreover, this construction commutes with $\mathrm{SL}(2,\mathbb{R})$–deformations: for all $A\in\mathrm{SL}(2,\mathbb{R})$ we have $P_\alpha(A\cdot q)=A\cdot P_\alpha(q)$.

Proposition 2.1 ([TW15])**.** *Given a curve $\alpha\in\mathscr{S}$ and $q\in QD^1(S)$, the auxiliary polygon $P_\alpha(q)$ defined above satisfies the following:*

(i) *The perimeter of $P_\alpha(q)$ is $2l_\alpha(q)$,*

(ii) $\mathrm{height}(P_\alpha(q))=l_\alpha^V(q)$ *and* $\mathrm{width}(P_\alpha(q))=l_\alpha^H(q)$,

(iii) $\pi\,\mathrm{Area}(P_\alpha(q))\le l_\alpha^{\mathrm{SL}}(q)^2\le 8\,\mathrm{Area}(P_\alpha(q))$,

(iv) $\mathrm{Area}(P_\alpha(q))=0$ *if and only if* $\alpha\in\widehat{\mathsf{cyl}}(q)$. □

Here, the *perimeter* of a polygon P is the length of its boundary ∂P. In the situation where P degenerates to a line segment, we view ∂P as a closed path traversing the line segment once in each direction.

Let $\mathscr{E}$ be the set of parallelism classes of saddle connections appearing on α^q. For each $E\in\mathscr{E}$, let $v_E=\sum_{e\in E}v_e$. If we orient $\partial P_\alpha(q)$ in the anti-clockwise direction, say, then the sides of $P_\alpha(q)$, viewed as directed line segments, are exactly the vectors $\pm v_E$ appearing in order of increasing anti-clockwise direction. We leave the following as an exercise to the reader.

Lemma 2.2. *There exists a tiling of $P_\alpha(q)$ using exactly one copy of each parallelogram spanned by the vectors v_E and $v_{E'}$, taken over all unordered pairs of distinct $E,E'\in\mathscr{E}$.* □

Noting that

$$|v_E \wedge v_{E'}| = \sum_{e \in E, e' \in E'} m_e m_{e'} |v_e \wedge v_{e'}|,$$

we immediately deduce:

Lemma 2.3. *For any $\alpha \in \mathscr{S}$ and $q \in QD^1(S)$, we have*

$$\mathrm{Area}(P_\alpha(q)) = \sum m_e m_{e'} |v_e \wedge v_{e'}|,$$

where the sum is taken over all unordered pairs of distinct saddle connections e, e' appearing on α^q. □

2.4 Rigidity

A key tool that we use is the following result of Duchin, Leininger, and Rafi. Our formulation of the statement is slightly different from theirs, but their proof still works without modification.

Proposition 2.4 ([DLR10], Lemma 22)**.** *Let $q, q' \in QD^1(S)$ be half-translation surfaces. If $\widehat{\mathsf{cyl}}(q) \subset \widehat{\mathsf{cyl}}(q')$ then $\mathrm{SL}(2,\mathbb{R}) \cdot q = \mathrm{SL}(2,\mathbb{R}) \cdot q'$.* □

Therefore, if $\mathrm{SL}(2,\mathbb{R}) \cdot q \neq \mathrm{SL}(2,\mathbb{R}) \cdot q'$ then both $\widehat{\mathsf{cyl}}(q) - \widehat{\mathsf{cyl}}(q')$ and $\widehat{\mathsf{cyl}}(q') - \widehat{\mathsf{cyl}}(q)$ are non-empty.

Remark 2.5. One consequence of Proposition 2.4 is that when S has genus at least 2, every $q \in \mathrm{QD}^1(S)$ has a crooked curve. This is true because $\mathrm{QD}^1(S)$ has dimension greater than 3, and hence cannot be a single $\mathrm{SL}(2,\mathbb{R})$–orbit.

Remark 2.6. Theorem 1 of [DLR10] states that the marked length spectrum of simple closed curves determines the half-translation surface q. Associated to q is the *marked polygonal area spectrum*, which is the $\mathrm{SL}(2,\mathbb{R})$–invariant function $\mathscr{S} \to \mathbb{R}$ given by $\alpha \mapsto \mathrm{Area}(P_\alpha(q))$. We observe that, by Proposition 2.1(iv) and Proposition 2.4, the marked polygonal area spectrum of q determines its $\mathrm{SL}(2,\mathbb{R})$–orbit.

