An Informal Introduction to Automorphic Representations

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Preface

You are a graduate student interested in Number Theory, and generally familiar with the basic concepts. Perhaps you have seen the Riemann zeta function, and read that it is an example for what is called an "L-function". Maybe your professor mentioned that "modular forms" are really important, so you pick up a book and start reading. It looks as if this is all just a special case of something called "automorphic forms" and "automorphic representations". Sounds fascinating, but really difficult to get into. So many new theories to learn! Some of the available books look very nice, but it seems 500 pages will barely cover the background material.

These notes are written for you. They offer a few prominent examples in an attempt to demystify the subject. We won't be shy about sweeping all kinds of difficulties under the rug. You won't *really* learn the subject from here; eventually, you *do* have to master all the technicalities. But maybe you get enough of a flavor to make you want to read the more serious literature – or not.

Good luck. Let me know how it goes.

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Chapter 1

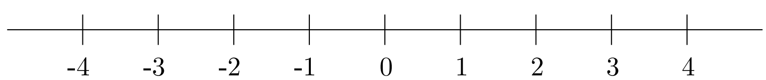
Basic Concepts

1.1 Adeles and ideles

It is impossible to talk about automorphic representations without the notions of ***adeles*** and ***ideles***, so our first order of business will be to introduce these concepts.

1.1.1 Adeles

In number theory, we study the ring of integers $\mathbb{Z}$. We may visualize the integers as equidistant points on a line:



When we do this, we really embed the ring $\mathbb{Z}$ as a discrete subring in the bigger ring $\mathbb{R}$. While $\mathbb{R}$ has many more points than $\mathbb{Z}$, it is not "too big", in the sense that the quotient $\mathbb{R}/\mathbb{Z}$ is compact. The use of $\mathbb{R}$ as a surrounding "universe" for integers introduces new possibilities. For example, $\mathbb{R}$ is a Lie group, and in particular is a locally compact topological group on which there exists a Haar measure and an integration theory. To understand $\mathbb{Z}$ better, we may then study functions on $\mathbb{R}$ that are invariant under translations by $\mathbb{Z}$ (i.e., periodic functions). This leads to the theory of Fourier series. The quotient $\mathbb{R}/\mathbb{Z}$ is in fact the ***dual group*** of $\mathbb{Z}$, in the sense that it parametrizes the ***characters*** of $\mathbb{Z}$ (homomorphisms into $\mathbb{C}^\times$).

We would like to do something similar with the rational numbers $\mathbb{Q}$, the field of fractions of $\mathbb{Z}$. Can we find a "universe" for $\mathbb{Q}$, consisting of a nice, locally compact topological ring $\mathbb{A}$ that contains $\mathbb{Q}$ as a discrete subring in a way such that $\mathbb{A}/\mathbb{Q}$ is compact? This is indeed possible, and this ring $\mathbb{A}$ is called the ring of ***adeles*** of $\mathbb{Q}$. Here is how it is constructed.

First, given a prime p, recall the definition of the p-adic numbers $\mathbb{Q}_p$. The ***p-adic absolute value*** $|\cdot|_p$ on $\mathbb{Q}$ is defined by

$$|x|_p = p^{-r}, \qquad \text{where } x = \frac{a}{b}\,p^r \text{ with } a, b, r \in \mathbb{Z},\ p \nmid ab, \tag{1.1}$$

for a non-zero $x \in \mathbb{Q}$, and $|x|_p = 0$ for $x = 0$. Then $\mathbb{Q}_p$ is the ***completion*** of $\mathbb{Q}$ with respect to $|\cdot|_p$ (constructed as Cauchy sequences modulo null sequences). This is analogous to $\mathbb{R}$ being the completion of $\mathbb{Q}$ with respect to the usual (archimedean) absolute value. In this context the latter is often denoted by $|\cdot|_\infty$, and $\mathbb{R}$ is denoted by $\mathbb{Q}_\infty$. One can show that, up to a natural notion of equivalence, the $|\cdot|_p$, for all primes p, and $|\cdot|_\infty$, are all the absolute values admitted by $\mathbb{Q}$ (Ostrowski's theorem). For a prime p, the ring of ***p-adic integers*** $\mathbb{Z}_p$ consists of all $x \in \mathbb{Q}_p$ with $|x|_p \leq 1$. It is easy to see that $\mathbb{Z}_p$ is a compact and open subset of $\mathbb{Q}_p$.

The ring of adeles is going to be defined as a subring of the product of all completions,

$$\mathbb{A} \subset \prod_{p \leq \infty} \mathbb{Q}_p = \mathbb{R} \times \mathbb{Q}_2 \times \mathbb{Q}_3 \times \mathbb{Q}_5 \times \ldots .$$

Since $\mathbb{Q}$ is a subset of each completion, we have a diagonal embedding $x \mapsto (x, x, \ldots)$ of $\mathbb{Q}$ into this direct product. However, the full direct product is too big and does not have the desired properties. To see why the direct product is unnecessarily large, note that a given $x \in \mathbb{Q}$ lies in $\mathbb{Z}_p$ for almost all p (because only finitely many primes "participate" in x). We therefore define

$$\mathbb{A} = \Big\{ a = (a_p)_{p \leq \infty} \in \prod_{p \leq \infty} \mathbb{Q}_p \;\Big|\; a_p \in \mathbb{Z}_p \text{ for almost all } p \Big\}. \tag{1.2}$$

Evidently, $\mathbb{A}$ is a subring of the full direct product. We call $\mathbb{A}$ the ***restricted direct product*** of the $\mathbb{Q}_p$ (with respect to the open-compact subsets $\mathbb{Z}_p$ for $p < \infty$) and write

$$\mathbb{A} = \prod_{p \leq \infty}{}' \mathbb{Q}_p. \tag{1.3}$$

The natural topology on such a restricted direct product is *not* the subspace topology inherited from the full direct product. Instead, we define the topology on $\mathbb{A}$ to be the one generated by the sets

$$\prod_{p \leq \infty} U_p, \qquad U_p \text{ open in } \mathbb{Q}_p,\ U_p = \mathbb{Z}_p \text{ for almost all } p. \tag{1.4}$$

With this ***restricted direct product topology***, $\mathbb{A}$ becomes a locally compact topological ring. As we saw, $\mathbb{A}$ is large enough to contain $\mathbb{Q}$ diagonally embedded. It is not difficult to see that $\mathbb{Q}$ is discrete and that $\mathbb{A}/\mathbb{Q}$ is compact. The adeles turn out to be the correct "universe" in which to study rational numbers. The fact that $\mathbb{A}$ is made up of all completions nicely captures the archimedean as well as the p-adic aspects of $\mathbb{Q}$.

The fact that $\mathbb{A}/\mathbb{Q}$ is compact can be made more precise. It is easy to see that, for $x \in \mathbb{A}$, there exists $\gamma \in \mathbb{Q}$ such that

$$x - \gamma \in \mathbb{R} \times \prod_{p < \infty} \mathbb{Z}_p$$

By further adding an appropriate integer, one may even assume that the archimedean component of $x - \gamma$ lies in $[0, 1)$. Requiring this condition, γ is uniquely determined. This way one obtains a topological group isomorphism

$$\mathbb{A}/\mathbb{Q} \cong \mathbb{R}/\mathbb{Z} \times \prod_{p < \infty} \mathbb{Z}_p. \tag{1.5}$$

By Tychonoff, the group on the right is compact. The set $[0, 1) \times \prod \mathbb{Z}_p$ is a convenient ***fundamental domain*** for $\mathbb{A}/\mathbb{Q}$. For example, a function $f : \mathbb{A} \to \mathbb{C}$ with the property

$f(x+\gamma+\kappa)=f(x)$ for $\gamma\in\mathbb{Q}$ and $\kappa\in\prod\mathbb{Z}_p$ may be integrated as follows,

$$\int\limits_{\mathbb{A}/\mathbb{Q}} f(x)\,dx=\int\limits_0^1 f(x)\,dx. \tag{1.6}$$

Here, the x variable on the left is an adele, but the x variable on the right is a real number (viewed as an adele with all non-archimedean components equal to zero). This way adelic integration often gets reduced to integration familiar from Calculus.

1.1.2 Ideles

Let us also introduce a multiplicative version of the adeles. This multiplicative version is called the ***ideles*** and is obtained by replacing all rings in the above construction by their respective unit groups. Hence, we define the group of ideles by

$$\mathbb{I}=\Big\{a=(a_p)_{p\le\infty}\in\prod_{p\le\infty}\mathbb{Q}_p^\times\;\Big|\;a_p\in\mathbb{Z}_p^\times\text{ for almost all }p\Big\}. \tag{1.7}$$

This is the restricted direct product of the $\mathbb{Q}_p^\times$ with respect to the $\mathbb{Z}_p^\times$. The topology on $\mathbb{I}$ is the one generated by the sets

$$\prod_{p\le\infty}U_p,\qquad U_p\text{ open in }\mathbb{Q}_p^\times,\ U_p=\mathbb{Z}_p^\times\text{ for almost all }p. \tag{1.8}$$

With this topology $\mathbb{I}$ becomes a locally compact topological group. Note that, algebraically, $\mathbb{I}$ is the unit group of $\mathbb{A}$, which is why often people write $\mathbb{A}^\times$ for $\mathbb{I}$. However, the topology on $\mathbb{I}$ is *not* the subspace topology inherited from $\mathbb{A}$.

Just as in the additive case, we may embed $\mathbb{Q}^\times$ diagonally into $\mathbb{I}$. Then $\mathbb{Q}^\times$ is a discrete subgroup of $\mathbb{I}$. The quotient $\mathbb{I}/\mathbb{Q}^\times$ is no longer compact, but not too far away from that. A theorem called ***strong approximation*** implies that

$$\mathbb{I}=\mathbb{Q}^\times\times\mathbb{R}_{>0}\times\prod_{p<\infty}\mathbb{Z}_p^\times. \tag{1.9}$$

Note that, among the subgroups of $\mathbb{I}$ on the right, $\mathbb{Q}^\times$ is diagonally embedded, while $\mathbb{R}_{>0}$ and $\mathbb{Z}_p^\times$ are locally embedded at their respective places. We see from (1.9) that $\mathbb{I}/\mathbb{Q}^\times$ is not compact, but $\mathbb{I}/(\mathbb{Q}^\times\mathbb{R}_{>0})$ is. The fundamental domain for $\mathbb{I}/\mathbb{Q}^\times$ resulting from (1.9) is a multiplicative version of (1.5).

1.1.1 Exercise. *Prove* (1.9).

The ideles are really an example for an ***adelized matrix group***. To explain this, note that we could have written the definition (1.7) as

$$\mathbb{I}=\Big\{a=(a_p)_{p\le\infty}\in\prod_{p\le\infty}\mathrm{GL}(1,\mathbb{Q}_p)\;\Big|\;a_p\in\mathrm{GL}(1,\mathbb{Z}_p)\text{ for almost all }p\Big\}. \tag{1.10}$$

Here, we consider the matrix group $\mathrm{GL}(1)$ (a very simple matrix group: 1×1 invertible matrices). Into this group we can simply "plug in" any commutative ring R (what we are really doing is considering the R-rational points of an algebraic variety called $\mathrm{GL}(1)$). If we replace $\mathrm{GL}(1)$ by another appropriate matrix group G, like $\mathrm{GL}(n)$, then instead of the ideles we obtain the ***adelization of*** G, denoted by $G(\mathbb{A})$. Hence, $G(\mathbb{A})$ is the restricted direct product of the groups $G(\mathbb{Q}_p)$ (for all p) with respect to the subgroups $G(\mathbb{Z}_p)$ (for all finite p). The topology on $G(\mathbb{A})$ is defined analogously to (1.8). We obtain a locally compact topological group, which will always contain the $\mathbb{Q}$-points $G(\mathbb{Q})$ as a discrete subgroup via the diagonal embedding.

Remarks:

i) Adeles and ideles were introduced by *Claude Chevalley*. He used the term "ideal element", abbreviated as "id. el.". The term adele derives from "additive idele".

ii) One of the best introductions into restricted direct products is still Sect. 3 of [19].

iii) The group $\mathbb{I}/\mathbb{Q}^\times$ is called the group of ***idele classes*** and plays a central role in ***global class field theory***. See Sects. VII.3 and X.3 of [11].

FAQ

- *I know what* $\mathrm{GL}(n, R)$ *means for a commutative ring* R*: It means invertible* $n \times n$ *matrices with entries in* R*. But I am still confused what the symbol* $\mathrm{GL}(n)$ *without any argument means. Is it an actual mathematical object, or is it just a convenient symbol that only becomes a "thing" when we "plug in" a ring* R *as its argument?*

  $\mathrm{GL}(n)$ is in fact an actual mathematical object, namely a *linear algebraic group*. Basically this means that it is a matrix group defined by polynomial conditions (in this case determinant $\neq 0$) such that the group operations are also polynomial. When viewed in this conceptual way, one should really say "$\mathrm{GL}(n)$ over $\mathbb{Q}$", or another field F instead of $\mathbb{Q}$, to indicate that the polynomials have coefficients in this field, and that the entries of the matrix come from the algebraic closure $\bar{F}$ of this field. You can look at a good book like [7] to learn more about algebraic groups. For the purpose of these notes, the naive view of the symbol "$\mathrm{GL}(n)$" as something that only becomes incarnated once we "plug in" a ring R is perfectly sufficient[1].

- *Some people write* GL_n *instead of* $\mathrm{GL}(n)$ *and* $\mathrm{GL}_n(R)$ *instead of* $\mathrm{GL}(n, R)$*. Is it the same thing?*

  Yes. These are just alternative notations. There is no difference.

1.1.3 Our first automorphic representation

As usual, a ***representation*** of a group G is a linear action of G on a vector space V. Equivalently, a representation of G is a homomorphism $\pi : G \to \mathrm{GL}(V)$. Even though, strictly speaking, a representation is really a pair (π, V), we will often say "the representation π" or

[1] Avoid the temptation of writing $\mathrm{GL}(n)$ when you really mean $\mathrm{GL}(n, R)$ for a ring R; it is an abuse of notation!

even "the representation V". Another convenient abuse of terminology is to talk of "a vector v in π", when we really mean a vector v in the space V of the representation.

In automorphic representation theory, V will always be a *complex* vector space, and G will be the adelization of a matrix group, as explained in the previous section. We agree that all automorphic representations (to be defined later) have to be ***irreducible***. As usual, irreducible means that there are no G-invariant subspaces other than 0 and V.

The simplest case is that of the ideles. Since this is an *abelian* group, every *irreducible* representation will be one-dimensional. Therefore, an automorphic representation will be a ***character*** of $\mathbb{I}$, i.e., a group homomorphism $\chi : \mathbb{I} \to \mathbb{C}^\times$. We impose the condition that χ be continuous.

