Tables for Representations of GSp(4)

Brooks Roberts, Ralf Schmidt

1/2007

Contents

1 Explanations

The following pages contain several tables with information on the irreducible, admissible, non–supercuspidal representations of $\mathrm{GSp}(4,F)$, where F is a non-archimedean local field. The basis for these tables is the paper

[ST] Sally, P.; Tadić, M.: *Induced representations and classifications for* $\mathrm{GSp}(2,F)$ *and* $\mathrm{Sp}(2,F)$. Société Mathématique de France, Mémoire 52 (1993), 75–133,

where a complete list of all such representations is given. Moreover, [ST] classifies all the unitary, tempered and square–integrable representations. Our notation is the same as in [ST], except that we write $\mathrm{GSp}(2n)$ instead of $\mathrm{GSp}(n)$.

2.1: This table lists all the irreducible, admissible, non-supercuspidal representations of $\mathrm{GSp}(4,F)$. Representations in the same group I – XI are constituents of the same induced representation. The "tempered" column gives the precise condition for a representation to be tempered. The "L^2" column indicates which of the tempered representations are square–integrable. The "g" column indicates the generic representations; in each case, these are subrepresentations of the full induced representation given in the third column. The "P", "B" and "Q" columns indicate those representations occurring in global CAP representations. Here P stands for the Siegel parabolic subgroup, B stands for the minimal parabolic subgroup, and Q stands for the Klingen parabolic subgroup (whose unipotent radical is a three-dimensional Heisenberg group). A • in the P–column means that the representation appears as a local component in cusp forms that are CAP with respect to P (these are the Saito–Kurokawa representations). A $\circ$ in the P–column means that the representation appears as a local component in non-cuspidal liftings from PGL(2) × PGL(2). Similarly for the B and Q columns (this will be explained in more detail elsewhere).

Representations in groups I – VI are supported in the minimal parabolic subgroup, i.e., they are constituents of representations induced from a character of $B(F)$. Representations in groups VII – IX are supported in Q, and representations in groups X and XI are supported in P; here π stands for a supercuspidal representation of $\mathrm{GL}(2,F)$, and ω_π is the central character of π. Two types of supercuspidal representations are also listed in this table, since they share (conjectural) local Langlands parameters with some of the other representations: Va* is in the same L-packet as Va, and XIa* is in the same L-packet as XIa. The only other cases of L-indistinguishability in this table are the representations VIa and VIb, which constitute a two-element L-packet, and VIIIa, VIIIb, which do also constitute a two-element L-packet.

2.2 has more precise information on how the representations in groups IV, V and VI decompose.

2.3: This table lists the exact conditions for a representation to be unitary.

3.1: This table shows an explicit form of the conjectural Langlands parameters from the Weil–Deligne group W_F' into the L-group $\mathrm{GSp}(4,\mathbb{C})$ for the non-supercuspidal representations. The parameters are given as a pair $\varphi = (\rho, N)$, where ρ is a homomorphism from the Weil group W_F to the L-group $\mathrm{GSp}(4,\mathbb{C})$ and N is a unipotent element in the Lie algebra of GSp(4) such that $\rho(x)N\rho(x)^{-1} = |x|N$. For groups I – VI, an entry $\tau_1, \ldots, \tau_4$ in the "ρ" column stands for the map $W_F \ni x \mapsto \mathrm{diag}(\tau_1(x), \ldots, \tau_4(x))$. For groups VII – XI, the "ρ" column has to be read as block diagonal entries. The character ν denotes the normalized absolute value on $F^\times$, identified with a character of W_F. Note that the representations VIa,b constitute an L-packet. The same is true for the representations VIIIa,b. The representation Va shares its L-parameter with a supercuspidal representation Va* of θ_{10} type, and the representation XIa shares its L-parameter with a supercuspidal representation XIa*. All other representations in the table constitute singleton L-packets. Finally, the last column lists the number of elements of

$$\mathcal{C}(\varphi) = \mathrm{Cent}(\varphi)/\mathrm{Cent}(\varphi)^0\,\mathbb{C}^\times,$$

where $\mathrm{Cent}(\varphi)$ denotes the centralizer of the image of φ, where $\mathrm{Cent}(\varphi)^0$ denotes its identity component, and where $\mathbb{C}^\times$ stands for the center of $\mathrm{GSp}(4,\mathbb{C})$.

3.2 and 3.4: These are the degree 4 and degree 5 L-functions resulting from the L-parameters listed in Table 3.1.

3.3 and 3.5: These are the degree 4 and degree 5 ε-factors resulting from the L-parameters listed in Table 3.1, assuming that the central character of each representation is trivial.

4.1 and 4.2: These tables show the semisimplifications of the Jacquet modules with respect to the unipotent radical of the Siegel resp. Klingen parabolic. The Jacquet modules were computed with the formulas in Sect. 2 of [ST].

5.1: This table contains, for the Iwahori-spherical representations, the dimensions of the spaces of invariant vectors under each parahoric subgroup. All the characters in this table are assumed to be unramified. Under the assumption that the representation has trivial central character, the signs under the dimensions indicate how the space splits into ± 1 eigenspaces under the Atkin–Lehner involution. The ε–factor shown is computed from the local parameter given in the previous table; we give the value at 1/2 of the ε–factor under the assumption of trivial central character.

5.2: This table contains, for the same representations as the previous table, the dimensions of the spaces of invariant vectors under the paramodular group (of any level).

2 General tables

2.1 Irreducible non-supercuspidal representations of $\mathrm{GSp}(4,F)$

		constituent of	representation	tempered	L^2	g	P	B	Q
I		$\chi_1\times\chi_2\rtimes\sigma$ (irreducible)		χ_i,σ unit.		•			
II	a	$\nu^{1/2}\chi\times\nu^{-1/2}\chi\rtimes\sigma$	$\chi\mathrm{St}_{\mathrm{GL}(2)}\rtimes\sigma$	χ,σ unit.		•	○		
	b	$(\chi^2\neq\nu^{\pm1},\chi\neq\nu^{\pm3/2})$	$\chi\mathbf{1}_{\mathrm{GL}(2)}\rtimes\sigma$				•		
III	a	$\chi\times\nu\rtimes\nu^{-1/2}\sigma$	$\chi\rtimes\sigma\mathrm{St}_{\mathrm{GSp}(2)}$	π,σ unit.		•			○
	b	$(\chi\notin\{1,\nu^{\pm2}\})$	$\chi\rtimes\sigma\mathbf{1}_{\mathrm{GSp}(2)}$						•
IV	a	$\nu^2\times\nu\rtimes\nu^{-3/2}\sigma$	$\sigma\mathrm{St}_{\mathrm{GSp}(4)}$	σ unit.	•	•			
	b		$L(\nu^2,\nu^{-1}\sigma\mathrm{St}_{\mathrm{GSp}(2)})$						
	c		$L(\nu^{3/2}\mathrm{St}_{\mathrm{GL}(2)},\nu^{-3/2}\sigma)$						
	d		$\sigma\mathbf{1}_{\mathrm{GSp}(4)}$						
V	a	$\nu\xi\times\xi\rtimes\nu^{-1/2}\sigma$	$\delta([\xi,\nu\xi],\nu^{-1/2}\sigma)$	σ unit.	•	•	○	○	○
	b	$(\xi^2=1,\ \xi\neq1)$	$L(\nu^{1/2}\xi\mathrm{St}_{\mathrm{GL}(2)},\nu^{-1/2}\sigma)$				•	○	
	c		$L(\nu^{1/2}\xi\mathrm{St}_{\mathrm{GL}(2)},\xi\nu^{-1/2}\sigma)$				•	○	
	d		$L(\nu\xi,\xi\rtimes\nu^{-1/2}\sigma)$					•	•
VI	a	$\nu\times1_{F^\times}\rtimes\nu^{-1/2}\sigma$	$\tau(S,\nu^{-1/2}\sigma)$	σ unit.		•	○	○	○
	b		$\tau(T,\nu^{-1/2}\sigma)$	σ unit.			•	○	○
	c		$L(\nu^{1/2}\mathrm{St}_{\mathrm{GL}(2)},\nu^{-1/2}\sigma)$				•	•	
	d		$L(\nu,1_{F^\times}\rtimes\nu^{-1/2}\sigma)$					•	•
VII		$\chi\rtimes\pi$ (irreducible)		χ,π unit.		•			
VIII	a	$1_{F^\times}\rtimes\pi$	$\tau(S,\pi)$	π unit.		•			
	b		$\tau(T,\pi)$	π unit.					
IX	a	$\nu\xi\rtimes\nu^{-1/2}\pi$	$\delta(\nu\xi,\nu^{-1/2}\pi)$	π unit.	•	•			○
	b	$(\xi\neq1,\ \xi\pi=\pi)$	$L(\nu\xi,\nu^{-1/2}\pi)$						•
X		$\pi\rtimes\sigma$ (irreducible)		π,σ unit.		•			
XI	a	$\nu^{1/2}\pi\rtimes\nu^{-1/2}\sigma$	$\delta(\nu^{1/2}\pi,\nu^{-1/2}\sigma)$	π,σ unit.	•	•	○		
	b	$(\omega_\pi=1)$	$L(\nu^{1/2}\pi,\nu^{-1/2}\sigma)$				•		
Va*		(supercuspidal)	$\delta^*([\xi,\nu\xi],\nu^{-1/2}\sigma)$	σ unit.	•		•	•	•
XIa*		(supercuspidal)	$\delta^*(\nu^{1/2}\pi,\nu^{-1/2}\sigma)$	π,σ unit.	•		•		

The induced representation $\chi_1\times\chi_2\rtimes\sigma$ is irreducible if and only if $\chi_1\neq\nu^{\pm1}$, $\chi_2\neq\nu^{\pm1}$ and $\chi_1\neq\nu^{\pm1}\chi_2^{\pm1}$.