2.5 $\mathrm{SL}(2,\mathbb{R})$–orbit closures

The space $\mathrm{QD}^1(S)$ of unit-area half translation structures on S is naturally partitioned into strata $\mathrm{Q}(\kappa)$, where κ is a partition of $4g-4$ specifying the orders of singularities. The mapping class group $\mathrm{MCG}(S)$ acts on $\mathrm{QD}^1(S)$ and each stratum $\mathrm{Q}(\kappa)$ by change of marking. The $\mathrm{SL}(2,\mathbb{R})$–action on $\mathrm{QD}^1(S)$ preserves each stratum, and also descends naturally to the moduli space of half-translation surfaces $\mathrm{MQD}(S) = \mathrm{QD}^1(S)/\mathrm{MCG}(S)$, as well as each unmarked stratum $\mathrm{MQ}(\kappa) = \mathrm{Q}(\kappa)/\mathrm{MCG}(S)$. Let $\pi\colon \mathrm{QD}^1(S) \to \mathrm{MQD}(S)$ be the natural projection, which is an orbifold covering map. Given $q \in \mathrm{Q}(\kappa)$, let M_q be the closure of $\pi(\mathrm{SL}(2,\mathbb{R}) \cdot q)$ in $\mathrm{MQ}(\kappa)$. We call M_q the *orbit closure* associated to q.

The structure of M_q has been elucidated in the work of Eskin, Mirzakhani, and Mohammadi, as follows:

Theorem 2.7 ([EMM15], Theorem 2.1)**.** *For any $q \in Q(\kappa)$, the orbit closure M_q is an affine invariant submanifold of $MQ(\kappa)$.* □

The statement given in [EMM15] actually refers to abelian differentials rather than quadratic differentials. However, the result applies equally well to the setting of quadratic differentials, by considering an appropriate two-fold branched covering of the surface.

The precise definition of "affine invariant submanifold" is rather involved and we shall not repeat it here in its entirety. It suffices to note that it includes the following:

- M_q is $\mathrm{SL}(2,\mathbb{R})$–invariant,
- M_q is the image of a properly immersed orbifold $f\colon N \to \mathrm{MQ}(\kappa)$.

Since $\pi\colon \mathrm{Q}(\kappa) \to \mathrm{MQ}(\kappa)$ is an orbifold covering, the preimage $\pi^{-1}(\mathrm{M}_q)$ is also the image of a properly immersed orbifold (in fact, manifold) of the same dimension as N.

The main conclusion we need to draw from Theorem 2.7 is that $\pi^{-1}(\mathrm{M}_q)$ is locally path connected. Let $\widetilde{\mathrm{M}}_q$ be the connected component of $\pi^{-1}(\mathrm{M}_q)$ containing q. Then local path connectedness of $\pi^{-1}(\mathrm{M}_q)$ implies that $\widetilde{\mathrm{M}}_q$ is open in $\pi^{-1}(\mathrm{M}_q)$.

Now let $\Gamma_q \le \mathrm{MCG}(S)$ be the (setwise) stabilizer of $\widetilde{\mathrm{M}}_q$, and define

$$\mathcal{O}_q = \Gamma_q \cdot (\mathrm{SL}(2,\mathbb{R}) \cdot q).$$

Lemma 2.8. *$\widetilde{\mathrm{M}}_q$ is the closure of $\mathcal{O}_q$ in $Q(\kappa)$.*

Proof. Certainly, $\widetilde{\mathrm{M}}_q$ contains the closure of $\mathcal{O}_q$, by $\mathrm{SL}(2,\mathbb{R})$–invariance. Also, $\pi^{-1}(\mathrm{M}_q)$ is the closure of $\mathrm{MCG}(S)\cdot(\mathrm{SL}(2,\mathbb{R})\cdot q)$. Note that if $g \in \mathrm{MCG}(S) - \Gamma_q$ then $g\cdot(\mathrm{SL}(2,\mathbb{R})\cdot q)$ is contained in the component $g\widetilde{\mathrm{M}}_q$ of $\pi^{-1}(\mathrm{M}_q)$, which is disjoint from $\widetilde{\mathrm{M}}_q$. Now if $q' \in \widetilde{\mathrm{M}}_q$ is a limit of a sequence of points $q_i \in g_i \cdot (\mathrm{SL}(2,\mathbb{R})\cdot q)$, the open set $\widetilde{\mathrm{M}}_q$ must contain almost all q_i, and therefore $g_i \in \Gamma_q$ for almost all i, and q' is in the closure of $\mathcal{O}_q$. □

We will also make use of the following finiteness result.