But not every such χ is automorphic. Possibly motivated by the importance of idele classes in global class field theory, we say that χ *is automorphic if it is trivial on* $\mathbb{Q}^\times$. Hence, an ***automorphic representation of*** GL(1) will be a continuous character $\chi : \mathbb{I}/\mathbb{Q}^\times \to \mathbb{C}^\times$.

How can we construct characters of $\mathbb{I}$ in the first place? In view of the definition (1.7), we could simply take local characters $\chi_p : \mathbb{Q}_p^\times \to \mathbb{C}^\times$ at each place, and define

$$\chi(a) = \prod_{p\le\infty} \chi_p(a_p), \qquad a = (a_p)_{p\le\infty} \in \mathbb{I}.$$

To make this well-defined, we have to assume that χ_p is ***unramified***, i.e., trivial on $\mathbb{Z}_p^\times$ for almost all p. Then the above product will be finite, and χ is indeed a continuous character of $\mathbb{I}$. We write $\chi = \otimes'_{p\le\infty}\chi_p$, or more loosely $\chi = \otimes\chi_p$, and say that χ is the ***restricted tensor product*** of the local characters χ_p.

If we randomly choose local characters χ_p, then $\chi = \otimes\chi_p$ is very unlikely to be automorphic. There is one choice however, familiar from basic number theory, that leads to an automorphic character. This is if we let χ_p be the normalized absolute value $|\cdot|_p$ on $\mathbb{Q}_p^\times$, defined in (1.1) for finite p, and understood to be the usual absolute value on $\mathbb{R}^\times$ for $p = \infty$. Since $|\cdot|_p$ is unramified for all finite p, we can form the ***global absolute value*** $|\cdot| = \otimes|\cdot|_p$. Explicitly,

$$|a| = \prod_{p\le\infty} |a_p|_p, \qquad a = (a_p)_{p\le\infty} \in \mathbb{I}. \tag{1.11}$$

The well-known ***product formula***, which is easy to verify, states that

$$\prod_{p\le\infty} |a|_p = 1 \qquad \text{for all } a \in \mathbb{Q}^\times. \tag{1.12}$$

By our definitions, this formula means precisely that the global absolute value is automorphic. We found our first automorphic representation.

Thanks to the structure theorem (1.9), we can actually determine *all* automorphic representations of $\mathbb{I}$. Let us work this out. By (1.9),

$$\mathbb{I}/\mathbb{Q}^\times \cong \mathbb{R}_{>0} \times \prod_{p<\infty} \mathbb{Z}_p^\times. \tag{1.13}$$

The global absolute value is the identity map on $\mathbb{R}_{>0}$ and is trivial on $\prod_{p<\infty} \mathbb{Z}_p^\times$. Hence, the most general automorphic representation trivial on $\prod_{p<\infty} \mathbb{Z}_p^\times$ is $|\cdot|^s$ for some complex number s. It follows that, after multiplying by an appropriate power of $|\cdot|$, a given automorphic representation $\chi = \otimes\chi_p$ is really just a character of $\prod_{p<\infty}\mathbb{Z}_p^\times$. But recall that χ_p is unramified for almost all p. Let S be the set of primes for which χ_p is *not* unramified. We see that χ is determined on

$$\prod_{p\in S}\mathbb{Z}_p^\times. \tag{1.14}$$

For $p \in S$, since χ_p is continuous, there exists a positive integer n_p such that χ_p is trivial on the subgroup $1+p^{n_p}\mathbb{Z}_p$. We choose the minimal n_p with this property. Then χ descends to a character

$$\prod_{p\in S}\mathbb{Z}_p^\times/(1+p^{n_p}\mathbb{Z}_p) \longrightarrow \mathbb{C}^\times. \tag{1.15}$$

The group $\mathbb{Z}_p^\times/(1+p^{n_p}\mathbb{Z}_p)$ is isomorphic to $(\mathbb{Z}_p/p^{n_p}\mathbb{Z}_p)^\times$, and hence to $(\mathbb{Z}/p^{n_p}\mathbb{Z})^\times$. We see that χ induces a character

$$\prod_{p\in S}(\mathbb{Z}/p^{n_p}\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times. \tag{1.16}$$

By the Chinese Remainder Theorem, the group on the left is isomorphic to $(\mathbb{Z}/N\mathbb{Z})^\times$, where $N = \prod_{p\in S} p^{n_p}$. Combining the previous steps, we see that χ determines in a natural way a ***Dirichlet character***

$$\tilde{\chi}:\ (\mathbb{Z}/N\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times. \tag{1.17}$$

Since we chose the n_p to be *minimal*, this Dirichlet character is ***primitive***, i.e., it does not factor through the projection $(\mathbb{Z}/N\mathbb{Z})^\times \to (\mathbb{Z}/M\mathbb{Z})^\times$ for any proper divisor M of N.

We can reverse the steps that lead from χ to $\tilde{\chi}$. Given a primitive Dirichlet character $\tilde{\chi}$ modulo some positive integer N, decompose N into its prime factors and use the Chinese Remainder Theorem to obtain a character as in (1.16). Then use $\mathbb{Z}_p^\times/(1+p^{n_p}\mathbb{Z}_p) \cong (\mathbb{Z}/p^{n_p}\mathbb{Z})^\times$ to obtain a character as in (1.15). Inflate it to a character of the group (1.14), and then further to a character of $\mathbb{I}/\mathbb{Q}^\times$ via (1.13). These arguments prove the following result.

1.1.2 Proposition. *There is a natural one-one correspondence between primitive Dirichlet characters and characters of the group* $\mathbb{I}/(\mathbb{Q}^\times\mathbb{R}_{>0})$*. Every automorphic representation of* $\mathbb{I} = \mathrm{GL}(1,\mathbb{A})$ *is of the form* $|\cdot|^s\chi$*, where* $|\cdot|$ *is the global absolute value,* s *is a complex number, and* χ *corresponds to a primitive Dirichlet character. The number* s *and the character* χ *are uniquely determined.*

We see that automorphic representations of GL(1) correspond to classical number theoretic objects, namely Dirichlet characters. There are many instances where automorphic representations correspond to more familiar classical objects. We will see this in the next section when we consider automorphic representations of GL(2).

FAQ

- *Shouldn't you have written "automorphic representations of* $\mathrm{GL}(1,\mathbb{A})$*" instead of "automorphic representations of* $\mathrm{GL}(1)$*"?*

  Maybe, but in this case the abuse of notation is actually not uncommon. The word "automorphic" implies that we are talking about an *adelized* group; for example, "automorphic representation of $\mathrm{GL}(1, \mathbb{Q}_p)$" wouldn't make any sense. People also say "automorphic representation *on* $\mathrm{GL}(1)$".

- *I've also seen "automorphic representation of* $\mathrm{GL}(n, \mathbb{A}_{\mathbb{Q}})$*". What does the subindex* $\mathbb{Q}$ *mean?*

  When it comes to automorphic representations, there is always a base field. We haven't stressed this point, since for now the base field will always be $\mathbb{Q}$. But with a little more effort one can develop the whole theory over any algebraic number field (finite extension of $\mathbb{Q}$). For such a field F one can define adeles $\mathbb{A}_F$ and adelized algebraic groups like $\mathrm{GL}(n, \mathbb{A}_F)$. Remember that in our FAQ on page 8 we mentioned that $\mathrm{GL}(n)$ really means a linear algebraic group defined over a field F. This is the base field - the field over which the algebraic group is defined.

- *Why "automorphic"? What does this word mean?*

  From Greek αὐτός "self" and μορφή "shape", hence "self-shaped" or "patterned after itself". Let's think of it as follows: Let χ be any character of $\mathbb{I}$. For $\gamma \in \mathbb{Q}^\times$, we can define a new character χ^γ by $\chi^\gamma(a) = \chi(\gamma a)$. Obviously, χ is automorphic if and only if $\chi^\gamma = \chi$ for all $\gamma \in \mathbb{Q}^\times$. Hence, automorphy means the character doesn't change under this action of $\mathbb{Q}^\times$. Trying to change the "shape" of χ by defining a new character χ^γ leads back to χ itself.

- *I know the factor of automorphy from the theory of modular forms. Why are we using the same word "automorphic"? Are these instances of the same general phenomenon?*

  We will talk about modular forms in Sect. 1.5.3 further below. What you mean with factor of automorphy is the factor $(c\tau + d)^{-k}$ in (1.159). Only if the slash operation (1.158) is defined with this factor present is it true that $F\big|_k \gamma = F$ for $\gamma \in \mathrm{SL}(2, \mathbb{Z})$ and F a modular form of weight k. We see the analogy with the operation $\chi \mapsto \chi^\gamma$ from the previous question: Trying to produce a new function by looking at $F\big|_k \gamma$ leads back to F itself. In this sense "F is automorphic". This only works if we throw in the factor $(c\tau + d)^{-k}$, hence we call it a "factor of automorphy". Note also that, in both cases, the γ's are *rational* objects: elements of $\mathrm{GL}(1, \mathbb{Q})$ resp. $\mathrm{SL}(2, \mathbb{Z})$. Automorphy always refers to invariance under rational elements of a group.

- *Wait - you just said rational elements.* $\mathrm{SL}(2, \mathbb{Z})$ *consists of integral elements, not rational elements. Why don't we see the group* $\mathrm{SL}(2, \mathbb{Q})$ *in the definition of modular forms?*

  Well, this is because we are working classically, not adelically. We will see in Proposition 1.5.3 that classical modular forms correspond to functions on $\mathrm{GL}(2, \mathbb{A})$ that are left $\mathrm{GL}(2, \mathbb{Q})$ invariant - hence invariant under rational elements of the bigger group. This is the essence of automorphy. Once we work with functions on the upper half plane, the bigger group gets replaced by $\mathrm{SL}(2, \mathbb{R})$ and the smaller group by $\mathrm{SL}(2, \mathbb{Z})$. Recall that $\mathbb{Q}$ in $\mathbb{A}$ is analogous to $\mathbb{Z}$ in $\mathbb{R}$.

1.2 Real analytic Eisenstein series

In this section we consider the adelized group $\mathrm{GL}(2, \mathbb{A})$ and a very simple kind of ***automorphic representation*** of this group. We will see how this simple example is closely connected to a well-known *classical* object called the ***real analytic Eisenstein series***.

1.2.1 Global parabolic induction

The previous chapter was all about GL(1). Let us try to generalize the notion of automorphic representation to GL(2). Thus, we consider the adelized group GL(2). To define this group, recall that all we have to do is replace GL(1) by GL(2) in (1.10), i.e.,

$$\mathrm{GL}(2, \mathbb{A}) = \Big\{ g = (g_p)_{p \le \infty} \in \prod_{p \le \infty} \mathrm{GL}(2, \mathbb{Q}_p) \,\Big|\, g_p \in \mathrm{GL}(2, \mathbb{Z}_p) \text{ for almost all } p \Big\}. \tag{1.18}$$

To get first examples of representations of $\mathrm{GL}(2, \mathbb{A})$, we will apply the method of ***induction***. In general, if G is a group and B is a subgroup, induction is a way to produce representations of G out of representations of B. For our current purposes it is enough to consider one-dimensional representations of B, i.e., characters $\chi : B \to \mathbb{C}^\times$. Given such a χ, the representation of G induced by χ has representation space

$$V = \{ \varphi : G \to \mathbb{C} \mid \varphi(bg) = \chi(b)\varphi(g) \text{ for all } g \in G \text{ and } b \in B \}. \tag{1.19}$$

Hence, V consists of functions on G with a left transformation property. On this space the group G acts by ***right translation***. This means that we define a representation ρ of G on V by

$$(\rho(g)\varphi)(h) = \varphi(hg) \qquad \text{for all } \varphi \in V \text{ and } g, h \in G. \tag{1.20}$$

One can easily check that ρ is indeed a representation of G. We write

$$\rho = \mathrm{ind}_B^G(\chi), \tag{1.21}$$

and call it the representation of G induced by χ.

In the case of $\mathrm{GL}(2, \mathbb{A})$, what subgroup B should we induce from? Here are two properties that B should have:

- B should be defined by *algebraic conditions* on the matrix entries, so that we can form the local groups $B(\mathbb{Q}_p)$ and the adelized group $B(\mathbb{A})$.
- B should be as large as possible, so that induction has a better chance of producing *irreducible* representations.

There are not too many groups satisfying the first condition that come to mind. For example, we could take upper triangular matrices, lower triangular matrices, diagonal matrices, or upper triangular matrices with 1's on the diagonal. Among these, upper triangular matrices are maximal, so this is what we are going to take for B. We could as well take lower triangular matrices, but these two subgroups are conjugate, so it won't give anything essentially different.

So we are going to induce from a character of

$$B(\mathbb{A}) = \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \;\Big|\; a, d \in \mathbb{I},\ b \in \mathbb{A} \right\}. \tag{1.22}$$

The use of the letter "B" for this group is very common and is a reference to *Armand Borel*. What do characters of this group look like? It is an exercise to prove that every such character χ must be trivial on the subgroup

$$N(\mathbb{A}) = \left\{ \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \;\Big|\; b \in \mathbb{A} \right\}. \tag{1.23}$$

Hence, χ must be of the form

$$\chi(\begin{bmatrix} a & b \\ & d \end{bmatrix}) = \chi_1(a)\chi_2(d) \qquad \text{for all } a, d \in \mathbb{I},\ b \in \mathbb{A}, \tag{1.24}$$

where χ_1, χ_2 are characters of $\mathbb{I}$. Conversely, any choice of characters χ_1, χ_2 of the ideles defines a character χ of $B(\mathbb{A})$ via (1.24).

Next we have to decide on a choice for χ_1 and χ_2. We will employ the following

> **General Principle:** *Induction from something automorphic will produce something automorphic.* (1.25)

Hence, for χ_1 and χ_2 we should take automorphic characters, i.e., characters of $\mathbb{I}/\mathbb{Q}^\times$. We saw that the simplest such character is the global absolute value and its powers. Therefore we will make the following choice,

$$\chi_1 = |\cdot|^{s_1}, \qquad \chi_2 = |\cdot|^{s_2}, \qquad s_1, s_2 \in \mathbb{C}.$$

In order to simplify matters even further, let us assume that χ_2 is the inverse of χ_1; this has the effect that the character χ defined by (1.24) is trivial on scalar matrices $\mathrm{diag}(a, a)$. Hence, we will assume that $s_2 = -s_1$. Writing $s/2$ instead of s_1, we then have

$$\chi_1 = |\cdot|^{s/2}, \qquad \chi_2 = |\cdot|^{-s/2}, \qquad s \in \mathbb{C}.$$

According to the definition (1.19), the space of the resulting induced representation consists of functions $\varphi :\ \mathrm{GL}(2, \mathbb{A}) \to \mathbb{C}$ with the transformation property

$$\varphi(\begin{bmatrix} a & b \\ & d \end{bmatrix} g) = \left|\frac{a}{d}\right|^{s/2} \varphi(g) \qquad \text{for all } g \in \mathrm{GL}(2, \mathbb{A}),\ a, d \in \mathbb{I},\ b \in \mathbb{A} \tag{1.26}$$

(and additional regularity properties, which we will for the moment ignore). Now we are going to do something that seems strange at first. Instead of (1.26), we consider functions with the transformation property

$$\varphi(\begin{bmatrix} a & b \\ & d \end{bmatrix} g) = \left|\frac{a}{d}\right|^{(s+1)/2} \varphi(g) \qquad \text{for all } g \in \mathrm{GL}(2, \mathbb{A}),\ a, d \in \mathbb{I},\ b \in \mathbb{A}. \tag{1.27}$$

The introduction of the additional factor $|a/d|^{1/2}$ is a certain normalization that we will talk about more. For now we can think of it as a simple shift in s that will lead to better formulas later on.