L–Packets: [VIa | VIb] [VIIIa | VIIIb] [Va | Va*] [XIa | XIa*]

Aubert involution: IIa ↔ IIb, IVa ↔ IVd, Va ↔ Vd, VIa ↔ VId, VIIIa ↔ VIIIb, XIa ↔ XIb, IIIa ↔ IIIb, IVb ↔ IVc, Vb ↔ Vc, VIb ↔ VIc, IXa ↔ IXb

2.2 Decompositions for types IV, V and VI

Group IV: Constituents of $\nu^2\times\nu\rtimes\nu^{-3/2}\sigma$.

$$\begin{aligned}\nu^2\times\nu\rtimes\nu^{-3/2}\sigma &= \underbrace{\nu^{3/2}\mathrm{St}_{\mathrm{GL}(2)}\rtimes\nu^{-3/2}\sigma}_{\text{sub}}+\underbrace{\nu^{3/2}\mathbf{1}_{\mathrm{GL}(2)}\rtimes\nu^{-3/2}\sigma}_{\text{quot}}\\ &= \underbrace{\nu^2\rtimes\nu^{-1}\sigma\mathrm{St}_{\mathrm{GSp}(2)}}_{\text{sub}}+\underbrace{\nu^2\rtimes\nu^{-1}\sigma\mathbf{1}_{\mathrm{GSp}(2)}}_{\text{quot}}.\end{aligned}$$

Each of the four representations on the right is reducible and has two irreducible constituents as shown in the following table. The quotients are on the bottom resp. on the right.

	$\nu^{3/2}\mathrm{St}_{\mathrm{GL}(2)}\rtimes\nu^{-3/2}\sigma$	$\nu^{3/2}\mathbf{1}_{\mathrm{GL}(2)}\rtimes\nu^{-3/2}\sigma$
$\nu^2\rtimes\nu^{-1}\sigma\mathrm{St}_{\mathrm{GSp}(2)}$	$\sigma\mathrm{St}_{\mathrm{GSp}(4)}$	$L(\nu^2,\nu^{-1}\sigma\mathrm{St}_{\mathrm{GSp}(2)})$
$\nu^2\rtimes\nu^{-1}\sigma\mathbf{1}_{\mathrm{GSp}(2)}$	$L(\nu^{3/2}\mathrm{St}_{\mathrm{GL}(2)},\nu^{-3/2}\sigma)$	$\sigma\mathbf{1}_{\mathrm{GSp}(4)}$

(1)

Group V: Constituents of $\nu\xi\times\xi\rtimes\nu^{-1/2}\sigma$, where ξ is a non-trivial quadratic character.

$$\begin{aligned}\nu\xi\times\xi\rtimes\nu^{-1/2}\sigma &= \underbrace{\nu^{1/2}\xi\,\mathrm{St}_{\mathrm{GL}(2)}\rtimes\nu^{-1/2}\sigma}_{\text{sub}}+\underbrace{\nu^{1/2}\xi\,\mathbf{1}_{\mathrm{GL}(2)}\rtimes\nu^{-1/2}\sigma}_{\text{quot}}\\ &= \underbrace{\nu^{1/2}\xi\,\mathrm{St}_{\mathrm{GL}(2)}\rtimes\xi\nu^{-1/2}\sigma}_{\text{sub}}+\underbrace{\nu^{1/2}\xi\,\mathbf{1}_{\mathrm{GL}(2)}\rtimes\xi\nu^{-1/2}\sigma}_{\text{quot}}.\end{aligned}$$

Each of the representations on the right side has two constituents as indicated in the following table. The quotients appear on the bottom resp. on the right.

	$\nu^{1/2}\xi\,\mathrm{St}_{\mathrm{GL}(2)}\rtimes\xi\nu^{-1/2}\sigma$	$\nu^{1/2}\xi\,\mathbf{1}_{\mathrm{GL}(2)}\rtimes\xi\nu^{-1/2}\sigma$
$\nu^{1/2}\xi\,\mathrm{St}_{\mathrm{GL}(2)}\rtimes\nu^{-1/2}\sigma$	$\delta([\xi,\nu\xi],\nu^{-1/2}\sigma)$	$L(\nu^{1/2}\xi\,\mathrm{St}_{\mathrm{GL}(2)},\nu^{-1/2}\sigma)$
$\nu^{1/2}\xi\,\mathbf{1}_{\mathrm{GL}(2)}\rtimes\nu^{-1/2}\sigma$	$L(\nu^{1/2}\xi\,\mathrm{St}_{\mathrm{GL}(2)},\xi\nu^{-1/2}\sigma)$	$L(\nu\xi,\xi\rtimes\nu^{-1/2}\sigma)$

(2)

Here $\delta([\xi,\nu\xi],\nu^{-1/2}\sigma)$ is a square integrable representation.

Group VI: Constituents of $\nu\times 1_{F^\times}\rtimes\nu^{-1/2}\sigma$.

$$\begin{aligned}\nu\times 1_{F^\times}\rtimes\nu^{-1/2}\sigma &= \underbrace{\nu^{1/2}\,\mathrm{St}_{\mathrm{GL}(2)}\rtimes\nu^{-1/2}\sigma}_{\text{sub}}+\underbrace{\nu^{1/2}\,\mathbf{1}_{\mathrm{GL}(2)}\rtimes\nu^{-1/2}\sigma}_{\text{quot}}\\ &= \underbrace{1_{F^\times}\rtimes\sigma\mathrm{St}_{\mathrm{GSp}(2)}}_{\text{sub}}+\underbrace{1_{F^\times}\rtimes\sigma\mathbf{1}_{\mathrm{GSp}(2)}}_{\text{quot}},\end{aligned}$$

and each representation on the right side is again reducible. Their constituents are summarized in the following table, with the quotients appearing on the bottom resp. on the right.

	$\nu^{1/2}\,\mathrm{St}_{\mathrm{GL}(2)}\rtimes\nu^{-1/2}\sigma$	$\nu^{1/2}\,\mathbf{1}_{\mathrm{GL}(2)}\rtimes\nu^{-1/2}\sigma$
$1_{F^\times}\rtimes\sigma\mathrm{St}_{\mathrm{GSp}(2)}$	$\tau(S,\nu^{-1/2}\sigma)$	$\tau(T,\nu^{-1/2}\sigma)$
$1_{F^\times}\rtimes\sigma\mathbf{1}_{\mathrm{GSp}(2)}$	$L(\nu^{1/2}\,\mathrm{St}_{\mathrm{GL}(2)},\nu^{-1/2}\sigma)$	$L(\nu,1_{F^\times}\rtimes\nu^{-1/2}\sigma)$

(3)

The representations $\tau(S,\nu^{-1/2}\sigma)$ and $\tau(T,\nu^{-1/2}\sigma)$ are tempered but not square integrable.