Theorem 2.9 ([EMM15], Theorem 2.2)**.** *Any closed* $\mathrm{SL}(2,\mathbb{R})$*–invariant subset of $Q(\kappa)$ is a finite union of* $\mathrm{SL}(2,\mathbb{R})$*–orbit closures.* □

2.6 Veech surfaces

Recall that $q \in \mathrm{QD}(S)$ is a *Veech surface* if its group of affine automorphisms is a lattice in $\mathrm{SL}(2,\mathbb{R})$. We now state several characterizations of Veech surfaces due to Smillie and Weiss [SW10], which builds on work of Vorobets [Vor96]. By a *triangle* on q, we mean a Euclidean triangle on q with isometrically embedded interior, whose sides are saddle connections on q. Let $\mathrm{hol}(q)$ be the set of holonomy vectors arising from saddle connections on q.

Theorem 2.10 ([SW10])**.** *For any $q \in Q(\kappa)$, the following are equivalent.*

(i) *q is a Veech surface,*

(ii) *q has no small triangles: there is a lower bound $\epsilon > 0$ on the areas of all triangles on q,*

(iii) *q has no small virtual triangles: there exists $\epsilon > 0$ such that $|u \wedge v| > \epsilon$ for all pairs of non-parallel holonomy vectors $u, v \in \mathrm{hol}(q)$,*

(iv) *the virtual triangle area spectrum* $VT(q) = \{|u \wedge v| : u, v \in \mathrm{hol}(q)\}$ *is discrete,*

(v) $\pi(\mathrm{SL}(2,\mathbb{R}) \cdot q)$ *is closed in* $MQ(\kappa)$. □

Note that condition (v) is the same as saying that $\mathrm{M}_q = \pi(\mathrm{SL}(2,\mathbb{R}) \cdot q)$. Applying Lemma 2.8, we deduce the following.

Corollary 2.11. *A half-translation surface q is a Veech surface if and only if $\widetilde{\mathrm{M}}_q = \mathscr{O}_q$.* □

3 Continuity of polygonal area

We wish to prove that the area of the auxiliary polygon $P_\alpha(q)$ varies continuously with q. In fact, we will show that the polygon itself is continuous in q, with respect to the Hausdorff metric in the plane. First we recall some basic notions from convex geometry. See, for instance, Sections 1.7 and 1.8 of [Sch14].

For any non-empty compact convex subset $K \subset \mathbb{R}^2$, the *support function* $h_K \colon S^1 \to \mathbb{R}$ is defined by

$$h_K(u) = \sup\{\langle x, u\rangle : x \in K\}.$$

Here $\langle \cdot, \cdot \rangle$ is the usual inner product on $\mathbb{R}^2$. The *width function* $w_K \colon S^1 \to \mathbb{R}$ is defined by

$$w_K(u) = h_K(u) + h_K(-u).$$

Note that w_K is even, and descends to a function on $\mathbb{R}\mathrm{P}^1$ which we also denote by w_K. Now let d_H denote Hausdorff distance. We have the following standard fact:

Lemma 3.1 ([Sch14],Lemma 1.8.14)**.** *Suppose K and L are non-empty compact convex subsets of $\mathbb{R}^2$. Then*

$$d_H(K,L) = \sup_{u \in S^1} |h_K(u) - h_L(u)|.$$ □

Next consider *centrally symmetric* convex sets: these are convex sets K such that $K = -K$. Note that the auxiliary polygons $P_\alpha(q)$ are both convex and centrally symmetric. If K is centrally symmetric then $h_K(u) = h_K(-u)$ for all u, and therefore

$$w_K = 2h_K. \tag{3.2}$$

Now let $\|f\|$ denote the sup norm for functions $f \colon \mathbb{R}\mathrm{P}^1 \to \mathbb{R}$, and let $\mathscr{C}(\mathbb{R}\mathrm{P}^1, \mathbb{R})$ be the space of continuous functions, with the sup metric. Let $\mathscr{K}_0$ be the space of non-empty centrally symmetric compact convex sets in $\mathbb{R}^2$, with the Hausdorff metric. The next lemma follows directly from Lemma 3.1 and equation (3.2).