We note for later use that the transformation property (1.27) implies that

$$\varphi(\begin{bmatrix} a & \\ & a \end{bmatrix} g) = \varphi(g) \qquad \text{for all } g \in \mathrm{GL}(2, \mathbb{A}),\ a \in \mathbb{I}. \tag{1.28}$$

We say that φ is ***invariant under the center***, observing that the center of $\mathrm{GL}(2, \mathbb{A})$ consists of all scalar matrices $\mathrm{diag}(a, a)$ with $a \in \mathbb{I}$.

Now let $V(s)$ be the space of functions φ with the property (1.27). In order for $V(s)$ not to be too large of a space, we should also impose certain regularity conditions on the φ's. For example, at the very least we want all φ's to be continuous. However, we will be mostly concerned with *one particular element* of $V(s)$, and this element will be part of the space no matter what regularity conditions we impose (within reason). Hence, we will not further specify the precise conditions at this point.

According to the principle (1.25), we just constructed an automorphic representation $V(s)$. However, it is not clear what this *means*. In order to proceed further, we will take a close look at the one particular element of $V(s)$ already alluded to. Finding one particularly nice element in an otherwise infinite-dimensional space is in fact an important principle in automorphic representation theory. For the case of $V(s)$, finding this vector will be the topic of the next section.

1.2.2 A distinguished vector

The distinguished vector will be a particularly "simple" function in $V(s)$, which we will call φ_0. The existence of such a function is almost obvious from the ***Iwasawa decomposition***. This decomposition states that GL(2) can be written as "upper triangular times maximal compact", and it works over any local field. To state this precisely, let B denote the subgroup of GL(2) consisting of upper triangular matrices (over whatever commutative ring we are working with). Then, for a prime number p, we have

$$\mathrm{GL}(2, \mathbb{Q}_p) = B(\mathbb{Q}_p) \cdot K_p, \qquad \text{where } K_p = \mathrm{GL}(2, \mathbb{Z}_p), \tag{1.29}$$

and for $p = \infty$ we have

$$\mathrm{GL}(2, \mathbb{R}) = B(\mathbb{R}) \cdot K_\infty, \qquad \text{where } K_\infty = \mathrm{SO}(2). \tag{1.30}$$

The Iwasawa decomposition holds for much more general groups, but for GL(2) it is not a deep theorem:

1.2.1 Exercise. *Prove* (1.29) *and* (1.30).

Recall that SO(2) consists of all $g \in \mathrm{SL}(2, \mathbb{R})$ with ${}^tgg = 1_2$, the 2×2 identity matrix. The elements of SO(2) can be explicitly written as

$$r(\theta) = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \qquad \text{with } \theta \in \mathbb{R}. \tag{1.31}$$

Hence, $\mathrm{SO}(2)$ is an abelian group isomorphic to $\mathbb{R}/2\pi\mathbb{Z}$. (This is actually *not* a maximal compact subgroup of $\mathrm{GL}(2,\mathbb{R})$, but $\mathrm{O}(2)$ is. The difference between $\mathrm{SO}(2)$ and $\mathrm{O}(2)$ is absorbed into $B(\mathbb{R})$.)

Note that the decomposition of an element of $\mathrm{GL}(2,\mathbb{Q}_p)$ according to (1.29) and (1.30) is not quite unique: in either case there is some overlap between $B(\mathbb{Q}_p)$ and K_p.

1.2.2 Exercise. *Work out precisely what $B(\mathbb{Q}_p)\cap K_p$ is.*

Combining the local Iwasawa decompositions at all places, we obtain the ***global Iwasawa decomposition***

$$\mathrm{GL}(2,\mathbb{A}) = B(\mathbb{A})\cdot K, \qquad \text{where } K = \prod_{p\le\infty} K_p. \tag{1.32}$$

The relevance for our induced representation is that, since every $\varphi \in V(s)$ transforms under $B(\mathbb{A})$, it is determined on K. In other words, the operation $\varphi \mapsto \varphi|_K$ is injective. Now, if you've done the above exercise, you will have no difficulty verifying the following facts:

- $\varphi|_K$ is left invariant under $B(\mathbb{A})\cap K$.
- Any function on K that is left invariant under $B(\mathbb{A})\cap K$ can be extended (in a unique way) to a function $\varphi:\ \mathrm{GL}(2,\mathbb{A}) \to \mathbb{C}$ with the transformation property (1.27).

So now it is clear what our distinguished element φ_0 is: It is the element in $V(s)$ whose restriction to K is constantly 1. In terms of the global Iwasawa decomposition (1.32), the formula for φ_0 is

$$\varphi_0(\begin{bmatrix} a & b \\ & d \end{bmatrix}\kappa) = \left|\frac{a}{d}\right|^{(s+1)/2} \qquad \text{for all } a,d\in\mathbb{I},\ b\in\mathbb{A},\ \text{and } \kappa\in K. \tag{1.33}$$

Since the function φ_0, when restricted to K, is as regular as possible, it will be in $V(s)$ for any reasonable choice of regularity conditions on the space $V(s)$.

Let us take this opportunity and introduce some common terminology. Let ρ be the representation of $\mathrm{GL}(2,\mathbb{A})$ on $V(s)$ by right translation, as in (1.20). By its very definition, the function φ_0 has the properties

$$\rho(g)\varphi_0 = \varphi_0 \qquad \text{for all } g\in K_p \tag{1.34}$$

for any prime p, as well as

$$\rho(g)\varphi_0 = \varphi_0 \qquad \text{for all } g\in K_\infty. \tag{1.35}$$

In general, a vector φ_0 in a global representation ρ with the property (1.34) is called *unramified at p*. If there exists an integer k such that φ_0 has the property

$$\rho(r(\theta))\varphi_0 = e^{ik\theta}\varphi_0 \qquad \text{for all } \theta\in\mathbb{R}, \tag{1.36}$$

then φ_0 is said to have *weight k*. Hence, equation (1.35) says precisely that φ_0 has weight 0. To summarize, our distinguished vector is *unramified at all primes and has weight* 0. These are particularly nice properties.

1.2.3 An Eisenstein series

In this section we will get to the heart of the matter. Recall that the product formula (1.12) states that the global absolute value $|\cdot|$ is *automorphic*, i.e., invariant under rationals. The transformation property (1.27) therefore implies that

$$\varphi(bg) = \varphi(g) \qquad \text{for all } b \in B(\mathbb{Q}) \text{ and } g \in \mathrm{GL}(2, \mathbb{A}). \tag{1.37}$$

This is interesting, since being invariant under rational points is the nature of automorphy. We'd have something even more interesting if instead of $B(\mathbb{Q})$ above we would have the full group $\mathrm{GL}(2, \mathbb{Q})$. Then φ would be (left) invariant under all rational points, just like an idele class character is invariant under all of $\mathrm{GL}(1, \mathbb{Q})$.

However, (1.37) is not satisfied with $\mathrm{GL}(2, \mathbb{Q})$ instead of $B(\mathbb{Q})$. But what if we could force $\mathrm{GL}(2, \mathbb{Q})$-invariance by applying a summation? Something like this: Define a new function Φ on $\mathrm{GL}(2, \mathbb{A})$ by

$$\Phi(g) = \sum_{\gamma \in \mathrm{GL}(2,\mathbb{Q})} \varphi(\gamma g). \tag{1.38}$$

(Obviously, Φ depends on φ. The only reason we don't write Φ_φ is to ease the notation.) Certainly, if this makes sense, then Φ is left $\mathrm{GL}(2, \mathbb{Q})$-invariant. However, this naive kind of way to force a function to be invariant will hardly converge. But observe that φ already has *some* kind of invariance, namely (1.37). Hence, instead of (1.38), we should really sum over cosets only, i.e.,

$$\Phi(g) = \sum_{\gamma \in B(\mathbb{Q}) \backslash \mathrm{GL}(2,\mathbb{Q})} \varphi(\gamma g). \tag{1.39}$$

This has a much better chance of convergence, and will still produce a left $\mathrm{GL}(2, \mathbb{Q})$-invariant function.

Let us calculate Φ for $\varphi = \varphi_0$, the distinguished vector in $V(s)$, and see what happens. We will write Φ_0 for Φ, in order to remember that this function is constructed from φ_0. Now here is a nice simplification. A theorem called ***strong approximation for*** $\mathrm{GL}(2)$ says that

$$\mathrm{GL}(2, \mathbb{A}) = \mathrm{GL}(2, \mathbb{Q}) \cdot \mathrm{GL}(2, \mathbb{R})^+ \cdot \prod_{p<\infty} \mathrm{GL}(2, \mathbb{Z}_p). \tag{1.40}$$

Here, the "+" on $\mathrm{GL}(2, \mathbb{R})$ denotes the subgroup consisting of matrices with positive determinant. Note the analogy with (1.9); really all we do is replace $\mathrm{GL}(1)$ by $\mathrm{GL}(2)$. However, while (1.9) is a *direct* product of groups, the decomposition of an element of $\mathrm{GL}(2, \mathbb{A})$ according to (1.40) is *not unique*.

The immediate relevance of (1.40) for us is as follows. Recall that Φ_0 is left $\mathrm{GL}(2, \mathbb{Q})$-invariant. It is also right K-invariant, a property inherited from φ_0. Hence, by (1.40), Φ_0 *is completely determined on* $\mathrm{GL}(2, \mathbb{R})^+$. But even more is true: By (1.30), any $g \in \mathrm{GL}(2, \mathbb{R})^+$ can be written, in fact uniquely, as

$$g = \begin{bmatrix} 1 & x \\ & 1 \end{bmatrix} \begin{bmatrix} y^{1/2} & \\ & y^{-1/2} \end{bmatrix} \begin{bmatrix} t & \\ & t \end{bmatrix} r(\theta), \qquad x, y, t \in \mathbb{R},\ y, t > 0,\ \theta \in \mathbb{R}/2\pi\mathbb{Z}. \tag{1.41}$$

(It is not necessary, but will be convenient later, to have the $\sqrt{y}$ instead of just y in this decomposition.) Since φ_0 is invariant under scalar matrices (by (1.28)) and right-invariant under K_∞, it follows that

$$\varphi_0 \textit{ is completely determined on elements of the form } \begin{bmatrix} 1 & x \\ & 1 \end{bmatrix}\begin{bmatrix} y^{1/2} & \\ & y^{-1/2} \end{bmatrix}, x, y \in \mathbb{R},\ y > 0.$$

This suggests the following. Let us view x and y as the real and imaginary part of a complex variable τ. Then $\tau = x + iy$ lies in the *upper half plane* $y > 0$. The above arguments show that Φ_0 is completely determined by the function

$$E(\tau, s) := \Phi_0(g), \qquad g = \begin{bmatrix} 1 & x \\ & 1 \end{bmatrix}\begin{bmatrix} y^{1/2} & \\ & y^{-1/2} \end{bmatrix}, \qquad x, y \in \mathbb{R},\ y > 0. \tag{1.42}$$

(We indicate the dependence on s, but think of s as fixed and τ as a variable.) Let us calculate this function on the upper half plane more explicitly. It is not difficult to see that $\mathrm{GL}(2, \mathbb{Q}) = B(\mathbb{Q})\mathrm{SL}(2, \mathbb{Z})$. Therefore the summation in (1.39) simplifies as follows,

$$\Phi_0(g) = \sum_{\gamma \in \left[\begin{smallmatrix} \pm 1 & * \\ & \pm 1 \end{smallmatrix}\right] \backslash \mathrm{SL}(2,\mathbb{Z})} \varphi_0(\gamma g). \tag{1.43}$$

Note that the element γ in (1.43) is diagonally embedded:

$$\gamma = (\gamma_\infty, \gamma_2, \gamma_3, \gamma_5, \ldots),$$

where each γ_p is the same matrix $\gamma \in \mathrm{SL}(2, \mathbb{Z})$, considered as an element of $\mathrm{GL}(2, \mathbb{Q}_p)$. Now for finite p, all the γ_p commute with the matrix g in (1.43), since our g has only an archimedean component; note that with g in (1.42) we really mean the element

$$g = (\begin{bmatrix} 1 & x \\ & 1 \end{bmatrix}\begin{bmatrix} y^{1/2} & \\ & y^{-1/2} \end{bmatrix}, 1, 1, 1, \ldots) \in \mathrm{GL}(2, \mathbb{A}).$$

Therefore, since φ_0 is right $\mathrm{GL}(2, \mathbb{Z}_p)$-invariant for all finite p, these γ_p disappear on the right. Hence, we may assume that the element γ in (1.43) only has an *archimedean* component. All the p-adic components have disappeared, and we are left with a purely archimedean calculation. To continue, we will use the following lemma, whose proof is left as an exercise.