2.3 Unitary representations of $\mathrm{GSp}(4,F)$

		representation	conditions for unitarity
I		$\chi_1 \times \chi_2 \rtimes \sigma$ (irreducible)	$e(\chi_1)=e(\chi_2)=e(\sigma)=0$
			$\chi_1=\nu^\beta\chi,\ \chi_2=\nu^\beta\chi^{-1},\ e(\sigma)=-\beta,$ $e(\chi)=0,\ \chi^2\neq 1,\ 0<\beta<1/2$
			$\chi_1=\nu^\beta,\ e(\chi_2)=0,\ e(\sigma)=-\beta/2,$ $\chi_2\neq 1,\ 0<\beta<1$
			$\chi_1=\nu^{\beta_1}\chi,\ \chi_2=\nu^{\beta_2}\chi,\ e(\sigma)=(-\beta_1-\beta_2)/2,$ $\chi^2=1,\ 0\le\beta_2\le\beta_1,\ 0<\beta_1<1,\ \beta_1+\beta_2<1$
II	a	$\chi\mathrm{St}_{\mathrm{GL}(2)}\rtimes\sigma$	$e(\sigma)=e(\chi)=0$
			$\chi=\xi\nu^\beta,\ e(\sigma)=-\beta,\ \xi^2=1,\ 0<\beta<1/2$
	b	$\chi\mathbf{1}_{\mathrm{GL}(2)}\rtimes\sigma$	$e(\sigma)=e(\chi)=0$
			$\chi=\xi\nu^\beta,\ e(\sigma)=-\beta,\ \xi^2=1,\ 0<\beta<1/2$
III	a	$\chi\rtimes\sigma\mathrm{St}_{\mathrm{GSp}(2)}$	$e(\sigma)=e(\chi)=0$
	b	$\chi\rtimes\sigma\mathbf{1}_{\mathrm{GSp}(2)}$	$e(\sigma)=e(\chi)=0$
IV	a	$\sigma\mathrm{St}_{\mathrm{GSp}(4)}$	$e(\sigma)=0$
	b	$L(\nu^2,\nu^{-1}\sigma\mathrm{St}_{\mathrm{GSp}(2)})$	never unitary
	c	$L(\nu^{3/2}\mathrm{St}_{\mathrm{GSp}(2)},\nu^{-3/2}\sigma)$	never unitary
	d	$\sigma\mathbf{1}_{\mathrm{GSp}(4)}$	$e(\sigma)=0$
V	a	$\delta([\xi,\nu\xi],\nu^{-1/2}\sigma)$	$e(\sigma)=0$
	b	$L(\nu^{1/2}\xi\mathrm{St}_{\mathrm{GL}(2)},\nu^{-1/2}\sigma)$	$e(\sigma)=0$
	c	$L(\nu^{1/2}\xi\mathrm{St}_{\mathrm{GL}(2)},\xi\nu^{-1/2}\sigma)$	$e(\sigma)=0$
	d	$L(\nu\xi,\xi\rtimes\nu^{-1/2}\sigma)$	$e(\sigma)=0$
VI	a	$\tau(S,\nu^{-1/2}\sigma)$	$e(\sigma)=0$
	b	$\tau(T,\nu^{-1/2}\sigma)$	$e(\sigma)=0$
	c	$L(\nu^{1/2}\mathrm{St}_{\mathrm{GL}(2)},\nu^{-1/2}\sigma)$	$e(\sigma)=0$
	d	$L(\nu,1_{F^\times}\rtimes\nu^{-1/2}\sigma)$	$e(\sigma)=0$
VII		$\chi\rtimes\pi$ (irreducible)	$e(\chi)=e(\pi)=0$
			$\chi=\nu^\beta\xi,\ \pi=\nu^{-\beta/2}\rho,\ 0<\beta<1,$ $\xi^2=1,\ \xi\neq 1,\ e(\rho)=0,\ \xi\rho=\rho$
VIII	a	$\tau(S,\pi)$	$e(\pi)=0$
	b	$\tau(T,\pi)$	$e(\pi)=0$
IX	a	$\delta(\nu\xi,\nu^{-1/2}\pi)$	$e(\pi)=0$
	b	$L(\nu\xi,\nu^{-1/2}\pi)$	$e(\pi)=0$
X		$\pi\rtimes\sigma$ (irreducible)	$e(\sigma)=e(\pi)=0$
			$\pi=\nu^\beta\rho,\ e(\sigma)=-\beta,\ 0<\beta<1/2,\ \omega_\rho=1$
XI	a	$\delta(\nu^{1/2}\pi,\nu^{-1/2}\sigma)$	$e(\sigma)=e(\pi)=0$
	b	$L(\nu^{1/2}\pi,\nu^{-1/2}\sigma)$	$e(\sigma)=e(\pi)=0$
π supercuspidal			$e(\omega_\pi)=0$

3 L-parameters and local factors

3.1 Local Langlands parameters

		representation	ρ	N	$\#\mathcal{C}$
I		$\chi_1 \times \chi_2 \rtimes \sigma$ (irreducible)	$\chi_1\chi_2\sigma,\ \chi_1\sigma,\ \chi_2\sigma,\ \sigma$	0	1
II	a	$\chi\mathrm{St}_{\mathrm{GL}(2)} \rtimes \sigma$	$\chi^2\sigma,\ \nu^{1/2}\chi\sigma,\ \nu^{-1/2}\chi\sigma,\ \sigma$	N_1	1
	b	$\chi\mathbf{1}_{\mathrm{GL}(2)} \rtimes \sigma$		0	1
III	a	$\chi \rtimes \sigma\mathrm{St}_{\mathrm{GSp}(2)}$	$\nu^{1/2}\chi\sigma,\ \nu^{-1/2}\chi\sigma,\ \nu^{1/2}\sigma,\ \nu^{-1/2}\sigma$	N_4	1
	b	$\chi \rtimes \sigma\mathbf{1}_{\mathrm{GSp}(2)}$		0	1
IV	a	$\sigma\mathrm{St}_{\mathrm{GSp}(4)}$	$\nu^{3/2}\sigma,\ \nu^{1/2}\sigma,\ \nu^{-1/2}\sigma,\ \nu^{-3/2}\sigma$	N_5	1
	b	$L(\nu^2, \nu^{-1}\sigma\mathrm{St}_{\mathrm{GSp}(2)})$		N_4	1
	c	$L(\nu^{3/2}\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-3/2}\sigma)$		N_1	1
	d	$\sigma\mathbf{1}_{\mathrm{GSp}(4)}$		0	1
V	a	$\delta([\xi, \nu\xi], \nu^{-1/2}\sigma)$	$\nu^{1/2}\sigma,\ \nu^{1/2}\xi\sigma,\ \nu^{-1/2}\xi\sigma,\ \nu^{-1/2}\sigma$	N_3	2
	b	$L(\nu^{1/2}\xi\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\sigma)$		N_1	1
	c	$L(\nu^{1/2}\xi\mathrm{St}_{\mathrm{GL}(2)}, \xi\nu^{-1/2}\sigma)$		N_2	1
	d	$L(\nu\xi, \xi \rtimes \nu^{-1/2}\sigma)$		0	1
VI	a	$\tau(S, \nu^{-1/2}\sigma)$	$\nu^{1/2}\sigma,\ \nu^{1/2}\sigma,\ \nu^{-1/2}\sigma,\ \nu^{-1/2}\sigma$	N_3	2
	b	$\tau(T, \nu^{-1/2}\sigma)$			
	c	$L(\nu^{1/2}\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\sigma)$		N_1	1
	d	$L(\nu, 1_{F^\times} \rtimes \nu^{-1/2}\sigma)$		0	1
VII		$\chi \rtimes \pi$	$\chi\omega_\pi\varphi_\pi',\ \varphi_\pi$	0	1
VIII	a	$\tau(S, \pi)$	$\omega_\pi\varphi_\pi',\ \varphi_\pi$	0	2
	b	$\tau(T, \pi)$			
IX	a	$\delta(\nu\xi, \nu^{-1/2}\pi)$	$\xi\nu^{1/2}\omega_\pi\varphi_\pi',\ \nu^{-1/2}\varphi_\pi$	N_6	1
	b	$L(\nu\xi, \nu^{-1/2}\pi)$		0	1
X		$\pi \rtimes \sigma$	$\sigma\omega_\pi,\ \sigma\varphi_\pi,\ \sigma$	0	1
XI	a	$\delta(\nu^{1/2}\pi, \nu^{-1/2}\sigma)$	$\nu^{1/2}\sigma,\ \sigma\varphi_\pi,\ \nu^{-1/2}\sigma$	N_2	2
	b	$L(\nu^{1/2}\pi, \nu^{-1/2}\sigma)$		0	1

Here, $\varphi_\pi : W_F \to \mathrm{GL}(2,\mathbb{C})$ is the L-parameter of the supercuspidal representation π of $\mathrm{GL}(2,F)$, and ω_π is its central character. Further,

$$N_1 = \begin{bmatrix} 0 &&& \\ & 0 & 1 & \\ && 0 & \\ &&& 0 \end{bmatrix}, \qquad N_2 = \begin{bmatrix} 0 &&& 1 \\ & 0 && \\ && 0 & \\ &&& 0 \end{bmatrix}, \qquad N_3 = \begin{bmatrix} 0 &&& 1 \\ & 0 & 1 & \\ && 0 & \\ &&& 0 \end{bmatrix},$$

$$N_4 = \begin{bmatrix} 0 & 1 && \\ & 0 && \\ && 0 & -1 \\ &&& 0 \end{bmatrix}, \qquad N_5 = \begin{bmatrix} 0 & 1 && \\ & 0 & 1 & \\ && 0 & -1 \\ &&& 0 \end{bmatrix}, \qquad N_6 = \begin{bmatrix} 0 && y & z \\ & 0 & x & y \\ && 0 & \\ &&& 0 \end{bmatrix},$$

where $S = \begin{bmatrix} x & y \\ y & z \end{bmatrix}$ is such that ${}^t\varphi_\pi(w)S\varphi_\pi(w) = \xi(w)\det(\varphi_\pi(w))S$ for all $w \in W_F$.