Lemma 3.3. *If $K, L \in \mathscr{K}_0$ then*

$$2d_H(K,L) = \sup_{\theta \in \mathbb{R}\mathrm{P}^1} |w_K(\theta) - w_L(\theta)| = \|w_K - w_L\|.$$ □

Corollary 3.4. *The map $W \colon \mathscr{K}_0 \to \mathscr{C}(\mathbb{R}\mathrm{P}^1, \mathbb{R})$ given by $K \mapsto \frac{1}{2} w_K$ is an isometric embedding.* □

Let us now return our attention to the auxiliary polygons.

Theorem 3.5. *The map $QD^1(S) \times \mathscr{S} \to \mathscr{K}_0$ defined by $(q,\alpha) \mapsto P_\alpha(q)$ is continuous in the first factor.*

Proof. Let $\alpha \in \mathscr{S}$ be fixed. Applying Proposition 2.1, we see that

$$w_{P_\alpha(q)}(\theta) = \mathrm{width}(e^{-i\theta} \cdot P_\alpha(q)) = l^H_\alpha(e^{-i\theta} \cdot q) = i\big(v_q^{\frac{\pi}{2}+\theta}, \alpha\big)$$

for all $q \in \mathrm{QD}^1(S)$ and $\theta \in \mathbb{RP}^1$. The map $\mathrm{QD}^1(S) \times \mathbb{RP}^1 \to \mathscr{MF}(S)$ given by $(q,\theta) \mapsto v_q^{\frac{\pi}{2}+\theta}$ is continuous. By continuity of intersection number on $\mathscr{MF}(S) \times \mathscr{MF}(S)$ and compactness of $\mathbb{RP}^1$, the map $q \mapsto w_{P_\alpha(q)}$ defines a continuous function from $\mathrm{QD}^1(S)$ to $\mathscr{C}(\mathbb{RP}^1, \mathbb{R})$. Moreover, its image is contained in $W(\mathscr{K}_0)$, and composing this map with $\frac{1}{2}W^{-1}$ yields the function $q \mapsto P_\alpha(q)$. This map is continuous by Corollary 3.4. □

Finally, applying continuity of $\mathrm{Area}\colon \mathscr{K}_0 \to \mathbb{R}$ [Sch14, Theorem 1.8.20] yields the desired result.

Corollary 3.6. *The function* $\mathrm{Area}\colon QD^1(S) \times \mathscr{S} \to \mathbb{R}_{\geq 0}$ *defined by* $\mathrm{Area}(q,\alpha) = \mathrm{Area}(P_\alpha(q))$ *is continuous and* $\mathrm{SL}(2,\mathbb{R})$*–invariant in the first factor.* □

4 The polygonal area spectrum

We are now ready to prove Theorem 1.1. The first step is Theorem 1.3, which says that $\mathrm{Poly}(q)$ is discrete if q is a Veech surface.

Proof of Theorem 1.3. By Theorem 2.10(iv), the virtual triangle area spectrum $\mathrm{VT}(q)$ is discrete. For each simple closed curve α, $\mathrm{Area}(P_\alpha(q))$ is a positive integer combination of numbers from the set $\mathrm{VT}(q)$, by Lemma 2.3. The result follows. □

Using Proposition 2.1(iii), we obtain:

Corollary 4.1. *If $q \in QD^1(S)$ is a Veech surface then*

$$\inf\{l^{\mathrm{SL}}_\alpha(q) : \ \alpha \text{ is crooked on } q\} > 0.$$ □

For the converse, it is worth remarking that the existence of short crooked curves is not an immediately obvious consequence of having small virtual triangles. From a given collection of saddle connections on q for which $|u \wedge v|$ can be taken to be arbitrarily small, it appears difficult to construct a sequence of saddle connections to satisfy the following:

- no two saddle connections intersect (in their interiors),
- consecutive saddle connections meet with an angle of at least π on both sides,
- their concatenation is homotopic to an essential simple closed curve.

(Small triangles on q are not particularly useful since their sides must meet at an angle of less than π.) In our proof below, the auxiliary polygon plays a key role in bypassing this difficulty.

For $q \in \mathrm{Q}(\kappa)$, recall that $\mathcal{O}_q = \Gamma_q \cdot (\mathrm{SL}(2,\mathbb{R}) \cdot q)$ is a dense subset of $\widetilde{\mathrm{M}}_q$ in $\mathrm{Q}(\kappa)$, where $\Gamma_q \le \mathrm{MCG}(S)$ is the stabilizer of $\widetilde{\mathrm{M}}_q$.