1.2.3 Lemma. *Let $x, y \in \mathbb{R}$ with $y > 0$. Let $\tau = x + iy$. Then, for* $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}(2, \mathbb{Z})$,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & x \\ & 1 \end{bmatrix}\begin{bmatrix} \sqrt{y} & \\ & \sqrt{y}^{-1} \end{bmatrix} = \begin{bmatrix} 1 & x' \\ & 1 \end{bmatrix}\begin{bmatrix} \sqrt{y'} & \\ & \sqrt{y'}^{-1} \end{bmatrix} r(\theta),$$

with

$$x' + iy' = \frac{a\tau + b}{c\tau + d}, \qquad e^{i\theta} = \frac{c\bar{\tau} + d}{|c\tau + d|}. \tag{1.44}$$

Using the notation of this lemma, we can now calculate

$$
\begin{aligned}
E(\tau,s) &= \Phi_0(g) && \text{(with } g \text{ as in (1.42))}\\
&= \sum_{\gamma\in\left[\begin{smallmatrix}\pm1&*\\&\pm1\end{smallmatrix}\right]\backslash\mathrm{SL}(2,\mathbb{Z})} \varphi_0(\begin{bmatrix}1&x'\\&1\end{bmatrix}\begin{bmatrix}\sqrt{y'}&\\&\sqrt{y'}^{-1}\end{bmatrix}r(\theta)) && \text{(by (1.43))}\\
&= \sum_{\gamma\in\left[\begin{smallmatrix}\pm1&*\\&\pm1\end{smallmatrix}\right]\backslash\mathrm{SL}(2,\mathbb{Z})} \varphi_0(\begin{bmatrix}1&x'\\&1\end{bmatrix}\begin{bmatrix}\sqrt{y'}&\\&\sqrt{y'}^{-1}\end{bmatrix}) && \text{(because } \varphi_0 \text{ has weight 0)}\\
&= \sum_{\gamma\in\left[\begin{smallmatrix}\pm1&*\\&\pm1\end{smallmatrix}\right]\backslash\mathrm{SL}(2,\mathbb{Z})} (y')^{(s+1)/2} && \text{(by (1.33))}\\
&= \sum_{\gamma\in\left[\begin{smallmatrix}\pm1&*\\&\pm1\end{smallmatrix}\right]\backslash\mathrm{SL}(2,\mathbb{Z})} \Big(\frac{y}{|c\tau+d|^2}\Big)^{(s+1)/2} && \text{(from (1.44))}\\
&= \sum_{\gamma\in\left[\begin{smallmatrix}\pm1&*\\&\pm1\end{smallmatrix}\right]\backslash\mathrm{SL}(2,\mathbb{Z})} \frac{y^{(s+1)/2}}{|c\tau+d|^{s+1}}.
\end{aligned}
$$

It is easy to see that the map $\begin{bmatrix}a&b\\c&d\end{bmatrix}\mapsto(c,d)$ induces a bijection

$$
\begin{bmatrix}\pm1&*\\&\pm1\end{bmatrix}\Big\backslash\mathrm{SL}(2,\mathbb{Z}) \xrightarrow{\sim} \{\pm1\}\backslash\{(c,d)\in\mathbb{Z}\times\mathbb{Z}\mid\ \gcd(c,d)=1\}. \tag{1.45}
$$

(We understand that the condition $\gcd(c,d)=1$ implies that c and d cannot both be zero.) Therefore,

$$
E(\tau,s)=\frac{1}{2}\sum_{\substack{(c,d)\in\mathbb{Z}\times\mathbb{Z}\\ \gcd(c,d)=1}}\frac{y^{(s+1)/2}}{|c\tau+d|^{s+1}}. \tag{1.46}
$$

We recognize $E(\tau,s)$ as the well-known ***real analytic Eisenstein series***; see Example 2.5 of [6], or Sect. 1.6 of [3], or §3 of Maass's classical paper [13]. Standard arguments show that the summation in (1.46) is absolutely convergent for $\mathrm{Re}(s)$ large enough (for $\mathrm{Re}(s)>1$, to be precise). However, one can prove that the Eisenstein series can be analytically continued to all values of s except $s=\pm1$ (see Theorem 1.6.1 of [3]).

Remark: In the work [13], Maass introduced a class of functions on the upper half plane which he called ***wave forms*** (*Wellenfunktionen*). The Eisenstein series $E(\tau,s)$ are examples for this kind of function. For a modern treatment of Maass wave forms, see Sect. 1.9 of [3].

1.2.4 Automorphic forms and automorphic representations

We saw in the previous section that the convergence of the summation (1.39), at least for $\varphi=\varphi_0$, depends on s. With not much more effort one can prove that, for $\mathrm{Re}(s)$ large enough, the summation (1.39) is convergent for *any* φ in the space $V(s)$. Let us assume this is the case.

Let $W(s)$ be space of all functions Φ, where φ runs through $V(s)$. By definition, we have a surjective map

$$\begin{aligned} V(s) &\longrightarrow W(s), \\ \varphi &\longmapsto \Phi. \end{aligned} \tag{1.47}$$

Since the summation (1.39) is happening on the *left* side of the argument, the map $\varphi \mapsto \Phi$ commutes with *right* translation by elements of $\mathrm{GL}(2,\mathbb{A})$. This means that the map (1.47) is an ***intertwining operator*** for the action of $\mathrm{GL}(2,\mathbb{A})$ by right translation on both sides.

Assume for the moment that $V(s)$ is *irreducible* as a $\mathrm{GL}(2,\mathbb{A})$-representation (we will see in the next chapter that this is the case for most values of s). Then the map (1.47) is automatically injective, and hence the representations $V(s)$ and $W(s)$ are ***equivalent*** (i.e., isomorphic as $\mathrm{GL}(2,\mathbb{A})$-representations). Hence, $W(s)$ is simply a different ***model*** for the representation $V(s)$. By its definition, $W(s)$ consists of functions Φ that are left-invariant under $\mathrm{GL}(2,\mathbb{Q})$:

$$\Phi(\gamma g) = \Phi(g) \qquad \text{for all } g \in \mathrm{GL}(2,\mathbb{A}),\ \gamma \in \mathrm{GL}(2,\mathbb{Q}). \tag{1.48}$$

This is the essential property of an automorphic form. The functions Φ satisfy additional regularity conditions inherited from the regularity conditions of the functions φ. We have not specified these conditions exactly, but among the additional properties satisfied by Φ are continuity and "moderate growth". The definition of an ***automorphic form on*** $\mathrm{GL}(2,\mathbb{A})$ is a function Φ satisfying (1.48) and these additional regularity conditions. We will give the precise definition in Sect. 1.4.5.

Still assuming that $V(s)$ is irreducible, the isomorphism $V(s) \cong W(s)$ means that we can *realize* the representation $V(s)$ as a space of automorphic forms. This is the official definition of an automorphic representation:

1.2.4 Definition. *Let V be an irreducible representation of* $\mathrm{GL}(2,\mathbb{A})$. *We say that V is an* ***automorphic representation*** *if there exists a space W consisting of automorphic forms, invariant under right translations, such that $V \cong W$ as* $\mathrm{GL}(2,\mathbb{A})$*-representations.*[2]

So we see that, while the principle (1.25) is valid, the space $V(s)$ (the result of parabolic induction) is not itself a space of automorphic forms. To realize it as a space of automorphic forms, we associate with each φ the function Φ. Instead of $\Phi(g)$, one usually writes $E(g,\varphi)$, and calls all functions $E(\cdot,\varphi)$ ***Eisenstein series***. We can summarize by saying that the construction of the Eisenstein series provides an intertwining operator of the parabolically induced representation with a space of automorphic forms.

As for the classical function $E(\tau,s)$, it follows from the definition (1.42), Lemma 1.2.3 and the properties of Φ_0 that

$$E\Big(\frac{a\tau+b}{c\tau+d},s\Big) = E(\tau,s) \qquad \text{for all } \gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}(2,\mathbb{Z}). \tag{1.49}$$

Let us write $\gamma\tau$ for the argument of $E(\cdot,s)$ on the left hand side:

$$E(\gamma\tau,s) = E(\tau,s) \qquad \text{for all } \gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}(2,\mathbb{Z}). \tag{1.50}$$

[2] In reality, one also has to allow the possiblity that W is a *quotient* of spaces of automorphic forms. In most cases, however, W will be a space of automorphic forms itself.

The map $(\gamma,\tau)\mapsto\gamma\tau$ is an action of $\mathrm{SL}(2,\mathbb{Z})$ on the upper half plane $\mathcal{H}$, and (1.50) states that $E(\cdot,s)$ is invariant under this action. This invariance is the classical incarnation of the key automorphic property (1.48).

1.3 Holomorphic Eisenstein series

The real analytic Eisenstein series are interesting enough, but maybe you are more familiar with the *holomorphic* Eisenstein series E_4, E_6, ..., which are defined in just about every text on classical modular forms. In this chapter we will explain how these objects also arise naturally as distinguished vectors in certain automorphic representations of $\mathrm{GL}(2,\mathbb{A})$.

1.3.1 Local parabolic induction

Recall that the space $V(s)$ consists of functions $\varphi:\ \mathrm{GL}(2,\mathbb{A})\to\mathbb{C}$ with the transformation property

$$\varphi(\begin{bmatrix} a & b \\ & d \end{bmatrix} g) = \Big|\frac{a}{d}\Big|^{(s+1)/2}\varphi(g) \qquad \text{for all } g\in\mathrm{GL}(2,\mathbb{A}),\ a,d\in\mathbb{I},\ b\in\mathbb{A}. \tag{1.51}$$

One question we have not settled yet is the irreducibility of the space $V(s)$. In order to investigate this question, we will consider the local analogue of $V(s)$. Hence, we fix a prime p or $p=\infty$, and consider functions $\varphi:\ \mathrm{GL}(2,\mathbb{Q}_p)\to\mathbb{C}$ with the analogous local transformation property, i.e.,

$$\varphi(\begin{bmatrix} a & b \\ & d \end{bmatrix} g) = \Big|\frac{a}{d}\Big|_p^{(s+1)/2}\varphi(g) \qquad \text{for all } g\in\mathrm{GL}(2,\mathbb{Q}_p),\ a,d\in\mathbb{Q}_p^\times,\ b\in\mathbb{Q}_p. \tag{1.52}$$

In the global case we did not precisely specify the regularity conditions to be satisfied by the functions in $V(s)$. In the local case we want to be more precise and specify these conditions exactly. We will distinguish the p-adic and the archimedean case.

The non-archimedean case

Let p be a prime number, and let $s\in\mathbb{C}$. Then we define $V_p(s)$ to be space of all functions $\varphi:\ \mathrm{GL}(2,\mathbb{Q}_p)\to\mathbb{C}$ that a) satisfy the transformation property (1.52), and b) are *locally constant*. This is a standard definition in the p-adic case, and results in $V_p(s)$ being a so-called ***admissible representation***.

It is easy to see that $V_p(s)$ is not irreducible for all values of s, since for $s=-1$ the space $V_p(s)$ contains the one-dimensional space of *constant* functions as a $\mathrm{GL}(2,\mathbb{Q}_p)$-invariant subspace; this is obvious from the defining property (1.52). This one-dimensional subspace carries the trivial representation of $\mathrm{GL}(2,\mathbb{Q}_p)$, which we denote by $1_{\mathrm{GL}(2)}$. The quotient of $V_p(-1)$ by this one-dimensional subspace turns out to be irreducible and is called the ***Steinberg representation*** $\mathrm{St}_{\mathrm{GL}(2)}$ of $\mathrm{GL}(2,\mathbb{Q}_p)$. For $V_p(1)$ we have a similar situation, except that $\mathrm{St}_{\mathrm{GL}(2)}$ is the subrepresentation and $1_{\mathrm{GL}(2)}$ is the quotient. One can prove that these are the only reducibilities that can occur for the representations $V_p(s)$:

1.3.1 Proposition. *Let p be a prime number and $s\in\mathbb{C}$.*

i) If $s\neq\pm1$, then $V_p(s)$ is an irreducible representation of $\mathrm{GL}(2,\mathbb{Q}_p)$.

ii) If $s=-1$, then there is an exact sequence

$$0 \longrightarrow 1_{\mathrm{GL}(2)} \longrightarrow V_p(s) \longrightarrow \mathrm{St}_{\mathrm{GL}(2)} \longrightarrow 0. \tag{1.53}$$

iii) If $s=1$, then there is an exact sequence

$$0 \longrightarrow \mathrm{St}_{\mathrm{GL}(2)} \longrightarrow V_p(s) \longrightarrow 1_{\mathrm{GL}(2)} \longrightarrow 0. \tag{1.54}$$

For a proof of this result, see §3 of [8], or Sect. 4.5 of [3].

We note for later use that $V_p(s)$ contains a *distinguished vector* φ_0. Just like in the global case discussed in Sect. 1.2.2, this distinguished vector is characterized by the property that it is constantly 1 when restricted to K_p. Hence, in terms of the Iwasawa decomposition it is given by

$$\varphi_0(\begin{bmatrix} a & b \\ & d \end{bmatrix} g) = \Big|\frac{a}{d}\Big|_p^{(s+1)/2} \qquad \text{for all } g \in K_p,\ a,d \in \mathbb{Q}_p^\times,\ b \in \mathbb{Q}_p. \tag{1.55}$$

This function is well-defined even though there is overlap between $B(\mathbb{Q}_p)$ and $K_p = \mathrm{GL}(2,\mathbb{Z}_p)$.

The archimedean case

Now consider $p=\infty$. Hence, we consider functions $\varphi:\ \mathrm{GL}(2,\mathbb{R}) \to \mathbb{C}$ with the transformation property

$$\varphi(\begin{bmatrix} a & b \\ & d \end{bmatrix} g) = \Big|\frac{a}{d}\Big|_\infty^{(s+1)/2} \varphi(g) \qquad \text{for all } g \in \mathrm{GL}(2,\mathbb{R}),\ a,d \in \mathbb{R}^\times,\ b \in \mathbb{R}. \tag{1.56}$$

By the Iwasawa decomposition (1.30), any such function is determined on $K_\infty = \mathrm{SO}(2)$. What weight k vectors does $V_\infty(s)$ contain? Recall that a vector has weight k if it satisfies (1.36). Hence, φ has weight k if and only if it satisfies

$$\varphi(gr(\theta)) = e^{ik\theta}\varphi(g) \qquad \text{for all } g \in \mathrm{GL}(2,\mathbb{R}),\ \theta \in \mathbb{R}. \tag{1.57}$$

This property is reconcilable with the property (1.56) if and only if k is *even* (since the overlap of $B(\mathbb{R})$ with K_∞ is $\mathrm{diag}(\pm1,\pm1)$). Thus, there exists a vector of weight k in $V_\infty(s)$ exactly if k is even, and up to multiples this vector is given in terms of the Iwasawa decomposition by

$$\varphi_k(\begin{bmatrix} a & b \\ & d \end{bmatrix} r(\theta)) = \Big|\frac{a}{d}\Big|_\infty^{(s+1)/2} e^{ik\theta} \qquad \text{for all } a,d \in \mathbb{R}^\times,\ b,\theta \in \mathbb{R}. \tag{1.58}$$

In particular, the subspace of $V_\infty(s)$ consisting of vectors of weight k is one-dimensional.

We still have not specified the regularity conditions of the functions in $V_\infty(s)$, but now we will. We could specify $V_\infty(s)$ to consist of continuous functions, or of smooth functions, or maybe functions that are L^2 in some sense. But whatever we do, at the very least $V_\infty(s)$ should contain the basic functions φ_k for k even, as these weight vectors form a sort of "skeleton" of the representation. It proves advantageous to make a quite radical choice and admit nothing else but the skeleton into the representation. Hence, we *define*

$$V_\infty(s) = \bigoplus_{\substack{k\in\mathbb{Z} \\ k \text{ even}}} \mathbb{C}\varphi_k. \tag{1.59}$$

Every vector in $V_\infty(s)$ is a finite linear combination of the basic functions φ_k. This is the most *algebraic* definition of $V_\infty(s)$.