3.2 L-factors (degree 4)

		representation	$L(s,\varphi)$
I		$\chi_1\times\chi_2\rtimes\sigma$ (irreducible)	$L(s,\chi_1\chi_2\sigma)L(s,\sigma)L(s,\chi_1\sigma)L(s,\chi_2\sigma)$
II	a	$\chi\mathrm{St}_{\mathrm{GL}(2)}\rtimes\sigma$	$L(s,\chi^2\sigma)L(s,\sigma)L(s,\nu^{1/2}\chi\sigma)$
	b	$\chi\mathbf{1}_{\mathrm{GL}(2)}\rtimes\sigma$	$L(s,\chi^2\sigma)L(s,\sigma)L(s,\nu^{1/2}\chi\sigma)L(s,\nu^{-1/2}\chi\sigma)$
III	a	$\chi\rtimes\sigma\mathrm{St}_{\mathrm{GSp}(2)}$	$L(s,\nu^{1/2}\chi\sigma)L(s,\nu^{1/2}\sigma)$
	b	$\chi\rtimes\sigma\mathbf{1}_{\mathrm{GSp}(2)}$	$L(s,\nu^{1/2}\chi\sigma)L(s,\nu^{1/2}\sigma)L(s,\nu^{-1/2}\chi\sigma)L(s,\nu^{-1/2}\sigma)$
IV	a	$\sigma\mathrm{St}_{\mathrm{GSp}(4)}$	$L(s,\nu^{3/2}\sigma)$
	b	$L(\nu^2,\nu^{-1}\sigma\mathrm{St}_{\mathrm{GSp}(2)})$	$L(s,\nu^{3/2}\sigma)L(s,\nu^{-1/2}\sigma)$
	c	$L(\nu^{3/2}\mathrm{St}_{\mathrm{GL}(2)},\nu^{-3/2}\sigma)$	$L(s,\nu^{3/2}\sigma)L(s,\nu^{1/2}\sigma)L(s,\nu^{-3/2}\sigma)$
	d	$\sigma\mathbf{1}_{\mathrm{GSp}(4)}$	$L(s,\nu^{3/2}\sigma)L(s,\nu^{1/2}\sigma)L(s,\nu^{-1/2}\sigma)L(s,\nu^{-3/2}\sigma)$
V	a	$\delta([\xi,\nu\xi],\nu^{-1/2}\sigma)$	$L(s,\nu^{1/2}\sigma)L(s,\nu^{1/2}\xi\sigma)$
	b	$L(\nu^{1/2}\xi\mathrm{St}_{\mathrm{GL}(2)},\nu^{-1/2}\sigma)$	$L(s,\nu^{1/2}\sigma)L(s,\nu^{1/2}\xi\sigma)L(s,\nu^{-1/2}\sigma)$
	c	$L(\nu^{1/2}\xi\mathrm{St}_{\mathrm{GL}(2)},\xi\nu^{-1/2}\sigma)$	$L(s,\nu^{1/2}\sigma)L(s,\nu^{1/2}\xi\sigma)L(s,\nu^{-1/2}\xi\sigma)$
	d	$L(\nu\xi,\xi\rtimes\nu^{-1/2}\sigma)$	$L(s,\nu^{1/2}\sigma)L(s,\nu^{1/2}\xi\sigma)L(s,\nu^{-1/2}\sigma)L(s,\nu^{-1/2}\xi\sigma)$
VI	a	$\tau(S,\nu^{-1/2}\sigma)$	$L(s,\nu^{1/2}\sigma)^2$
	b	$\tau(T,\nu^{-1/2}\sigma)$	$L(s,\nu^{1/2}\sigma)^2$
	c	$L(\nu^{1/2}\mathrm{St}_{\mathrm{GL}(2)},\nu^{-1/2}\sigma)$	$L(s,\nu^{1/2}\sigma)^2L(s,\nu^{-1/2}\sigma)$
	d	$L(\nu,1_{F^\times}\rtimes\nu^{-1/2}\sigma)$	$L(s,\nu^{1/2}\sigma)^2L(s,\nu^{-1/2}\sigma)^2$
VII		$\chi\rtimes\pi$	1
VIII	a	$\tau(S,\pi)$	1
	b	$\tau(T,\pi)$	1
IX	a	$\delta(\nu\xi,\nu^{-1/2}\pi)$	1
	b	$L(\nu\xi,\nu^{-1/2}\pi)$	1
X		$\pi\rtimes\sigma$	$L(s,\sigma)L(s,\omega_\pi\sigma)$
XI	a	$\delta(\nu^{1/2}\pi,\nu^{-1/2}\sigma)$	$L(s,\nu^{1/2}\sigma)$
	b	$L(\nu^{1/2}\pi,\nu^{-1/2}\sigma)$	$L(s,\nu^{1/2}\sigma)L(s,\nu^{-1/2}\sigma)$

$\varphi=(\rho,N)$ is the L-parameter of the representation, as listed in Table 3.1.

3.3 ε-factors (degree 4)

		inducing data	$a(\varphi)$	$\varepsilon(1/2,\varphi)$
I			$a(\chi_1\sigma)+a(\chi_2\sigma)+2a(\sigma)$	$\chi_1(-1) \quad (=\chi_2(-1))$
II	a	$\sigma\chi$ unr.	$2a(\sigma)+1$	$-\sigma(-1)(\sigma\chi)(\varpi)$
		$\sigma\chi$ ram.	$2a(\chi\sigma)+2a(\sigma)$	$\chi(-1)$
	b	$\sigma\chi$ unr.	$2a(\sigma)$	$\chi(-1)$
		$\sigma\chi$ ram.	$2a(\chi\sigma)+2a(\sigma)$	$\chi(-1)$
III	a	σ unr.	2	1
		σ ram.	$4a(\sigma)$	1
	b	σ unr.	0	1
		σ ram.	$4a(\sigma)$	1
IV	a	σ unr.	3	$-\sigma(\varpi)$
		σ ram.	$4a(\sigma)$	1
	b	σ unr.	2	1
		σ ram.	$4a(\sigma)$	1
	c	σ unr.	1	$-\sigma(\varpi)$
		σ ram.	$4a(\sigma)$	1
	d	σ unr.	0	1
		σ ram.	$4a(\sigma)$	1
V	a	σ, ξ unr.	2	-1
		σ unr., ξ ram.	$2a(\xi)+1$	$-\sigma(\varpi)\xi(-1)$
		σ ram., $\sigma\xi$ unr.	$2a(\sigma)+1$	$-\sigma(-1)(\sigma\xi)(\varpi)$
		σ, $\sigma\xi$ ram.	$2a(\xi\sigma)+2a(\sigma)$	$\xi(-1)$
	b	σ, ξ unr.	1	$\sigma(\varpi)$
		σ unr., ξ ram.	$2a(\xi)$	$\xi(-1)$
		σ ram., $\sigma\xi$ unr.	$2a(\sigma)+1$	$-\xi(-1)(\sigma\xi)(\varpi)$
		σ, $\sigma\xi$ ram.	$2a(\xi\sigma)+2a(\sigma)$	$\xi(-1)$

		inducing data	$a(\varphi)$	$\varepsilon(1/2,\varphi)$
V	c	σ, ξ unr.	1	$-\sigma(\varpi)$
		σ unr., ξ ram.	$2a(\xi)+1$	$-\sigma(\varpi)$
		σ ram., $\sigma\xi$ unr.	$2a(\sigma)$	$\xi(-1)$
		σ, $\sigma\xi$ ram.	$2a(\xi\sigma)+2a(\sigma)$	$\xi(-1)$
	d	σ,ξ unr.	0	1
		σ or ξ ram.	$2a(\xi\sigma)+2a(\sigma)$	$\xi(-1)$
VI	a	σ unr.	2	1
		σ ram.	$4a(\sigma)$	1
	b	σ unr.	2	1
		σ ram.	$4a(\sigma)$	1
	c	σ unr.	1	$-\sigma(\varpi)$
		σ ram.	$4a(\sigma)$	1
	d	σ unr.	0	1
		σ ram.	$4a(\sigma)$	1
VII			$2a(\pi)$	$\chi(-1) \quad (=\omega_\pi(-1))$
VIII	a		$2a(\pi)$	1
	b		$2a(\pi)$	1
IX	a		$2a(\pi)$	$\xi(-1)$
	b		$2a(\pi)$	$\xi(-1)$
X			$a(\sigma\pi)+2a(\sigma)$	$\sigma(-1)\varepsilon(1/2,\sigma\pi)$
XI	a	σ unr.	$a(\sigma\pi)+1$	$-\sigma(\varpi)\varepsilon(1/2,\sigma\pi)$
		σ ram.	$a(\sigma\pi)+2a(\sigma)$	$\sigma(-1)\varepsilon(1/2,\sigma\pi)$
	b	σ unr.	$a(\sigma\pi)$	$\varepsilon(1/2,\sigma\pi)$
		σ ram.	$a(\sigma\pi)+2a(\sigma)$	$\sigma(-1)\varepsilon(1/2,\sigma\pi)$