Proposition 4.2. *Suppose $q \in QD^1(S)$ is not a Veech surface. Then there is a number $a > 0$ such that the polygonal area spectrum* $\mathrm{Poly}(q)$ *contains a dense subset of* $[0,a]$.

Proof. Applying Corollary 2.11, we have $\widetilde{\mathrm{M}}_q \neq \mathcal{O}_q$ and so we may choose $q' \in \widetilde{\mathrm{M}}_q - \mathcal{O}_q$. Then $\mathrm{SL}(2,\mathbb{R}) \cdot q \neq \mathrm{SL}(2,\mathbb{R}) \cdot q'$, and so by Proposition 2.4, there exists a curve $\alpha \in \widehat{\mathrm{cyl}}(q) - \widehat{\mathrm{cyl}}(q')$. By Proposition 2.1(iv), we have $\mathrm{Area}(q,\alpha) = 0$ and $\mathrm{Area}(q',\alpha) = a > 0$. Since $\widetilde{\mathrm{M}}_q$ is connected and $\mathrm{Area}(\boldsymbol{\cdot},\alpha)$ is continuous, by Corollary 3.6, we deduce that $[0,a] \subseteq \mathrm{Area}(\widetilde{\mathrm{M}}_q,\alpha)$. It follows that $\mathrm{Area}(\mathcal{O}_q,\alpha)$, and hence $\mathrm{Area}(\mathcal{O}_q,\mathcal{S})$, contains a dense subset of $[0,a]$. Finally, the polygonal area spectrum $\mathrm{Poly}(q) = \mathrm{Area}(q,\mathcal{S})$ is invariant under $\mathrm{SL}(2,\mathbb{R})$–deformations and changes of markings, and therefore $\mathrm{Area}(q,\mathcal{S}) = \mathrm{Area}(\mathcal{O}_q,\mathcal{S})$. □

Finally, combining the preceding result with Proposition 2.1(iii), we deduce:

Corollary 4.3. *If $q \in QD^1(S)$ is not a Veech surface then*

$$\inf\left\{ l_\alpha^{\mathrm{SL}}(q) : \ \alpha \textit{ is crooked on } q \right\} = 0.$$ □

We conclude this section with a proof of Theorem 1.4, which is restated below. For $a > 0$, recall that

$$\mathrm{PA}(a) = \left\{ q \in \mathrm{QD}^1(S) : \ \mathrm{Area}(\mathrm{P}_\alpha(q)) \ge a \text{ for every crooked curve } \alpha \text{ on } q \right\}.$$

Theorem 4.4. *For any $a > 0$, the set $PA(a)$ contains only finitely many affine equivalence classes of half-translation surfaces.*

Proof. Applying Corollary 3.6, we deduce that $\mathrm{PA}(a)$ is a closed and $\mathrm{SL}(2,\mathbb{R})$–invariant subset of $\mathrm{QD}^1(S)$. Moreover, $\mathrm{PA}(a)$ is invariant under the action of $\mathrm{MCG}(S)$, and so $\mathrm{PA}(a)$ descends to a closed $\mathrm{SL}(2,\mathbb{R})$–invariant subset C in $\mathrm{MQD}(S)$. It follows that $C \cap \mathrm{Q}(\kappa)$ is closed and $\mathrm{SL}(2,\mathbb{R})$–invariant in each stratum $\mathrm{Q}(\kappa)$ under the subspace topology. By Theorem 2.9, $C \cap \mathrm{Q}(\kappa)$ is a finite union of $\mathrm{SL}(2,\mathbb{R})$–orbit closures. Since elements of $\mathrm{PA}(a)$ are necessarily Veech surfaces by Theorem 1.1, $C \cap \mathrm{Q}(\kappa)$ must be a finite union of $\mathrm{SL}(2,\mathbb{R})$–orbits. Finally, there are only finitely many strata for a given genus, and so the desired result follows. □

5 The virtual triangle area spectrum

We are almost ready to prove Theorem 1.2. In the introduction we defined the virtual triangle area spectrum $\mathrm{VT}(q)$. Let $\mathrm{VT}_0(q) \subset \mathrm{VT}(q)$ be the subset consisting of the numbers $|u \wedge v|$ such that u and v are the holonomy vectors of a pair of saddle connections with a common endpoint (that is, a *based* virtual triangle). Note that saddle connections forming a based virtual triangle need not form a triangle, since they may have angle π or more on both sides. We also define $\mathrm{VT}_0(X) = \bigcup_{q\in X} \mathrm{VT}_0(q)$ for any set $X \subset \mathrm{QD}^1(S)$.