However, being so radical comes with a price: *The space $V_\infty(s)$ is not invariant under right translations by* $\mathrm{GL}(2,\mathbb{R})$. It is of course invariant under right translations by K_∞, but not under the full group. Hence, $V_\infty(s)$ is not actually a representation of $\mathrm{GL}(2,\mathbb{R})$.

But we can still act on $V_\infty(s)$ with the ***Lie algebra*** $\mathfrak{g} = \mathfrak{gl}(2,\mathbb{R})$, which consists of all 2×2 real matrices. Given $X \in \mathfrak{g}$, its action on $\varphi \in V_\infty(s)$ is given by

$$(X\varphi)(g) = \frac{d}{dt}\Big|_0 \varphi(g\exp(tX)), \qquad g \in \mathrm{GL}(2,\mathbb{R}). \tag{1.60}$$

This is the ***derived action*** of the Lie algebra (being the derivative of right translation).[3] We will say much more about the Lie algebra and its action in Section 1.4.

It is not difficult to see that $V_\infty(s)$, as defined in (1.59), is invariant under this action. Hence on $V_\infty(s)$ we have both the $\mathfrak{g}$-action and the K_∞-action, and these two actions are in a certain sense compatible. The space $V_\infty(s)$ is what is called a $(\mathfrak{g}, K_\infty)$-*module*. The category of $(\mathfrak{g}, K_\infty)$-modules is convenient to work with, and we prefer it over other categories of $\mathrm{GL}(2,\mathbb{R})$-representations.

What about irreducibility of $V_\infty(s)$ as a $(\mathfrak{g}, K_\infty)$-module? Just like in the p-adic case we see that if $s = -1$ then the constant functions (spanned by φ_0) constitute an invariant subspace (carrying the trivial $(\mathfrak{g}, K_\infty)$-module). But now there are additional cases of reducibility:

1.3.2 Proposition. *Let p be a prime number and $s \in \mathbb{C}$.*

i) If s is not an odd integer, then $V_\infty(s)$ is an irreducible representation of $\mathrm{GL}(2,\mathbb{R})$.

ii) If $s = -\ell$, where ℓ is an odd positive integer, then there is an exact sequence

$$0 \longrightarrow \mathcal{F}_\ell \longrightarrow V_\infty(-\ell) \longrightarrow \mathcal{D}_\ell \longrightarrow 0. \tag{1.61}$$

Here $\mathcal{F}_\ell$ is a finite-dimensional representation with weight structure

$$[-\ell+1, -\ell+3, \ldots, \ell-3, \ell-1], \tag{1.62}$$

and $\mathcal{D}_\ell$ is an infinite-dimensional representation with weight structure

$$[\ldots, -\ell-3,\ -\ell-1\,] \,\sqcup\, [\,\ell+1,\ \ell+3, \ldots]. \tag{1.63}$$

iii) If $s = \ell$, where ℓ is an odd positive integer, then there is an exact sequence

$$0 \longrightarrow \mathcal{D}_\ell \longrightarrow V_\infty(\ell) \longrightarrow \mathcal{F}_\ell \longrightarrow 0. \tag{1.64}$$

[3] Formula (1.60) is a particular instance of the following principle: Whenever we have an action of a Lie group, we have a derived action of the corresponding Lie algebra. Recall that the Lie algebra, which in our case has a very concrete realization as 2×2 matrices, is really the tangent space at the identity of the group. The velocity vector to the curve $t \mapsto \exp(tX)$ at $t = 0$ is precisely the tangent vector X.

Here, by "weight structure" we simply mean a list of all the weights occurring in a representation. Interpreted as integer points on a line, it is a way to visualize the "skeleton" of the representation. (For the small group $\mathrm{GL}(2,\mathbb{R})$ this is all happening on a line. For higher rank groups the weights would be represented by lattice points in a higher-dimensional Euclidean space – this is the theory of ***roots*** and ***weights***.)

For a proof of Proposition 1.3.2, see §5 of [8], or Sect. 2.5 of [3], or our considerations in Sect. 1.4.2. The $(\mathfrak{g}, K_\infty)$-modules $\mathcal{D}_\ell$ are called ***discrete series representations*** and are irreducible.

1.3.2 Infinite tensor products

In the previous section we defined the local induced representations $V_p(s)$ and explored their reducibilities. This can be used to understand the reducibilities of the global induced representation $V(s)$ considered in Sect. 1.2.1. First we have to understand the precise relationship between the local and global representations. Roughly speaking, $V(s)$ is an "infinite tensor product" of all the $V_p(s)$.

To understand this, let us choose arbitrary vectors $\varphi_p \in V_p(s)$ for all $p \leq \infty$, but subject to the condition that φ_p equals the distinguished vector $\varphi_{p,0}$ given by (1.55) for almost all p. Then we can piece the φ_p together to obtain a global function φ, defined as follows:

$$\varphi(g) = \prod_{p\leq\infty} \varphi_p(g_p), \qquad g = (g_p) \in \mathrm{GL}(2,\mathbb{A}). \tag{1.65}$$

Note that this product is finite, since $g_p \in K_p$ for almost all p, and $\varphi_p|_{K_p} = 1$ for almost all p. Hence φ is well-defined. We write

$$\varphi = \bigotimes_{p\leq\infty} \varphi_p, \tag{1.66}$$

and speak of φ as a "pure tensor" composed of the local functions φ_p. The important thing to note is that φ has the transformation property (1.27); this is immediate by its definition and the properties of the local functions φ_p. We still have not specified the precise properties of the functions in $V(s)$, but certainly we would like all pure tensors of the above form to be in $V(s)$. In fact, *we want $V(s)$ to consist precisely of all linear combinations of such pure tensors.* It is not very difficult to come up with regularity conditions on the functions in $V(s)$ such that this is the case. But, for simplicity, let us simply think of $V(s)$ as being *defined* as the vector space generated by all functions φ of the form (1.66).

It is tempting to then write

$$V(s) = \bigotimes_{p\leq\infty} V_p(s) \tag{1.67}$$

but one has to make sense out of the infinite tensor product. This can be done by defining the right hand side as a *direct limit* of ordinary (finite) tensor products, with inclusions between these defined using the distinguished vectors $\varphi_{p,0}$. The resulting space is called a ***restricted tensor product*** with respect to the $\varphi_{p,0}$. The details of this construction can be found in §9 of [8], or in Sect. 3.3 of [3].

But observe that $V(s) = \otimes V_p(s)$ is *not* a representation of $\mathrm{GL}(2,\mathbb{A})$, since the archimedean component $V_\infty(s)$ is *not* a representation of $\mathrm{GL}(2,\mathbb{R})$. If $\varphi \in V(s)$ and $g \in \mathrm{GL}(2,\mathbb{R})$, then the right translate $\varphi(\,\cdot\,g)$ need not be in $V(s)$ again. We still have the action of the ***finite adelic group***

$$\mathrm{GL}(2,\mathbb{A}_f)=\Big\{g=(g_p)_{p<\infty}\in\prod_{p<\infty}\mathrm{GL}(2,\mathbb{Q}_p)\;\Big|\;g_p\in\mathrm{GL}(2,\mathbb{Z}_p)\text{ for almost all }p\Big\} \tag{1.68}$$

on $V(s)$ by right translation. $V(s)$ is also a $(\mathfrak{g},K_\infty)$-module, with K_∞ acting by right translation, and $\mathfrak{g}$ acting as in (1.60). The actions of $\mathrm{GL}(2,\mathbb{A}_f)$ and of $(\mathfrak{g},K_\infty)$ commute. For the sake of brevity, we will still address $V(s)$ as a *representation of* $\mathrm{GL}(2,\mathbb{A})$, even though what we really mean is that $V(s)$ carries a commuting $\mathrm{GL}(2,\mathbb{A}_f)$- and $(\mathfrak{g},K_\infty)$-module structure.

The irreducibility question

Once a global representation π is realized as a restricted tensor product of local representations, $\pi=\otimes\pi_p$, the irreducibility of π is easy to decide: π is irreducible if and only if all π_p are irreducible. Even more is true: The irreducible constituents of π are obtained by choosing irreducible constituents of the π_p at every place. This is a result of Langlands; see Lemma 1 of [12].

Applied to our situation $V(s)=\otimes V_p(s)$, we now obtain from the local statements in Proposition 1.3.1 and 1.3.2 the following result: *$V(s)$ is irreducible if and only if s is not an odd integer*. Recall that s can be any complex number; only for $s\in 2\mathbb{Z}+1$ does reducibility occur.

We will investigate the cases of reducibility more closely. Something special happens if $s=\pm1$, since then, by Proposition 1.3.1 and 1.3.2, *every* local component $V_p(s)$ is reducible. By Langlands' result, this implies that $V(s)$ has *infinitely many* irreducible constitutents. In fact, one of these constituents is the ***global trivial representation***,

$$1_{\mathrm{GL}(2,\mathbb{A})}=\bigotimes_{p\le\infty}1_{\mathrm{GL}(2,\mathbb{Q}_p)},$$

obtained by piecing together all the local trivial representations. Of course, for $s=-1$ we can see directly from (1.27) that $V(s)$ contains the constant functions as a one-dimensional invariant subspace.

Let us consider the more manageable cases where the only reducibility comes from the archimedean place. Hence, we will assume that $s=k-1$, where k is an *even* positive integer ≥ 4. Then, by Proposition 1.3.1, all the $V_p(s)$ for $p<\infty$ are irreducible. By iii) of Proposition 1.3.2, there is an exact sequence

$$0\longrightarrow\mathcal{D}_{k-1}\otimes\bigotimes_{p<\infty}V_p(k-1)\longrightarrow V(k-1)\longrightarrow\mathcal{F}_{k-1}\otimes\bigotimes_{p<\infty}V_p(k-1)\longrightarrow 0. \tag{1.69}$$

The principle (1.25) is to be understood in the sense that all irreducible subquotients of a representation induced from something automorphic should be automorphic. Hence, the irreducible representations on the left and on the right of (1.69) should be automorphic. In the next section we will concentrate on the constituent on the left.

1.3.3 Another Eisenstein series

In this section we continue to let $s = k-1$, where k is an even positive integer ≥ 4. We claimed above that the representation

$$\pi := \mathcal{D}_{k-1} \otimes \bigotimes_{p<\infty} V_p(k-1), \tag{1.70}$$

which is an irreducible subrepresentation of $V(k-1)$, is automorphic. By Definition 1.2.4, we should be able to find a model of $V(s)$ (meaning an isomorphic representation) consisting of automorphic forms. This is actually not hard to do: All we have to do is to associate with each $\varphi \in \pi$ the function Φ defined in (1.39). We stated before that this summation converges as long as $\mathrm{Re}(s)$ is large enough, and it turns out $s = k-1$ with $k \geq 4$ is large enough. Hence, the representation π defined in (1.70) is indeed an automorphic representation of $\mathrm{GL}(2,\mathbb{A})$.

Just as in Sects. 1.2.2 and 1.2.3, let us calculate the function Φ explicitly for a distinguished vector φ in π. It is pretty clear what this distinguished vector is: It will be a pure tensor made up of distinguished vectors in each local representation. For each finite prime, we will take the special vector $\varphi_{p,0}$ given explicitly in (1.55). For $p = \infty$ we will take the function $\varphi_{\infty,k}$ of weight k given by (1.58). Then our global distinguished vector is

$$\varphi_k := \varphi_{\infty,k} \otimes \bigotimes_{p<\infty} \varphi_{p,0}. \tag{1.71}$$

When we worked with the full induced representation $V(s)$, our distinguished vector defined in Sect. 1.2.2 was defined similarly, but with $\varphi_{\infty,k}$ replaced by $\varphi_{\infty,0}$. We couldn't do that now because $\mathcal{D}_{k-1}$ does not contain the weight 0. Keeping in mind the weight structure (1.63) of $\mathcal{D}_{k-1}$, namely

$$[\ldots, -k-2,\ -k\,] \sqcup [\,k,\ k+2, \ldots], \tag{1.72}$$

the most natural thing is to take the vector of weight k, which is what we did.

We will now calculate the function Φ_k obtained from φ_k via the summation process (1.39) and see if we get anything interesting. As in Sect. 1.2.3 we see that Φ_k is completely determined by the function

$$E_k(\tau) := y^{-k/2}\,\Phi_k(g), \qquad g = \begin{bmatrix} 1 & x \\ & 1 \end{bmatrix} \begin{bmatrix} y^{1/2} & \\ & y^{-1/2} \end{bmatrix}, \qquad x, y \in \mathbb{R},\ y > 0. \tag{1.73}$$

Comparing with (1.42), note the additional factor $y^{-k/2}$. We may think of this factor also being present in (1.42), but with $k = 0$. Its usefulness will become apparent in the calculation below; let's just carry it along for the moment.