$\varphi=(\rho,N)$ is the L-parameter of the representation, as listed in Table 3.1. In this table we assume that all the representations have trivial central character. Under this assumption the ε-factors do not depend on the choice of an additive character, as long as this character has conductor $\mathfrak{o}$. We have

$$\varepsilon(s,\varphi)=\varepsilon(1/2,\varphi)q^{-a(\varphi)(s-1/2)}$$

where $\varepsilon(1/2,\varphi)\in\{\pm1\}$.

3.4 L-factors (degree 5)

		representation	$L(s,\rho_5\circ\varphi)$
I		$\chi_1\times\chi_2\rtimes\sigma$ (irreducible)	$L(s,\chi_1)L(s,\chi_1^{-1})L(s,\chi_2)L(s,\chi_2^{-1})L(s,1_{F^\times})$
II	a	$\chi\mathrm{St}_{\mathrm{GL}(2)}\rtimes\sigma$	$L(s,\nu^{1/2}\chi)L(s,\nu^{1/2}\chi^{-1})L(s,1_{F^\times})$
	b	$\chi\mathbf{1}_{\mathrm{GL}(2)}\rtimes\sigma$	$L(s,\nu^{1/2}\chi)L(s,\nu^{-1/2}\chi)$ $L(s,\nu^{1/2}\chi^{-1})L(s,\nu^{-1/2}\chi^{-1})L(s,1_{F^\times})$
III	a	$\chi\rtimes\sigma\mathrm{St}_{\mathrm{GSp}(2)}$	$L(s,\chi)L(s,\chi^{-1})L(s,\nu)$
	b	$\chi\rtimes\sigma\mathbf{1}_{\mathrm{GSp}(2)}$	$L(s,\chi)L(s,\chi^{-1})L(s,\nu)L(s,\nu^{-1})L(s,1_{F^\times})$
IV	a	$\sigma\mathrm{St}_{\mathrm{GSp}(4)}$	$L(s,\nu^2)$
	b	$L(\nu^2,\nu^{-1}\sigma\mathrm{St}_{\mathrm{GSp}(2)})$	$L(s,\nu^2)L(s,\nu)L(s,\nu^{-2})$
	c	$L(\nu^{3/2}\mathrm{St}_{\mathrm{GL}(2)},\nu^{-3/2}\sigma)$	$L(s,\nu^2)L(s,1_{F^\times})L(s,\nu^{-1})$
	d	$\sigma\mathbf{1}_{\mathrm{GSp}(4)}$	$L(s,\nu^2)L(s,\nu)L(s,1_{F^\times})L(s,\nu^{-1})L(s,\nu^{-2})$
V	a	$\delta([\xi,\nu\xi],\nu^{-1/2}\sigma)$	$L(s,\nu\xi)L(s,\xi)L(s,1_{F^\times})$
	b	$L(\nu^{1/2}\xi\mathrm{St}_{\mathrm{GL}(2)},\nu^{-1/2}\sigma)$	$L(s,\nu\xi)L(s,\xi)L(s,1_{F^\times})$
	c	$L(\nu^{1/2}\xi\mathrm{St}_{\mathrm{GL}(2)},\xi\nu^{-1/2}\sigma)$	$L(s,\nu\xi)L(s,\xi)L(s,1_{F^\times})$
	d	$L(\nu\xi,\xi\rtimes\nu^{-1/2}\sigma)$	$L(s,\nu\xi)L(s,\nu^{-1}\xi)L(s,\xi)^2L(s,1_{F^\times})$
VI	a	$\tau(S,\nu^{-1/2}\sigma)$	$L(s,\nu)L(s,1_{F^\times})^2$
	b	$\tau(T,\nu^{-1/2}\sigma)$	
	c	$L(\nu^{1/2}\mathrm{St}_{\mathrm{GL}(2)},\nu^{-1/2}\sigma)$	$L(s,\nu)L(s,1_{F^\times})^2$
	d	$L(\nu,1_{F^\times}\rtimes\nu^{-1/2}\sigma)$	$L(s,\nu)L(s,\nu^{-1})L(s,1_{F^\times})^3$
VII		$\chi\rtimes\pi$	$L(s,\chi)L(s,\chi^{-1})L(s,\mathrm{Sym}^2\circ\mu)$
VIII	a	$\tau(S,\pi)$	$L(s,1_{F^\times})^2L(s,\mathrm{Sym}^2\circ\mu)$
	b	$\tau(T,\pi)$	
IX	a	$\delta(\nu\xi,\nu^{-1/2}\pi)$	$L(s,\nu\xi)L(s,\mathrm{Sym}^2\circ\mu)L(s,\xi)^{-1}$
	b	$L(\nu\xi,\nu^{-1/2}\pi)$	$L(s,\nu\xi)L(s,\nu^{-1}\xi)L(s,\mathrm{Sym}^2\circ\mu)$
X		$\pi\rtimes\sigma$	$L(s,\mu)L(s,\det(\mu)^{-1}\mu)L(s,1_{F^\times})$
XI	a	$\delta(\nu^{1/2}\pi,\nu^{-1/2}\sigma)$	$L(s,\nu^{1/2}\mu)L(s,1_{F^\times})$
	b	$L(\nu^{1/2}\pi,\nu^{-1/2}\sigma)$	$L(s,\nu^{1/2}\mu)L(s,\nu^{-1/2}\mu)L(s,1_{F^\times})$

$\varphi=(\rho,N)$ is the L-parameter of the representation, as listed in Table 3.1. The homomorphism $\rho_5:\mathrm{GSp}(4,\mathbb{C})\to\mathrm{SO}(5,\mathbb{C})$ is a five-dimensional representation of the L-group that induces an isomorphism $\mathrm{PGSp}(4,\mathbb{C})\cong\mathrm{SO}(5,\mathbb{C})$.

3.5 ε-factors (degree 5)

		representation	$a(\rho_5 \circ \varphi)$	$\varepsilon(1/2, \rho_5 \circ \varphi)$
I		$\chi_1 \times \chi_2 \rtimes \sigma$ (irreducible)	$2a(\chi_1) + 2a(\chi_2)$	$\chi_1(-1)\chi_2(-1)$
II	a	$\chi \mathrm{St}_{\mathrm{GL}(2)} \rtimes \sigma$	χ unr. : 2, χ ram. : $4a(\chi)$	1
	b	$\chi \mathbf{1}_{\mathrm{GL}(2)} \rtimes \sigma$	$4a(\chi)$	1
III	a	$\chi \rtimes \sigma \mathrm{St}_{\mathrm{GSp}(2)}$	$2a(\chi) + 2$	$\chi(-1)$
	b	$\chi \rtimes \sigma \mathbf{1}_{\mathrm{GSp}(2)}$	$2a(\chi)$	$\chi(-1)$
IV	a	$\sigma \mathrm{St}_{\mathrm{GSp}(4)}$	4	1
	b	$L(\nu^2, \nu^{-1}\sigma \mathrm{St}_{\mathrm{GSp}(2)})$	2	1
	c	$L(\nu^{3/2}\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-3/2}\sigma)$	2	1
	d	$\sigma \mathbf{1}_{\mathrm{GSp}(4)}$	0	1
V	a	$\delta([\xi, \nu\xi], \nu^{-1/2}\sigma)$	ξ unr. : 2, ξ ram. : $4a(\xi)$	1
	b	$L(\nu^{1/2}\xi \mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\sigma)$	ξ unr. : 2, ξ ram. : $4a(\xi)$	1
	c	$L(\nu^{1/2}\xi \mathrm{St}_{\mathrm{GL}(2)}, \xi\nu^{-1/2}\sigma)$	ξ unr. : 2, ξ ram. : $4a(\xi)$	1
	d	$L(\nu\xi, \xi \rtimes \nu^{-1/2}\sigma)$	$4a(\xi)$	1
VI	a	$\tau(S, \nu^{-1/2}\sigma)$	2	1
	b	$\tau(T, \nu^{-1/2}\sigma)$		
	c	$L(\nu^{1/2}\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\sigma)$	2	1
	d	$L(\nu, 1_{F^\times} \rtimes \nu^{-1/2}\sigma)$	0	1
VII		$\chi \rtimes \pi$	$2a(\chi) + a(\mathrm{Sym}^2 \circ \mu)$	$\chi(-1)\varepsilon(\frac{1}{2}, \mathrm{Sym}^2 \circ \mu)$
VIII	a	$\tau(S, \pi)$	$a(\mathrm{Sym}^2 \circ \mu)$	$\varepsilon(\frac{1}{2}, \mathrm{Sym}^2 \circ \mu)$
	b	$\tau(T, \pi)$		
IX	a	$\delta(\nu\xi, \nu^{-1/2}\pi)$	ξ unr. : $a(\mathrm{Sym}^2 \circ \mu) + 2$	$\xi(-1)\varepsilon(\frac{1}{2}, \mathrm{Sym}^2 \circ \mu)$
			ξ ram. : $2a(\xi) + a(\mathrm{Sym}^2 \circ \mu)$	
	b	$L(\nu\xi, \nu^{-1/2}\pi)$	$2a(\xi) + a(\mathrm{Sym}^2 \circ \mu)$	$\xi(-1)\varepsilon(\frac{1}{2}, \mathrm{Sym}^2 \circ \mu)$
X		$\pi \rtimes \sigma$	$2a(\mu)$	$\det(\mu)(-1)$
XI	a	$\delta(\nu^{1/2}\pi, \nu^{-1/2}\sigma)$	$2a(\mu)$	1
	b	$L(\nu^{1/2}\pi, \nu^{-1/2}\sigma)$	$2a(\mu)$	1