We know from the implication (i) ⇒ (iii) of Theorem 2.10 that if q is a Veech surface then $\mathrm{VT}(q)$ has a gap above zero. The remainder of Theorem 1.2 follows from the next proposition:

Proposition 5.1. *Suppose $q \in QD^1(S)$ is not a Veech surface. Then there is a number $a > 0$ such that $VT_0(q)$ contains a dense subset of* $[0, a]$.

Proof. As in the proof of Proposition 4.2, there exist a half-translation surface $q' \in \widetilde{\mathrm{M}}_q - \mathscr{O}_q$ and a curve $\alpha \in \widehat{\mathrm{cyl}}(q) - \widehat{\mathrm{cyl}}(q')$, and we know that $\mathrm{Area}(q, \alpha) = 0$ and $\mathrm{Area}(q', \alpha) > 0$. Let q_t be a path in $\widetilde{\mathrm{M}}_q$ from $q_0 = q$ to $q_1 = q'$. Consider the set

$$\left\{t \in [0,1] : \ \mathrm{Area}(q_t, \alpha) > 0\right\},$$

which is an open neighborhood of 1 in $[0,1]$ that does not contain 0. It has a connected component $(t_0, 1]$. Replacing the path q_t by its restriction to $[t_0, 1]$ and reparametrizing over $[0,1]$, we have $\mathrm{Area}(q_0, \alpha) = 0$ and $\mathrm{Area}(q_t, \alpha) > 0$ for all $t \in (0,1]$.

Now consider the geodesic representative α^{q_0} and express it as a concatenation of saddle connections $e_1 \cdots e_k$. Let α_i be the topological arc represented by e_i; it is an isotopy class rel endpoints, where the interior of the arc is required to avoid the singularities. Consider the set

$$\left\{t \in [0,1] : \ \text{each } \alpha_i \text{ is represented by a saddle connection in } q_t\right\}.$$

This set is an open neighborhood of 0 in $[0,1]$. Again, replacing the path q_t by its restriction to an interval $[0, \epsilon]$ and reparametrizing over $[0,1]$, we may assume that the arcs $\alpha_1, \ldots, \alpha_k$ are represented by saddle connections for all $t \in [0,1]$. Let $e_i(t)$ denote the saddle connection in q_t representing α_i. Let $v_i(t)$ be the holonomy vector of $e_i(t)$.

Define functions $\phi_i(t) = |v_i(t) \wedge v_{i+1}(t)|$ for each i (with indices taken mod k). These are continuous because holonomy vectors vary continuously where defined. Since all e_i are parallel on q_0, we have $\phi_i(0) = 0$ for all i. We claim that $\phi_i(t_1) > 0$ for some $t_1 > 0$ and some i. If not, then for all $t \in [0,1]$ the saddle connections $e_i(t)$ are all parallel. The angles between consecutive saddle connections (on either side) must remain constant, since they are constrained to lie in the discrete set $\pi\mathbb{Z}$, and therefore the concatenation $e_1(t) \cdots e_k(t)$ remains a geodesic representative for α on q_t. But this contradicts the fact that $\mathrm{Area}(q_t, \alpha) > 0$ for all $t \in (0,1]$.

Restricting q_t to $[0, t_1]$ and reparametrizing over $[0,1]$ one last time, we have a path q_t in $\widetilde{\mathrm{M}}_q$ and a pair of saddle connections $e_i(t)$, $e_{i+1}(t)$ which persist on q_t throughout the path, such that $\phi_i(0) = |v_i(0) \wedge v_{i+1}(0)| = 0$ and $\phi_i(1) = |v_i(1) \wedge v_{i+1}(1)| = a > 0$. The function ϕ_i is defined and continuous on the open set $U \subset \widetilde{\mathrm{M}}_q$ where e_i and e_{i+1} persist. This set contains the path q_t and hence $\phi_i(U)$ contains $[0, a]$. Since $U \cap \mathscr{O}_q$ is dense in U, it follows that $\phi_i(U \cap \mathscr{O}_q)$, and hence $\mathrm{VT}_0(\mathscr{O}_q)$, contains a dense subset of $[0, a]$. Finally, the based virtual triangle area spectrum is invariant under change of marking and $\mathrm{SL}(2,\mathbb{R})$, and so $\mathrm{VT}_0(q) = \mathrm{VT}_0(\mathscr{O}_q)$. □

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