Arguing exactly as in Sect. 1.2.3, we see that

$$\Phi_k(g) = \sum_{\gamma \in \left[\begin{smallmatrix} \pm 1 & * \\ & \pm 1 \end{smallmatrix}\right] \backslash \mathrm{SL}(2,\mathbb{Z})} \varphi_k(\gamma g), \tag{1.74}$$

where γ has an archimedean component only; here, g is still as in (1.73). By the definition (1.71) of φ_k, we therefore have

$$\Phi_k(g) = \sum_{\gamma \in \left[\begin{smallmatrix} \pm 1 & * \\ & \pm 1 \end{smallmatrix}\right] \backslash \mathrm{SL}(2,\mathbb{Z})} \varphi_{\infty,k}(\gamma g), \tag{1.75}$$

thus being reduced to a purely archimedean calculation. Using the notation of Lemma 1.2.3, we now have

$$
\begin{aligned}
E_k(\tau) &= y^{-k/2}\,\Phi_k(g) && \text{(with } g \text{ as in (1.73))}\\
&= y^{-k/2} \sum_{\gamma\in\left[\begin{smallmatrix}\pm1 & *\\ & \pm1\end{smallmatrix}\right]\backslash \mathrm{SL}(2,\mathbb{Z})} \varphi_{\infty,k}\Big(\begin{bmatrix}1 & x'\\ & 1\end{bmatrix}\begin{bmatrix}\sqrt{y'} & \\ & \sqrt{y'}^{-1}\end{bmatrix} r(\theta)\Big) && \text{(by (1.75))}\\
&= y^{-k/2} \sum_{\gamma\in\left[\begin{smallmatrix}\pm1 & *\\ & \pm1\end{smallmatrix}\right]\backslash \mathrm{SL}(2,\mathbb{Z})} e^{ik\theta}\,\varphi_{\infty,k}\Big(\begin{bmatrix}1 & x'\\ & 1\end{bmatrix}\begin{bmatrix}\sqrt{y'} & \\ & \sqrt{y'}^{-1}\end{bmatrix}\Big) && \text{(because } \varphi_{\infty,k} \text{ has weight } k)\\
&= y^{-k/2} \sum_{\gamma\in\left[\begin{smallmatrix}\pm1 & *\\ & \pm1\end{smallmatrix}\right]\backslash \mathrm{SL}(2,\mathbb{Z})} (y')^{k/2}\,e^{ik\theta} && \text{(by (1.56))}\\
&= y^{-k/2} \sum_{\gamma\in\left[\begin{smallmatrix}\pm1 & *\\ & \pm1\end{smallmatrix}\right]\backslash \mathrm{SL}(2,\mathbb{Z})} \Big(\frac{y}{|c\tau+d|^2}\Big)^{k/2}\Big(\frac{c\bar\tau+d}{|c\tau+d|}\Big)^{k} && \text{(from (1.44))}\\
&= \sum_{\gamma\in\left[\begin{smallmatrix}\pm1 & *\\ & \pm1\end{smallmatrix}\right]\backslash \mathrm{SL}(2,\mathbb{Z})} \Big(\frac{1}{|c\tau+d|^2}\Big)^{k/2}\Big(\frac{c\tau+d}{|c\tau+d|}\Big)^{-k} && \text{(now the } y^{-k/2} \text{ is handy!)}\\
&= \sum_{\gamma\in\left[\begin{smallmatrix}\pm1 & *\\ & \pm1\end{smallmatrix}\right]\backslash \mathrm{SL}(2,\mathbb{Z})} \frac{1}{(c\tau+d)^k}.
\end{aligned}
$$

Using (1.45), we may rewrite this as

$$
E_k(\tau) = \frac{1}{2}\sum_{\substack{(c,d)\in\mathbb{Z}\times\mathbb{Z}\\ \gcd(c,d)=1}} \frac{1}{(c\tau+d)^k}. \tag{1.76}
$$

Up to possible normalizations, this is precisely the ***classical holomorphic Eisenstein series*** of weight k, as defined in Sect. I.2 of [10], or §4.1 of [16], or Sect. 1.3 of [3], or many other places. It is well known, and may easily be deduced from our formulas, that E_k satisfies

$$
E_k\Big(\frac{a\tau+b}{c\tau+d}\Big)(c\tau+d)^{-k} = E_k(\tau) \qquad \text{for all } \gamma = \begin{bmatrix}a & b\\ c & d\end{bmatrix} \in \mathrm{SL}(2,\mathbb{Z}). \tag{1.77}
$$

Comparing with (1.49), we note the presence of the additional factor $(c\tau+d)^{-k}$. The left hand side of (1.77) is often written as $(E_k\big|_k\gamma)(\tau)$. The property (1.77) then reads

$$
E_k\big|_k\gamma = E_k \qquad \text{for all } \gamma\in\mathrm{SL}(2,\mathbb{Z}). \tag{1.78}
$$

This is the typical transformation property of a *modular form of weight* k.

1.3.4 The growth condition and Fourier expansions

We mentioned in Sect. 1.2.4 that one of the defining properties of an automorphic form is that of "moderate growth". Let us now explain what exactly this means. Instead of automorphic forms, first consider a simpler kind of functions, namely functions on the real line. One reasonable way to impose a growth condition on such a function $\Phi : \mathbb{R} \to \mathbb{C}$ is to require that there exist an appropriate positive constant C and some non-negative integer n such that

$$|\Phi(x)| \le C|x|^n \qquad \text{for all } x \in \mathbb{R}. \tag{1.79}$$

In this case we say that Φ has ***polynomial growth***. It is a good way of allowing some moderate growth, but ensures that Φ does not get out of hand.

Next assume our functions would not live on $\mathbb{R}$, but on $\mathbb{R}_{>0}$. To impose a polynomial growth condition on $\Phi : \mathbb{R}_{>0} \to \mathbb{C}$, it would be too restrictive to require that

$$|\Phi(x)| \le C|x|^n \qquad \text{for all } x \in \mathbb{R}_{>0}. \tag{1.80}$$

This condition implies $\Phi(x) \to 0$ as $x \to 0$, but we want to allow polynomial growth as $x \to 0$ as well. It seems the correct condition as $x \to 0$ should be

$$|\Phi(x)| \le C|x|^{-n} \qquad \text{for all } x \in \mathbb{R}_{>0} \tag{1.81}$$

for appropriate C and n. But now (1.81) is too restrictive for $x \to \infty$. So let's take the "intersection" of (1.80) and (1.81), namely the one condition

$$|\Phi(x)| \le C(|x| + |x|^{-1})^n \qquad \text{for all } x \in \mathbb{R}_{>0} \tag{1.82}$$

for appropriate C and n. If we define the "norm" or "height" function on $\mathbb{R}_{>0}$ by $\|x\| = |x| + |x|^{-1}$, then (1.82) becomes

$$|\Phi(x)| \le C\|x\|^n \qquad \text{for all } x \in \mathbb{R}_{>0} \tag{1.83}$$

for appropriate C and n. Note that, with this definition, $\Phi(x)$ is of moderate growth if and only if $\Phi(x^{-1})$ is. This makes sense; after all, our functions live on a *group*, and interchanging x and x^{-1} should lead to the same condition.

Next consider functions $\Phi : \mathrm{GL}(2, \mathbb{R}) \to \mathbb{C}$. Let $|g|$ be the usual Euclidean norm on $\mathrm{GL}(2, \mathbb{R})$, i.e.,

$$|g| = \sqrt{a^2 + b^2 + c^2 + d^2} \qquad \text{for } g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}(2, \mathbb{R}). \tag{1.84}$$

As in the $\mathbb{R}_{>0}$ example, for Φ to have moderate growth, it would be too restrictive to require that

$$|\Phi(g)| \le C|g|^n \qquad \text{for all } g \in \mathrm{GL}(2, \mathbb{R}). \tag{1.85}$$

For example, (1.84) implies that $\Phi(\begin{bmatrix} 1 & \\ & y \end{bmatrix})$ is bounded as $y \to 0$, whereas we want to allow polynomial growth. Also, we want a condition that is invariant under interchanging g and g^{-1}. Hence, guided by the previous example, we define the "height" function

$$\|g\| = |g| + |g^{-1}|, \tag{1.86}$$

with $|g|$ as in (1.84), and require that

$$|\Phi(g)| \le C\|g\|^n \qquad \text{for all } g \in \mathrm{GL}(2, \mathbb{R}) \tag{1.87}$$

for appropriate C and n. Using the ***sub-multiplicativity*** of the Euclidean norm (1.84), you can easily prove the following useful fact:

$$\text{If } h_1, h_2 \in \mathrm{GL}(2,\mathbb{R}), \text{ and } \Phi \text{ is slowly increasing in the sense of (1.87),} \\ \text{then so is the function } g \mapsto \Phi(h_1 g h_2). \tag{1.88}$$

Hence, our growth condition is not only invariant under taking inverses, but also under translations. Again, this makes sense since our functions live on a group.

We are now in a position to define moderate growth for adelic functions. Recall from (1.68) the definition of the finite adelic group $\mathrm{GL}(2,\mathbb{A}_f)$.

1.3.3 Definition. *Let $\Phi:\ \mathrm{GL}(2,\mathbb{A}) \to \mathbb{C}$ be a continuous function. We say that Φ has **moderate growth**, or is **slowly increasing**, if, for each $h \in \mathrm{GL}(2,\mathbb{A}_f)$, the function on $\mathrm{GL}(2,\mathbb{R})$ given by*

$$g \longmapsto \Phi(gh) \tag{1.89}$$

is of moderate growth in the sense of (1.87).

This is the growth condition imposed on all automorphic forms. Let us verify that the Eisenstein series Φ_k defined in Sect. 1.3.3 satisfies this condition. First, we use strong approximation (1.40) to write the element $h \in \mathrm{GL}(2,\mathbb{A}_f)$ in (1.89) as

$$h = \rho\,\rho_\infty^{-1}\,\kappa, \qquad \rho \in \mathrm{GL}(2,\mathbb{Q}),\ \kappa \in \prod_{p<\infty} \mathrm{GL}(2,\mathbb{Z}_p). \tag{1.90}$$

Here, ρ is diagonally embedded into $\mathrm{GL}(2,\mathbb{A})$, and ρ_∞ is the archimedean component of ρ, so that the element on the right hand side of (1.90) has trivial archimedean component. Then we can use the invariance properties of Φ_k to write, for $g \in \mathrm{GL}(2,\mathbb{R})$,

$$\begin{aligned} \Phi_k(gh) &= \Phi_k(g\rho\,\rho_\infty^{-1}\,\kappa) \\ &= \Phi_k(g\rho\,\rho_\infty^{-1}) \\ &= \Phi_k(\rho^{-1}\,g\rho\,\rho_\infty^{-1}) \\ &= \Phi_k(\rho_\infty^{-1}\,g). \end{aligned} \tag{1.91}$$

Now using the fact (1.88), we see that it is enough to verify that the function $g \to \Phi_k(g)$ is slowly increasing. In other words, we were able to get rid of the element $h \in \mathrm{GL}(2,\mathbb{A}_f)$, and only need to verify (1.87).

Since the matrix (1.31) satisfies $|r(\theta)| = \sqrt{2}$, it follows from the sub-multiplicativity of the euclidean norm that

$$\frac{1}{\sqrt{2}}\,\|g\| \le \|gr(\theta)\| \le \sqrt{2}\,\|g\| \qquad \text{for all } \theta \in \mathbb{R}. \tag{1.92}$$

The left hand side of (1.87) satisfies $|\Phi_k(gr(\theta))| = |\Phi_k(g)|$, since Φ_k has weight k. From these facts and the Iwasawa decomposition (1.30) it follows that we need to verify (1.87) only for upper triangular matrices g.

Moreover, since Φ_k is invariant under the center (see (1.28)), we may assume that g is the matrix in (1.73). Hence, it all comes down to estimating the function $E_k(\tau)$ on the upper half plane. For this we invoke the classical ***Fourier expansion*** of the Eisenstein series E_k:

$$E_k(\tau) = 1 + \frac{(2\pi i)^k}{\zeta(k)(k-1)!} \sum_{n=1}^{\infty} \sigma_{k-1}(n) e^{2\pi i n\tau}, \tag{1.93}$$

where

$$\sigma_\ell(n) = \sum_{0<d|n} d^\ell. \tag{1.94}$$

The value $\zeta(k)$ denotes $\sum_{n=1}^{\infty} n^{-k}$ and can be expressed in terms of Bernoulli numbers. The proof of the expansion (1.93) is standard; see Theorem X.3.2 of [10], or Lemma 4.1.6 of [16], or Sect. 1.3 of [3]. Simply using the triangle inequality on the (absolutely convergent) series (1.93), we get

$$|E_k(\tau)| \le 1 + \frac{(2\pi)^k}{\zeta(k)(k-1)!} \sum_{n=1}^{\infty} \sigma_{k-1}(n) e^{-2\pi n y}. \tag{1.95}$$

Hence we get a uniform estimate in x as $y \to \infty$. For $y \to 0$ the inequality (1.95) is less useful, but observe that, by (1.77),

$$E_k\Big(\frac{i}{y}\Big)(iy)^{-k} = E_k(iy) \qquad \text{for all } y > 0. \tag{1.96}$$

This identity shows that if we have a polynomial estimate in y for $y > 1$, then we have such an estimate for $y < 1$ as well. Combining everything, we see that Φ_k satisfies the moderate growth condition 1.3.3.

Imagine that in (1.93) there would be a term $e^{2\pi i n\tau}$ for some *negative* n. Then, setting $\tau = iy$, we would have an exponentially growing term $e^{-2\pi n y}$. In this case the function Φ_k would not be of moderate growth. We see that, in classical terms, *the moderate growth condition means that there are no negative terms in the Fourier expansions of the functions on the upper half plane.*

FAQ

- *It seems that "strong approximation" is a frequently used concept. We have seen it for ideles in* (1.9) *and for* $\mathrm{GL}(2,\mathbb{A})$ *in* (1.40)*. What is the general principle here?*

  Strong approximation is indeed an important concept. While (1.9) is elementary, (1.40) may be deduced from the following statement about the group $G = \mathrm{SL}(2)$:

  $$\mathrm{SL}(2,\mathbb{Q})\mathrm{SL}(2,\mathbb{R}) \text{ is dense in } \mathrm{SL}(2,\mathbb{A}). \tag{1.97}$$

  Here, as usual, $\mathrm{SL}(2,\mathbb{Q})$ is diagonally embedded into $\mathrm{SL}(2,\mathbb{A})$, and $\mathrm{SL}(2,\mathbb{R})$ is locally embedded at the archimedean place. In this form the statement generalizes to a larger class of groups G:

  $$G(\mathbb{Q})G(\mathbb{R}) \text{ is dense in } G(\mathbb{A}). \tag{1.98}$$

  Groups G for which this holds must be *simply connected* in the algebraic group sense (which is why it holds for $\mathrm{SL}(2)$ and not $\mathrm{GL}(2)$). For precise statements see [9]. The most general form of strong approximation replaces $\mathbb{Q}$ by an algebraic number field K, and $G(\mathbb{R})$ by $G(K_v)$, where K_v is a completion of K such that $G(K_v)$ is not compact.

1.4 Differential operators

Maybe you've wondered about the following while studying the Eisenstein series examples in the previous chapter. The motivation for introducing the function $E_k(\tau)$ in (1.73) on the upper half plane was that this function determines the adelic function Φ_k completely. After some calculation, we ended up with the formula (1.76), which shows that $E_k(\tau)$ is in fact *holomorphic*. This is as good a property as a function defined on a complex domain can have.

On the other hand, the function $E(\tau, s)$ in (1.42), defined with the same purpose in mind of capturing all the values of the adelic function Φ_0, turns out to be *not* holomorphic; see (1.46). Both $E_k(\tau)$ and $E(\tau, s)$ were constructed from distinguished vectors in an automorphic representation. The only difference in these distinguished vectors happened at the archimedean place: For $E_k(\tau)$, we took the vector of weight k in the discrete series representation $\mathcal{D}_{k-1}$; for $E(\tau, s)$, we took the vector of weight 0 in $V_\infty(s)$. What is the reason that these two choices lead to different analytic properties for the resulting functions on the upper half plane?

In this chapter we will see that the answer comes from the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ and the mechanism of raising and lowering the weight.[4] We will also see that $E(\tau, s)$, while not holomorphic, still satisfies a nice differential equation (eventually justifying the name *real analytic* Eisenstein series). At the end of the chapter we will have all the ingredients in place to give the precise definition of an automorphic form on $\mathrm{GL}(2,\mathbb{A})$.