$\varphi = (\rho, N)$ is the L-parameter of the representation, as listed in Table 3.1. The homomorphism $\rho_5 : \mathrm{GSp}(4, \mathbb{C}) \to \mathrm{SO}(5, \mathbb{C})$ is a five-dimensional representation of the L-group that induces an isomorphism $\mathrm{PGSp}(4, \mathbb{C}) \cong \mathrm{SO}(5, \mathbb{C})$. In this table we do not require that the representations have trivial central character.

4 Jacquet modules

4.1 Jacquet modules: The Siegel parabolic

		representation	s.s.$(s_2(\pi))$	#
I		$\chi_1\times\chi_2\rtimes\sigma$ (irreducible)	$(\chi_1\times\chi_2)\otimes\sigma+(\chi_1^{-1}\times\chi_2^{-1})\otimes\chi_1\chi_2\sigma$ $+(\chi_1\times\chi_2^{-1})\otimes\chi_2\sigma+(\chi_2\times\chi_1^{-1})\otimes\chi_1\sigma$	4
II	a	$\chi\mathrm{St}_{\mathrm{GL}(2)}\rtimes\sigma$	$\chi\mathrm{St}_{\mathrm{GL}(2)}\otimes\sigma+\chi^{-1}\mathrm{St}_{\mathrm{GL}(2)}\otimes\chi^2\sigma$ $+(\chi\nu^{1/2}\times\chi^{-1}\nu^{1/2})\otimes\chi\nu^{-1/2}\sigma$	3
	b	$\chi\mathbf{1}_{\mathrm{GL}(2)}\rtimes\sigma$	$\chi\mathbf{1}_{\mathrm{GL}(2)}\otimes\sigma+\chi^{-1}\mathbf{1}_{\mathrm{GL}(2)}\otimes\chi^2\sigma$ $+(\chi\nu^{-1/2}\times\chi^{-1}\nu^{-1/2})\otimes\chi\nu^{1/2}\sigma$	3
III	a	$\chi\rtimes\sigma\mathrm{St}_{\mathrm{GSp}(2)}$	$(\chi\times\nu)\otimes\sigma\nu^{-1/2}+(\nu\times\chi^{-1})\otimes\chi\sigma\nu^{-1/2}$	2
	b	$\chi\rtimes\sigma\mathbf{1}_{\mathrm{GSp}(2)}$	$(\chi\times\nu^{-1})\otimes\sigma\nu^{1/2}+(\nu^{-1}\times\chi^{-1})\otimes\chi\sigma\nu^{1/2}$	2
IV	a	$\sigma\mathrm{St}_{\mathrm{GSp}(4)}$	$\nu^{3/2}\mathrm{St}_{\mathrm{GL}(2)}\otimes\nu^{-3/2}\sigma$	1
	b	$L(\nu^2,\nu^{-1}\sigma\mathrm{St}_{\mathrm{GSp}(2)})$	$\nu^{3/2}\mathbf{1}_{\mathrm{GL}(2)}\otimes\nu^{-3/2}\sigma+(\nu\times\nu^{-2})\otimes\nu^{1/2}\sigma$	2
	c	$L(\nu^{3/2}\mathrm{St}_{\mathrm{GL}(2)},\nu^{-3/2}\sigma)$	$\nu^{-3/2}\mathrm{St}_{\mathrm{GL}(2)}\otimes\nu^{3/2}\sigma+(\nu^2\times\nu^{-1})\otimes\nu^{-1/2}\sigma$	2
	d	$\sigma\mathbf{1}_{\mathrm{GSp}(4)}$	$\nu^{-3/2}\mathbf{1}_{\mathrm{GL}(2)}\otimes\nu^{3/2}\sigma$	1
V	a	$\delta([\xi,\nu\xi],\nu^{-1/2}\sigma)$	$\nu^{1/2}\xi\mathrm{St}_{\mathrm{GL}(2)}\otimes\nu^{-1/2}\sigma+\nu^{1/2}\xi\mathrm{St}_{\mathrm{GL}(2)}\otimes\xi\nu^{-1/2}\sigma$	2
	b	$L(\nu^{1/2}\xi\mathrm{St}_{\mathrm{GL}(2)},\nu^{-1/2}\sigma)$	$\nu^{-1/2}\xi\mathrm{St}_{\mathrm{GL}(2)}\otimes\nu^{1/2}\sigma+\nu^{1/2}\xi\mathbf{1}_{\mathrm{GL}(2)}\otimes\xi\nu^{-1/2}\sigma$	2
	c	$L(\nu^{1/2}\xi\mathrm{St}_{\mathrm{GL}(2)},\xi\nu^{-1/2}\sigma)$	$\nu^{-1/2}\xi\mathrm{St}_{\mathrm{GL}(2)}\otimes\xi\nu^{1/2}\sigma+\nu^{1/2}\xi\mathbf{1}_{\mathrm{GL}(2)}\otimes\nu^{-1/2}\sigma$	2
	d	$L(\nu\xi,\xi\rtimes\nu^{-1/2}\sigma)$	$\nu^{-1/2}\xi\mathbf{1}_{\mathrm{GL}(2)}\otimes\xi\nu^{1/2}\sigma+\nu^{-1/2}\xi\mathbf{1}_{\mathrm{GL}(2)}\otimes\nu^{1/2}\sigma$	2
VI	a	$\tau(S,\nu^{-1/2}\sigma)$	$2\cdot(\nu^{1/2}\mathrm{St}_{\mathrm{GL}(2)}\otimes\nu^{-1/2}\sigma)+\nu^{1/2}\mathbf{1}_{\mathrm{GL}(2)}\otimes\nu^{-1/2}\sigma$	3
	b	$\tau(T,\nu^{-1/2}\sigma)$	$\nu^{1/2}\mathbf{1}_{\mathrm{GL}(2)}\otimes\nu^{-1/2}\sigma$	1
	c	$L(\nu^{1/2}\mathrm{St}_{\mathrm{GL}(2)},\nu^{-1/2}\sigma)$	$\nu^{-1/2}\mathrm{St}_{\mathrm{GL}(2)}\otimes\nu^{1/2}\sigma$	1
	d	$L(\nu,1_{F^\times}\rtimes\nu^{-1/2}\sigma)$	$2\cdot(\nu^{-1/2}\mathbf{1}_{\mathrm{GL}(2)}\otimes\nu^{1/2}\sigma)+\nu^{-1/2}\mathrm{St}_{\mathrm{GL}(2)}\otimes\nu^{1/2}\sigma$	3
VII		$\chi\rtimes\pi$	0	0
VIII	a	$\tau(S,\pi)$	0	0
	b	$\tau(T,\pi)$	0	0
IX	a	$\delta(\nu\xi,\nu^{-1/2}\pi)$	0	0
	b	$L(\nu\xi,\nu^{-1/2}\pi)$	0	0
X		$\pi\rtimes\sigma$	$\pi\otimes\sigma+\tilde{\pi}\otimes\omega_\pi\sigma$	2
XI	a	$\delta(\nu^{1/2}\pi,\nu^{-1/2}\sigma)$	$\nu^{1/2}\pi\otimes\nu^{-1/2}\sigma$	1
	b	$L(\nu^{1/2}\pi,\nu^{-1/2}\sigma)$	$\nu^{-1/2}\pi\otimes\nu^{1/2}\sigma$	1

Listed are the semisimplifications of the Jacquet modules of all non-supercuspidal representations with respect to the unipotent radical of the Siegel parabolic subgroup. The Jacquet modules are representations of $\mathrm{GL}(2,F)\times F^\times$. The last column lists the number of irreducible constituents.

4.2 Jacquet modules: The Klingen parabolic

		representation	s.s.$(s_1(\pi))$	#
I		$\chi_1 \times \chi_2 \rtimes \sigma$ (irreducible)	$\chi_1 \otimes (\chi_2 \rtimes \sigma) + \chi_2 \otimes (\chi_1 \rtimes \sigma)$ $+\chi_2^{-1} \otimes (\chi_1 \rtimes \chi_2\sigma) + \chi_1^{-1} \otimes (\chi_2 \rtimes \chi_1\sigma)$	4
II	a	$\chi \mathrm{St}_{\mathrm{GL}(2)} \rtimes \sigma$	$\chi\nu^{1/2} \otimes (\chi\nu^{-1/2} \rtimes \sigma) + \chi^{-1}\nu^{1/2} \otimes (\chi\nu^{1/2} \rtimes \chi\nu^{-1/2}\sigma)$	2
	b	$\chi \mathbf{1}_{\mathrm{GL}(2)} \rtimes \sigma$	$\chi\nu^{-1/2} \otimes (\chi\nu^{1/2} \rtimes \sigma) + \chi^{-1}\nu^{-1/2} \otimes (\chi\nu^{-1/2} \rtimes \chi\nu^{1/2}\sigma)$	2
III	a	$\chi \rtimes \sigma \mathrm{St}_{\mathrm{GSp}(2)}$	$\chi \otimes \sigma \mathrm{St}_{\mathrm{GSp}(2)} + \chi^{-1} \otimes \chi\sigma \mathrm{St}_{\mathrm{GSp}(2)}$ $+\nu \otimes (\chi \rtimes \sigma\nu^{-1/2})$	3
	b	$\chi \rtimes \sigma \mathbf{1}_{\mathrm{GSp}(2)}$	$\chi \otimes \sigma \mathbf{1}_{\mathrm{GSp}(2)} + \chi^{-1} \otimes \chi\sigma \mathbf{1}_{\mathrm{GSp}(2)}$ $+\nu^{-1} \otimes (\chi \rtimes \sigma\nu^{1/2})$	3
IV	a	$\sigma \mathrm{St}_{\mathrm{GSp}(4)}$	$\nu^2 \otimes \nu^{-1}\sigma \mathrm{St}_{\mathrm{GSp}(2)}$	1
	b	$L(\nu^2, \nu^{-1}\sigma \mathrm{St}_{\mathrm{GSp}(2)})$	$\nu^{-2} \otimes \nu\sigma \mathrm{St}_{\mathrm{GSp}(2)} + \nu \otimes (\nu^2 \rtimes \nu^{-3/2}\sigma)$	2
	c	$L(\nu^{3/2} \mathrm{St}_{\mathrm{GL}(2)}, \nu^{-3/2}\sigma)$	$\nu^2 \otimes \nu^{-1}\sigma \mathbf{1}_{\mathrm{GSp}(2)} + \nu^{-1} \otimes (\nu^2 \rtimes \nu^{-1/2}\sigma)$	2
	d	$\sigma \mathbf{1}_{\mathrm{GSp}(4)}$	$\nu^{-2} \otimes \nu\sigma \mathbf{1}_{\mathrm{GSp}(2)}$	1
V	a	$\delta([\xi, \nu\xi], \nu^{-1/2}\sigma)$	$\nu\xi \otimes (\xi \rtimes \nu^{-1/2}\sigma)$	1
	b	$L(\nu^{1/2}\xi \mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\sigma)$	$\xi \otimes (\nu\xi \rtimes \xi\nu^{-1/2}\sigma)$	1
	c	$L(\nu^{1/2}\xi \mathrm{St}_{\mathrm{GL}(2)}, \xi\nu^{-1/2}\sigma)$	$\xi \otimes (\nu\xi \rtimes \nu^{-1/2}\sigma)$	1
	d	$L(\nu\xi, \xi \rtimes \nu^{-1/2}\sigma)$	$\nu^{-1/2}\xi \otimes (\xi \rtimes \nu^{1/2}\sigma)$	1
VI	a	$\tau(S, \nu^{-1/2}\sigma)$	$\nu \otimes (1_{F^\times} \rtimes \nu^{-1/2}\sigma) + 1_{F^\times} \otimes \sigma \mathrm{St}_{\mathrm{GSp}(2)}$	2
	b	$\tau(T, \nu^{-1/2}\sigma)$	$1_{F^\times} \otimes \sigma \mathrm{St}_{\mathrm{GSp}(2)}$	1
	c	$L(\nu^{1/2} \mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\sigma)$	$1_{F^\times} \otimes \sigma \mathbf{1}_{\mathrm{GSp}(2)}$	1
	d	$L(\nu, 1_{F^\times} \rtimes \nu^{-1/2}\sigma)$	$1_{F^\times} \otimes \sigma \mathbf{1}_{\mathrm{GSp}(2)} + \nu^{-1} \otimes (1_{F^\times} \rtimes \nu^{1/2}\sigma)$	2
VII		$\chi \rtimes \pi$	$\chi \otimes \pi + \chi^{-1} \otimes \chi\pi$	2
VIII	a	$\tau(S, \pi)$	$1_{F^\times} \otimes \pi$	1
	b	$\tau(T, \pi)$	$1_{F^\times} \otimes \pi$	1
IX	a	$\delta(\nu\xi, \nu^{-1/2}\pi)$	$\nu\xi \otimes \nu^{-1/2}\pi$	1
	b	$L(\nu\xi, \nu^{-1/2}\pi)$	$\nu^{-1}\xi \otimes \nu^{1/2}\pi$	1
X		$\pi \rtimes \sigma$	0	0
XI	a	$\delta(\nu^{1/2}\pi, \nu^{-1/2}\sigma)$	0	0
	b	$L(\nu^{1/2}\pi, \nu^{-1/2}\sigma)$	0	0

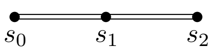
Listed are the semisimplifications of the Jacquet modules of all non-supercuspidal representations with respect to the unipotent radical of the Klingen parabolic subgroup. The Jacquet modules are representations of $F^\times \times \mathrm{GSp}(2, F)$. Note that $\mathrm{GSp}(2, F) = \mathrm{GL}(2, F)$; to translate into standard GL(2) notation, use the formula $\chi \rtimes \sigma = \chi\sigma \times \sigma$. The last column lists the number of irreducible constituents.