1.4.1 The real vs. the complex Lie algebra

In Sect. 1.3.2 we defined a representation of $\mathrm{GL}(2,\mathbb{A})$ to be a vector space V which is simultaneously a $\mathrm{GL}(2,\mathbb{A}_f)$-module as well as a $(\mathfrak{g}, K_\infty)$-module. Here $\mathfrak{g} = \mathfrak{gl}(2,\mathbb{R})$ is the Lie algebra of $\mathrm{GL}(2,\mathbb{R})$, consisting of all 2×2 real matrices. The Lie bracket on $\mathfrak{g}$ is simply the commutator $[X, Y] = XY - YX$. Since the action of the Lie algebra is important[5] for the theory of automorphic forms, we will take a closer look. For the time being, all our automorphic forms will be invariant under the center.[6] This implies that the center of $\mathfrak{gl}(2,\mathbb{R})$, consisting of all scalar matrices

$$\begin{bmatrix} x & \\ & x \end{bmatrix}, \qquad x \in \mathbb{R},$$

acts trivially. Hence, we will ignore this center and only investigate

$$\mathfrak{sl}(2,\mathbb{R}) = \{X \in \mathfrak{gl}(2,\mathbb{R}) \mid \mathrm{tr}(g) = 0\}. \tag{1.99}$$

Here, tr denotes the usual matrix trace. As a real vector space, $\mathfrak{sl}(2,\mathbb{R})$ is three-dimensional, an obvious basis given by

$$\hat{H} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad \hat{R} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad \hat{L} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}. \tag{1.100}$$

[4] Maybe you find it plausible that the Lie algebra, which acts via differentiation as in (1.60), has to do with holomorphy, a property characterized by differential equations.

[5] Recall that, when it comes to archimedean representations, we work in the category of $(\mathfrak{g}, K_\infty)$-modules. Hence, the Lie algebra action captures an essential part of the archimedean aspect of automorphic forms.

[6] You might wonder how much this assumption restricts generality. If you know about classical modular forms, this is what it amounts to: Being invariant under the center means we are only considering modular forms of *Haupttypus*, meaning modular forms "without character". To incorporate modular forms of *Nebentypus* into the framework of automorphic representations, one will have to allow for the center to act non-trivially.

The structure of $\mathfrak{sl}(2,\mathbb{R})$ is completely determined by the commutation relations

$$[\hat{H},\hat{R}]=2\hat{R},\qquad [\hat{H},\hat{L}]=-2\hat{L},\qquad [\hat{R},\hat{L}]=\hat{H}. \tag{1.101}$$

Assume that V is a complex vector space which is also an $\mathfrak{sl}(2,\mathbb{R})$-module. Since $\mathbb{C}$ and $\mathfrak{sl}(2,\mathbb{R})$ both act on V (the former simply by scalar multiplication), we get a combined action of

$$\mathbb{C}\otimes\mathfrak{sl}(2,\mathbb{R})\cong\mathfrak{sl}(2,\mathbb{C}), \tag{1.102}$$

where

$$\mathfrak{sl}(2,\mathbb{C})=\{X\in\mathfrak{gl}(2,\mathbb{C})\mid \mathrm{tr}(g)=0\} \tag{1.103}$$

(and $\mathfrak{gl}(2,\mathbb{C})$ consists of all 2×2 complex matrices, with Lie bracket given by the commutator). Conversely, given an $\mathfrak{sl}(2,\mathbb{C})$-module, we may always restrict the action to $\mathfrak{sl}(2,\mathbb{R})$. If V is irreducible as an $\mathfrak{sl}(2,\mathbb{C})$-module, then it is irreducible as an $\mathfrak{sl}(2,\mathbb{R})$-module, and vice versa. In this sense, $\mathfrak{sl}(2,\mathbb{R})$- and $\mathfrak{sl}(2,\mathbb{C})$-modules are pretty much the same thing, as long as we agree that such modules are always understood to be *complex* vector spaces.

The elements (1.100) are also a $\mathbb{C}$-basis of $\mathfrak{sl}(2,\mathbb{C})$. However, it turns out that a more convenient basis of $\mathfrak{sl}(2,\mathbb{C})$ is given by the elements

$$H=-i\begin{bmatrix}0&1\\-1&0\end{bmatrix},\qquad R=\frac{1}{2}\begin{bmatrix}1&i\\i&-1\end{bmatrix},\qquad L=\frac{1}{2}\begin{bmatrix}1&-i\\-i&-1\end{bmatrix}. \tag{1.104}$$

These satisfy the same structural relations as in (1.101), namely,

$$[H,R]=2R,\qquad [H,L]=-2L,\qquad [R,L]=H. \tag{1.105}$$

We will explain why the elements (1.104) are more convenient than the elements (1.100), even though they look more complicated. For this consider a $(\mathfrak{g},K_\infty)$-module (π,V). Let $v\in V$ be a vector of weight k, i.e.,

$$\pi(r(\theta))v=e^{ik\theta}v\qquad\text{for all }\theta\in\mathbb{R}. \tag{1.106}$$

Then, since $r(\theta)=\exp(\theta\begin{bmatrix}0&1\\-1&0\end{bmatrix})$, taking a derivative $\frac{d}{d\theta}$ on both sides gives

$$\pi(H)v=kv. \tag{1.107}$$

In fact, the conditions (1.106) and (1.107) can be shown to be equivalent: Being a weight k vector simply means being an eigenvector of $\pi(H)$ with eigenvalue k. Consider the subspace of all such eigenvectors,

$$V_k:=\{v\in V\mid \pi(H)v=kv\}. \tag{1.108}$$

Then it follows from the relations (1.105) that

$$\pi(R)V_k\subset V_{k+2}\qquad\text{and}\qquad \pi(L)V_k\subset V_{k-2}. \tag{1.109}$$

Hence, R *raises* the weight by 2 and L *lowers* the weight by 2, explaining the names of these operators. The relations (1.107) and (1.109) are simple and convenient. There is no equally simple analogue purely in terms of $\mathfrak{sl}(2,\mathbb{R})$, which is why working with $\mathfrak{sl}(2,\mathbb{C})$ makes life easier.

1.4.2 Raising and lowering in parabolic induction

This business about raising and lowering the weight is quite important, so let us work through an example. Consider the $(\mathfrak{g}, K_\infty)$-module $V_\infty(s)$ defined in Sect. 1.3.1. Recall from (1.59) that

$$V_\infty(s) = \bigoplus_{\substack{k\in\mathbb{Z}\\ k \text{ even}}} \mathbb{C}\varphi_k, \tag{1.110}$$

where

$$\varphi_k(\begin{bmatrix} a & b \\ & d \end{bmatrix} r(\theta)) = \Big|\frac{a}{d}\Big|_\infty^{(s+1)/2} e^{ik\theta} \qquad \text{for all } a,d \in \mathbb{R}^\times,\ b,\theta \in \mathbb{R}. \tag{1.111}$$

Hence, the weight-k space defined in (1.108) is spanned by φ_k if k is even, and is zero if k is odd.

1.4.1 Lemma. *With the above notations, the following formulas hold for any even integer k.*

$$H\varphi_k = k\varphi_k, \tag{1.112}$$

$$R\varphi_k = \frac{s+1+k}{2}\,\varphi_{k+2}, \tag{1.113}$$

$$L\varphi_k = \frac{s+1-k}{2}\,\varphi_{k-2}. \tag{1.114}$$

Property (1.112) is clear from (1.107). If you've never done this kind of calculation before, you might want to prove (1.113) and (1.114) as an exercise. By (1.109), it is enough to evaluate $(R\varphi_k)(1)$ and $(L\varphi_k)(1)$. The only somewhat tricky part is to calculate $\hat{L}\varphi_k$, for which the identity

$$\begin{bmatrix} 1 & \\ y & 1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{y}{1+y^2} \\ & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{1+y^2}} & \\ & \sqrt{1+y^2} \end{bmatrix} r(\theta), \qquad e^{i\theta} = \frac{1-iy}{\sqrt{1+y^2}}, \tag{1.115}$$

is helpful.

The formulas in Lemma 1.4.1 tell us a lot (in fact, everything) about the reducibilities of the representations $V_\infty(s)$. For example, if $s+1\pm k$ is never zero, the formulas show that we can hop from any of the φ_k to any other. This implies that $V_\infty(s)$ is irreducible, proving the statement of Proposition 1.3.2 i). In fact, you can now give a complete proof of this proposition!

We point out that the discrete series representation $\mathcal{D}_{k-1}$ defined in Proposition 1.3.2, where k is an even positive integer, has a ***lowest weight vector*** of weight k. By this we mean that there exists a non-zero vector v_k of weight k in the space of $\mathcal{D}_{k-1}$ with the property $Lv_k = 0$: The weight of v_k cannot be lowered without killing the vector entirely. Physicists like to speak of L as an ***annihilation operator***. Of course, the vector v_k is nothing but the function φ_k (or any non-zero multiple of it).

It is in fact true that $\mathcal{D}_{k-1}$ is *characterized* by the lowest weight property: If V is any $(\mathfrak{g}, K_\infty)$-module for which the center of $\mathfrak{g}$ acts trivially, and if there exists a non-zero vector $v_k \in V$ of some even positive weight k with the property $Lv_k = 0$, then $V \cong \mathcal{D}_{k-1}$.

FAQ

- *Is it just standard to have identities like (1.115) in your back pocket? That identity is kind of intimidating.*

It is one of those useful little items in the toolbox of the working automorphic forms researcher. The precise form of this identity is obviously not so easy to memorize, but maybe it is possible to remember that there is *some* explicit way of writing the Iwasawa decomposition of a lower triangular matrix. Then when the need arises one can come back to a place like this and look up the exact statement.

1.4.3 Functions on $\mathrm{SL}(2,\mathbb{R})$ and on the upper half plane

Twice now we found it convenient to consider functions on the upper half plane $\mathcal{H}$. First, when we defined the function $E(\tau,s)$ in (1.42); second, when we defined the function $E_k(\tau)$ in (1.73). The following mechanism underlies both constructions. Let $W(k)$ be the space of smooth functions on $\mathrm{SL}(2,\mathbb{R})$ of weight k. Hence, each $\Phi\in W(k)$ satisfies

$$\Phi(gr(\theta))=e^{ik\theta}\,\Phi(g)\qquad\text{for all } g\in\mathrm{SL}(2,\mathbb{R}),\ \theta\in\mathbb{R}. \tag{1.116}$$

Let W be the space of smooth functions $F:\mathcal{H}\to\mathbb{C}$. To $\Phi\in W(k)$ we associate $F\in W$ via

$$F(x+iy)=y^{-k/2}\,\Phi(\begin{bmatrix}1&x\\&1\end{bmatrix}\begin{bmatrix}y^{1/2}&\\&y^{-1/2}\end{bmatrix}),\qquad x,y\in\mathbb{R},\ y>0. \tag{1.117}$$

Note that this is precisely what we did in (1.42) (with $k=0$) and (1.73) (with $k\geq 4$ and even). In these cases the function Φ was the restriction to $\mathrm{SL}(2,\mathbb{R})$ of an automorphic form on $\mathrm{GL}(2,\mathbb{A})$.

Now the map $\Phi\mapsto F$ is in fact an *isomorphism* of $W(k)$ onto W. Given $F\in W$, we first define Φ on upper triangular matrices with positive diagonal elements using the formula (1.117). Then we use the fact that each $g\in\mathrm{SL}(2,\mathbb{R})$ can be uniquely written in the form

$$g=\begin{bmatrix}1&x\\&1\end{bmatrix}\begin{bmatrix}y^{1/2}&\\&y^{-1/2}\end{bmatrix}r(\theta),\qquad x,y\in\mathbb{R},\ y>0,\ \theta\in\mathbb{R}/2\pi\mathbb{Z} \tag{1.118}$$

to extend Φ to a weight k function on all of $\mathrm{SL}(2,\mathbb{R})$. This construction provides an inverse map to $\Phi\mapsto F$.

So now we have an isomorphism $W(k)\cong W$ for *each* integer k, given by the formula (1.117). By (1.109), under right translation, the operator R maps $W(k)$ to $W(k+2)$. Hence there must exist a corresponding operator $R_k:\ W\to W$. Similarly, to $L:\ W(k)\to W(k-2)$ there must exist a corresponding operator $L_k:\ W\to W$. Just to make it completely clear, the definition of the operators R_k and L_k is that the diagrams

$$\begin{array}{ccc}W(k)&\xrightarrow{\sim}&W\\ {\scriptstyle L}\downarrow&&\downarrow{\scriptstyle L_k}\\ W(k-2)&\xrightarrow{\sim}&W\end{array}\qquad\qquad\begin{array}{ccc}W(k)&\xrightarrow{\sim}&W\\ {\scriptstyle R}\downarrow&&\downarrow{\scriptstyle R_k}\\ W(k+2)&\xrightarrow{\sim}&W\end{array} \tag{1.119}$$

are commutative. The following result tells us what these operators are.

1.4.2 Lemma. *Define operators R_k,L_k on the space W of smooth functions on $\mathcal{H}$ by*

$$R_k=\frac{k}{y}+2i\frac{\partial}{\partial\tau},\qquad L_k=-2iy^2\frac{\partial}{\partial\bar\tau},$$

Then the diagrams (1.119) are commutative.

In Lemma 1.4.2, the operators

$$\frac{\partial}{\partial \tau} = \frac{1}{2}\Big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Big), \qquad \frac{\partial}{\partial \bar{\tau}} = \frac{1}{2}\Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big) \tag{1.120}$$

are the usual Wirtinger derivatives. Observe that L_k doesn't actually depend on k. Slight variations of the operators R_k and L_k were introduced by Maass; see [14], or page 152 of [15], or Sect. 2.1 of [3]. As a good exercise, you might want to prove Lemma 1.4.2; it is not completely trivial!

As an interesting consequence of Lemma 1.4.2 we note the following. Let $F \in W$ correspond to $\Phi \in W(k)$ via (1.117). Then

$$F \text{ is holomorphic if and only if } L\Phi = 0. \tag{1.121}$$

For F is holomorphic if and only if $\frac{\partial F}{\partial \bar{\tau}} = 0$, if and only if $L_k F = 0$, if and only if $L\Phi = 0$. Thus, *Φ being a lowest weight vector is equivalent to the corresponding function on $\mathcal{H}$ being holomorphic.* We now have a nice explanation for the fact that the Eisenstein series $E_k(\tau)$ defined in (1.73) is holomorphic, while the Eisenstein series $E(\tau, s)$ defined in (1.42) is not: The function Φ_k corresponding to $E_k(\tau)$ has the lowest weight property, but the function Φ_0 corresponding to $E(\tau, s)$ does not.

1.4.4 The Casimir element

We saw in the previous sections that the element $\varphi_k \in \mathcal{D}_k$ has the lowest weight property, i.e., it satisfies $L\varphi_k = 0$ (here, k is an even positive integer). In view of Lemma 1.4.2, if we write things out in coordinates on $\mathrm{SL}(2, \mathbb{R})$, this means that φ_k satisfies a first-order differential equation. In contrast, unless $s = -1$, the function $\varphi_0 \in V_\infty(s)$ satisfies neither $L\varphi_0 = 0$ nor $R\varphi_0 = 0$; this follows from Lemma 1.4.1. Let's see if we can find some other differential equation satisfied by φ_0.