5 Invariant vectors

5.1 Iwahori-spherical representations: Dimensions of spaces of parahori-invariant vectors

		representation	a	$\varepsilon(1/2,\pi)$	K	$\mathrm{K}(\mathfrak{p})$	$\mathrm{Kl}(\mathfrak{p})$	$\mathrm{Si}(\mathfrak{p})$	I
I		$\chi_1 \times \chi_2 \rtimes \sigma$ (irreducible)	0	1	1	$\underset{+-}{2}$	4	$\underset{\substack{++\\--}}{4}$	$\underset{\substack{++++\\----}}{8}$
II	a	$\chi\mathrm{St}_{\mathrm{GL}(2)} \rtimes \sigma$	1	$-(\sigma\chi)(\varpi)$	0	$\underset{-}{1}$	2	$\underset{-}{1}$	$\underset{+---}{4}$
	b	$\chi\mathbf{1}_{\mathrm{GL}(2)} \rtimes \sigma$	0	1	1	$\underset{+}{1}$	2	$\underset{++-}{3}$	$\underset{+++-}{4}$
III	a	$\chi \rtimes \sigma\mathrm{St}_{\mathrm{GSp}(2)}$	2	1	0	0	1	$\underset{+-}{2}$	$\underset{++--}{4}$
	b	$\chi \rtimes \sigma\mathbf{1}_{\mathrm{GSp}(2)}$	0	1	1	$\underset{+-}{2}$	3	$\underset{+-}{2}$	$\underset{++--}{4}$
IV	a	$\sigma\mathrm{St}_{\mathrm{GSp}(4)}$	3	$-\sigma(\varpi)$	0	0	0	0	$\underset{-}{1}$
	b	$L(\nu^2, \nu^{-1}\sigma\mathrm{St}_{\mathrm{GSp}(2)})$	2	1	0	0	1	$\underset{+-}{2}$	$\underset{++-}{3}$
	c	$L(\nu^{3/2}\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-3/2}\sigma)$	1	$-\sigma(\varpi)$	0	$\underset{-}{1}$	2	$\underset{-}{1}$	$\underset{+--}{3}$
	d	$\sigma\mathbf{1}_{\mathrm{GSp}(4)}$	0	1	1	$\underset{+}{1}$	1	$\underset{+}{1}$	$\underset{+}{1}$
V	a	$\delta([\xi, \nu\xi], \nu^{-1/2}\sigma)$	2	-1	0	0	1	0	$\underset{+-}{2}$
	b	$L(\nu^{1/2}\xi\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\sigma)$	1	$\sigma(\varpi)$	0	$\underset{+}{1}$	1	$\underset{+}{1}$	$\underset{++}{2}$
	c	$L(\nu^{1/2}\xi\mathrm{St}_{\mathrm{GL}(2)}, \xi\nu^{-1/2}\sigma)$	1	$-\sigma(\varpi)$	0	$\underset{-}{1}$	1	$\underset{-}{1}$	$\underset{--}{2}$
	d	$L(\nu\xi, \xi \rtimes \nu^{-1/2}\sigma)$	0	1	1	0	1	$\underset{+-}{2}$	$\underset{+-}{2}$
VI	a	$\tau(S, \nu^{-1/2}\sigma)$	2	1	0	0	1	$\underset{-}{1}$	$\underset{+--}{3}$
	b	$\tau(T, \nu^{-1/2}\sigma)$	2	1	0	0	0	$\underset{+}{1}$	$\underset{+}{1}$
	c	$L(\nu^{1/2}\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\sigma)$	1	$-\sigma(\varpi)$	0	$\underset{-}{1}$	1	0	$\underset{-}{1}$
	d	$L(\nu, 1_{F^\times} \rtimes \nu^{-1/2}\sigma)$	0	1	1	$\underset{+}{1}$	2	$\underset{+-}{2}$	$\underset{++-}{3}$

I: The Iwahori subgroup.
$\mathrm{Si}(\mathfrak{p})$: The Siegel congruence subgroup $(= P_1)$
$\mathrm{Kl}(\mathfrak{p})$: The Klingen congruence subgroup $(= P_2)$
$\mathrm{K}(\mathfrak{p})$: The paramodular group $(= P_{02})$
K: $\mathrm{GSp}(4, \mathfrak{o})$ $(= P_{12})$
a: Exponent of the conductor of the L-parameter
ϖ: Uniformizer



The Atkin–Lehner eigenvalues listed in this table are correct if one assumes that

- in group II, where the central character is $\chi^2\sigma^2$, the character $\chi\sigma$ is trivial.
- in groups IV, V and VI, where the central character is σ^2, the character σ itself is trivial.

If these assumptions are not met, then one has to interchange all plus and minus signs.

5.2 Iwahori-spherical representations: Dimensions of spaces of paramodular vectors

		representation	a	$\varepsilon(1/2,\pi)$	$V(0)$	$V(1)$	$V(2)$	$V(3)$	$V(n)$
I		$\chi_1\times\chi_2\rtimes\sigma$ (irreducible)	0	1	$\underset{+}{1}$	$\underset{+-}{2}$	$\underset{+++-}{4}$	$\underset{\substack{+++\\---}}{6}$	$\left[\frac{(n+2)^2}{4}\right]$
II	a	$\chi\mathrm{St}_{\mathrm{GL}(2)}\rtimes\sigma$	1	$-(\sigma\chi)(\varpi)$	0	$\underset{-}{1}$	$\underset{+-}{2}$	$\underset{++--}{4}$	$\left[\frac{(n+1)^2}{4}\right]$
	b	$\chi\mathbf{1}_{\mathrm{GL}(2)}\rtimes\sigma$	0	1	$\underset{+}{1}$	$\underset{+}{1}$	$\underset{++}{2}$	$\underset{++}{2}$	$\left[\frac{n+2}{2}\right]$
III	a	$\chi\rtimes\sigma\mathrm{St}_{\mathrm{GSp}(2)}$	2	1	0	0	$\underset{+}{1}$	$\underset{+-}{2}$	$\left[\frac{n^2}{4}\right]$
	b	$\chi\rtimes\sigma\mathbf{1}_{\mathrm{GSp}(2)}$	0	1	$\underset{+}{1}$	$\underset{+-}{2}$	$\underset{++-}{3}$	$\underset{++--}{4}$	$n+1$
IV	a	$\sigma\mathrm{St}_{\mathrm{GSp}(4)}$	3	$-\sigma(\varpi)$	0	0	0	$\underset{-}{1}$	$\left[\frac{(n-1)^2}{4}\right]$
	b	$L(\nu^2,\nu^{-1}\sigma\mathrm{St}_{\mathrm{GSp}(2)})$	2	1	0	0	$\underset{+}{1}$	$\underset{+}{1}$	$\left[\frac{n}{2}\right]$
	c	$L(\nu^{3/2}\mathrm{St}_{\mathrm{GL}(2)},\nu^{-3/2}\sigma)$	1	$-\sigma(\varpi)$	0	$\underset{-}{1}$	$\underset{+-}{2}$	$\underset{+--}{3}$	n
	d	$\sigma\mathbf{1}_{\mathrm{GSp}(4)}$	0	1	$\underset{+}{1}$	$\underset{+}{1}$	$\underset{+}{1}$	$\underset{+}{1}$	1
V	a	$\delta([\xi,\nu\xi],\nu^{-1/2}\sigma)$	2	-1	0	0	$\underset{-}{1}$	$\underset{+-}{2}$	$\left[\frac{n^2}{4}\right]$
	b	$L(\nu^{1/2}\xi\mathrm{St}_{\mathrm{GL}(2)},\nu^{-1/2}\sigma)$	1	$\sigma(\varpi)$	0	$\underset{+}{1}$	$\underset{+}{1}$	$\underset{++}{2}$	$\left[\frac{n+1}{2}\right]$
	c	$L(\nu^{1/2}\xi\mathrm{St}_{\mathrm{GL}(2)},\xi\nu^{-1/2}\sigma)$	1	$-\sigma(\varpi)$	0	$\underset{-}{1}$	$\underset{+}{1}$	$\underset{--}{2}$	$\left[\frac{n+1}{2}\right]$
	d	$L(\nu\xi,\xi\rtimes\nu^{-1/2}\sigma)$	0	1	$\underset{+}{1}$	0	$\underset{+}{1}$	0	$\frac{1+(-1)^n}{2}$
VI	a	$\tau(S,\nu^{-1/2}\sigma)$	2	1	0	0	$\underset{+}{1}$	$\underset{+-}{2}$	$\left[\frac{n^2}{4}\right]$
	b	$\tau(T,\nu^{-1/2}\sigma)$	2	1	0	0	0	0	0
	c	$L(\nu^{1/2}\mathrm{St}_{\mathrm{GL}(2)},\nu^{-1/2}\sigma)$	1	$-\sigma(\varpi)$	0	$\underset{-}{1}$	$\underset{-}{1}$	$\underset{--}{2}$	$\left[\frac{n+1}{2}\right]$
	d	$L(\nu,1_{F^\times}\rtimes\nu^{-1/2}\sigma)$	0	1	$\underset{+}{1}$	$\underset{+}{1}$	$\underset{++}{2}$	$\underset{++}{2}$	$\left[\frac{n+2}{2}\right]$

Here $V(n)$ is the subspace of vectors invariant under the paramodular group of level n, defined as the set of all $g\in\mathrm{GSp}(4,F)$ such that

$$g\in\begin{bmatrix}\mathfrak{o}&\mathfrak{o}&\mathfrak{o}&\mathfrak{p}^{-n}\\ \mathfrak{p}^n&\mathfrak{o}&\mathfrak{o}&\mathfrak{o}\\ \mathfrak{p}^n&\mathfrak{o}&\mathfrak{o}&\mathfrak{o}\\ \mathfrak{p}^n&\mathfrak{p}^n&\mathfrak{p}^n&\mathfrak{o}\end{bmatrix}\qquad\text{and}\qquad\det(g)\in\mathfrak{o}^\times.$$

The Atkin–Lehner eigenvalues listed in this table are correct if one assumes that

- in group II, where the central character is $\chi^2\sigma^2$, the character $\chi\sigma$ is trivial.
- in groups IV, V and VI, where the central character is σ^2, the character σ itself is trivial.

If these assumptions are not met, then one has to interchange the plus and minus signs in the $V(1)$ and the $V(3)$ column.