Now here is an interesting thing one can do with the elements H, R, L of $\mathfrak{sl}(2, \mathbb{C})$ defined in (1.104). We define a new element Ω, called the ***Casimir element***, by

$$\Omega = \frac{1}{4}H^2 + \frac{1}{2}RL + \frac{1}{2}LR. \tag{1.122}$$

Alternative ways of writing Ω are

$$\begin{aligned} \Omega &= \frac{1}{4}H^2 + \frac{1}{2}H + LR \\ &= \frac{1}{4}H^2 - \frac{1}{2}H + RL. \end{aligned} \tag{1.123}$$

The multiplications in (1.122) and (1.123) are really taking place in the ***universal enveloping algebra*** $\mathcal{U}(\mathfrak{sl}(2, \mathbb{C}))$; after all, there is no multiplication defined on $\mathfrak{sl}(2, \mathbb{C})$ itself, just a Lie bracket. If you are uncomfortable with the universal enveloping algebra, you may imagine a $(\mathfrak{g}, K_\infty)$-module V, think of H, R, L as operators on V, and of the multiplications in (1.122) and (1.123) as taking place in the endomorphism algebra of V.

The remarkable thing is that Ω *commutes with all of* $\mathfrak{sl}(2,\mathbb{C})$. That's right; using the commutation relations (1.105), you can check that

$$H\Omega = \Omega H, \qquad R\Omega = \Omega R, \qquad L\Omega = \Omega L. \tag{1.124}$$

Thus, Ω lies in the center of the algebra $\mathcal{U}(\mathfrak{sl}(2,\mathbb{C}))$. Of course, every polynomial in Ω then also lies in the center of $\mathcal{U}(\mathfrak{sl}(2,\mathbb{C}))$. With a little more effort, one can in fact show that

$$\text{The center of } \mathcal{U}(\mathfrak{sl}(2,\mathbb{C})) \text{ is precisely the polynomial ring } \mathbb{C}[\Omega]. \tag{1.125}$$

An important general principle in representation theory called ***Schur's Lemma*** implies that Ω must act as a scalar on irreducible $(\mathfrak{g}, K_\infty)$-modules. Let's see how Ω acts on $V_\infty(s)$. Using the formulas from Lemma 1.4.1, a quick calculation shows that

$$\Omega\varphi_k = \frac{s^2-1}{4}\,\varphi_k. \tag{1.126}$$

This is independent of k, so that Ω acts as the scalar $\frac{s^2-1}{4}$ on all of $V_\infty(s)$. Since the discrete series representations $\mathcal{D}_\ell$ and the finite-dimensional representations $\mathcal{F}_\ell$ are subrepresentations and quotients of certain $V_\infty(s)$ (see Proposition 1.3.2), this also gives us the scalars by which Ω acts on these kinds of representations. The following table summarizes the results.

$(\mathfrak{g}, K_\infty)$-module	Ω-eigenvalue	weight structure
$V_\infty(s)$	$\frac{s^2-1}{4}$	all even weights
$\mathcal{D}_\ell$	$\frac{\ell^2-1}{4}$	$[\ldots, -\ell-3,\ -\ell-1\,] \sqcup [\,\ell+1,\ \ell+3, \ldots]$
$\mathcal{F}_\ell$	$\frac{\ell^2-1}{4}$	$[-\ell+1, -\ell+3, \ldots, \ell-3, \ell-1]$

(1.127)

It is now easy to write down a differential equation for $\varphi_0 \in V_\infty(s)$: It is simply

$$\Omega\varphi_0 = \frac{s^2-1}{4}\,\varphi_0, \tag{1.128}$$

a second-order differential equation, in fact satisfied by *every* element of $V_\infty(s)$. The function Φ_0, related to φ_0 by the summation (1.43), then satisfies the same differential equation:

$$\Omega\Phi_0 = \frac{s^2-1}{4}\,\Phi_0. \tag{1.129}$$

Note here that, while Φ_0 is originally defined on $\mathrm{GL}(2,\mathbb{A})$, we considered it as a function on $\mathrm{GL}(2,\mathbb{R})$ only, simply via restriction.

We would also like to see the operator on the upper half plane corresponding to Ω, analogous to Lemma 1.4.2. Let the spaces W and $W(k)$ be defined as in Sect. 1.4.3. Then it is clear from the definition (1.122) and the properties of H, R, L that Ω maps $W(k)$ to itself, for any k. Let Ω_k be the operator $W \to W$ defined by the commutativity of the diagram

$$\begin{array}{ccc} W(k) & \xrightarrow{\sim} & W \\ {\scriptstyle\Omega}\downarrow & & \downarrow{\scriptstyle\Omega_k} \\ W(k) & \xrightarrow{\sim} & W \end{array} \tag{1.130}$$

The following result follows easily from Lemma 1.4.2.

1.4.3 Lemma. *Define an operator Ω_k on the space W of smooth functions on $\mathcal{H}$ by*

$$\Omega_k = y^2\Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\Big) - 2iky\frac{\partial}{\partial\bar\tau} + \frac{k}{2}\Big(\frac{k}{2} - 1\Big).$$

Then the diagram (1.130) is commutative.

Applying this to the Eisenstein series $E(\tau, s)$ defined in (1.42), we are supposed to set $k = 0$, since Φ_0 has weight 0. Hence $\Omega_0 E(\tau, s) = \frac{s^2-1}{4}E(\tau, s)$, i.e.,

$$y^2\Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\Big)E(\tau, s) = \frac{s^2-1}{4}E(\tau, s). \tag{1.131}$$

This is the ***wave equation*** (*Wellengleichung*) considered by Maass in [13].

Remark: The three types of representations listed in Table (1.127) are not quite *all* the irreducible, admissible $(\mathfrak{g}, K_\infty)$-modules for which the center acts trivially. (Here, *admissible* means that each weight occurs only finitely many times in the $(\mathfrak{g}, K_\infty)$-module.) What's missing are the *twists* of $V_\infty(s)$ and $\mathcal{F}_\ell$ by the one-dimensional representations $g \mapsto \mathrm{sgn}(\det(g))$ of $\mathrm{GL}(2, \mathbb{R})$. The discrete series representations $\mathcal{D}_\ell$ are invariant under such twisting. In particular, every irreducible, admissible representation of $\mathrm{GL}(2, \mathbb{R})$ for which the center acts trivially and which has a lowest weight vector of weight k must be isomorphic to $\mathcal{D}_{k-1}$.

1.4.5 Automorphic forms: The official definition

We are now in a position to precisely define the notion of automorphic form on $\mathrm{GL}(2, \mathbb{A})$. These will be continuous functions $\Phi : \mathrm{GL}(2, \mathbb{A}) \to \mathbb{C}$ with certain properties. For simplicity we will only consider functions that are invariant under the center, i.e., that satisfy

$$\Phi(g\begin{bmatrix} z & \\ & z \end{bmatrix}) = \Phi(g) \qquad \text{for all } g \in \mathrm{GL}(2, \mathbb{A}),\ z \in \mathbb{I}. \tag{1.132}$$

Hence, strictly speaking, what we are going to define is the notion of automorphic form on the *projective* group $\mathrm{PGL}(2) = \mathrm{GL}(2)/Z$, where Z is the center of $\mathrm{GL}(2)$.

One of the requirements for an automorphic form is smoothness. We say that $\Phi : \mathrm{GL}(2, \mathbb{A}) \to \mathbb{C}$ is ***smooth***, if the function

$$\mathrm{GL}(2, \mathbb{R}) \times \mathrm{GL}(2, \mathbb{A}_f) \longrightarrow \mathbb{C}, \tag{1.133}$$

$$(g,h) \longmapsto \Phi(gh),$$

is C^∞ in g for fixed h, and is locally constant in h for fixed g. Smoothness in the archimedean variable ensures that we can act on Φ with the Lie algebra $\mathfrak{g} = \mathfrak{gl}(2,\mathbb{R})$ via right translation, i.e.,

$$(X\Phi)(g) = \frac{d}{dt}\Big|_0 \Phi(g\exp(tX)), \qquad g \in \mathrm{GL}(2,\mathbb{A}),\ X \in \mathfrak{g}; \tag{1.134}$$

cf. (1.60). Note that the space of smooth functions on $\mathrm{GL}(2,\mathbb{A})$ is invariant under right translation by all of $\mathrm{GL}(2,\mathbb{A})$. By this we mean that if Φ is smooth and $g \in \mathrm{GL}(2,\mathbb{A})$, then the function $\mathrm{GL}(2,\mathbb{A}) \ni h \mapsto \Phi(hg)$ is also smooth.

Another requirement on automorphic forms is K_∞-finiteness. We say that Φ is ***K_∞-finite*** if the space spanned by all functions $\mathrm{GL}(2,\mathbb{A}) \ni h \mapsto \Phi(hg)$, where g runs through $K_\infty = \mathrm{SO}(2)$, is finite-dimensional. In the context of functions on $\mathrm{GL}(2,\mathbb{R})$ we encountered this property before without actually mentioning it: The functions φ_k in (1.59) are all K_∞-finite (this is trivial, since they all have a weight). In fact, the space $V_\infty(s)$ consists precisely of the K_∞-finite vectors in the smooth version of the induced representation.

The space consisting of all smooth and K_∞-finite functions on $\mathrm{GL}(2,\mathbb{A})$ is no longer invariant under right translation by all of $\mathrm{GL}(2,\mathbb{A})$. But we still have right translation by the groups $K_\infty = \mathrm{SO}(2)$ and $\mathrm{GL}(2,\mathbb{A}_f)$, and we still have the action (1.134) of the Lie algebra. This way the space becomes a $(\mathfrak{g}, K_\infty)$-module and a $\mathrm{GL}(2,\mathbb{A}_f)$-module, thus a representation of $\mathrm{GL}(2,\mathbb{A})$ in the sense of the definition made in Sect. 1.3.2.

An automorphic form is also supposed to satisfy a differential equation when considered as a function of the archimedean variable. As for the type of differential equation, the definition should be general enough so as to include the Φ_0 example in (1.129). Hence, a reasonable attempt would be to require that Φ be an eigenfunction of the Casimir operator:

$$\Omega\Phi = \lambda\Phi \qquad \text{for some } \lambda \in \mathbb{C}. \tag{1.135}$$

This is the same as saying that Φ is annihilated by the element $\Omega - \lambda$, which lies in the center of the universal enveloping algebra; see (1.125). It turns out there is no harm in allowing slightly more general differential equations and only require that

$$f(\Omega)\Phi = 0 \qquad \text{for } \textit{some} \text{ non-constant polynomial } f \in \mathbb{C}[X]. \tag{1.136}$$

Just to be clear, this condition means that $(f(\Omega)\Phi)(g) = 0$ for all elements of $\mathrm{GL}(2,\mathbb{A})$ (not only $\mathrm{GL}(2,\mathbb{R})$).

We can express (1.136) in a very elegant way by considering the center $\mathcal{Z}$ of the universal enveloping algebra $\mathcal{U}(\mathfrak{g}_\mathbb{C})$. In generalization of (1.125) we have $\mathcal{Z} \cong \mathbb{C}[Z,\Omega]$, where $Z = \left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right]$ lies in the center of $\mathfrak{g}_\mathbb{C}$. Consider the vector space $\mathcal{Z}\Phi$ obtained by applying *all* elements of $\mathcal{Z}$ to Φ. The element Z makes no contribution due to the fact that we only consider functions with the property (1.132). Hence $\mathcal{Z}\Phi = \mathbb{C}[\Omega]\Phi$. Evidently, the condition (1.136) means *precisely* that this space is finite-dimensional. We articulate this by saying that Φ is ***$\mathcal{Z}$-finite***.

We are ready to give the official definition of an automorphic form on $\mathrm{GL}(2,\mathbb{A})$. This time we won't be sweeping anything under the rug, but will state the precise list of requirements.

1.4.4 Definition. *Let $\Phi:\ \mathrm{GL}(2,\mathbb{A}) \to \mathbb{C}$ be a function with the property (1.132). Then Φ is an **automorphic form** if it satisfies the following properties.*

i) $\boxed{\Phi(\gamma g) = \Phi(g) \text{ for all } g \in \mathrm{GL}(2,\mathbb{A}) \text{ and } \gamma \in \mathrm{GL}(2,\mathbb{Q}).}$

ii) *Φ is smooth.*

iii) *Φ is right-invariant under $\mathrm{GL}(2,\mathbb{Z}_p)$ for almost all primes p.*

iv) *Φ is K_∞-finite.*

v) *Φ is $\mathcal{Z}$-finite.*

vi) *Φ has moderate growth in the sense of Definition 1.3.3.*

We boxed property i) because one should think of it as the actual automorphic property, and of all the rest as additional regularity conditions.

The classic reference for the definition of automorphic forms, for any group and over any number field, is [2]. We will talk later about groups other than GL(2) and number fields other than $\mathbb{Q}$.

1.5 Cusp forms

While our Eisenstein series examples are interesting enough, they are "easy" automorphic forms in the sense that they are obtained via the process of parabolic induction. Thus, Eisenstein series originate from a proper subgroup of $\mathrm{GL}(2,\mathbb{A})$, namely the Borel subgroup $B(\mathbb{A})$, and are not "native" to $\mathrm{GL}(2,\mathbb{A})$ itself. The most interesting automorphic forms are those that cannot be obtained via parabolic induction. These are called ***cusp forms***, and they are the topic of this section.

1.5.1 Cuspidal automorphic forms

Let's start with a brief review of our Eisenstein series examples. In Sect. 1.2.1 we defined the space $V(s)$ of parabolic induction, where s is a complex parameter. In Sect. 1.2.2 we picked a distinguished vector φ_0 in this infinite-dimensional space. This vector had weight 0 at the archimedean place. The Eisenstein series constructed from φ_0 via the summation (1.39) is an automorphic form, and is closely related to the classical real-analytic Eisenstein series (1.46).

Next we looked at special values of s for which the space $V(s)$ breaks up. More precisely, we looked at $s = k-1$, where k is an even integer $k \geq 4$. In this case $V(s)$ contains the irreducible subrepresentation (1.70). Again we picked a distinguished vector φ_k in this infinite-dimensional space. This time, φ_k had weight k at the archimedean place. The Eisenstein series constructed from φ_k is closely related to the classical holomorphic Eisenstein series (1.76).

What these examples have in common is that they are constructed via *parabolic induction.* By this we mean that the automorphic forms we obtained originated one way or another from a space $V(s)$, which is the space of an induced representation

$$\mathrm{ind}_{B(\mathbb{A})}^{G(\mathbb{A})}(\chi).$$

Here, G abbreviates GL(2), the group $B(\mathbb{A})$ is the ***Borel subgroup*** defined in (1.22), and χ is a certain character of $B(\mathbb{A})$. The principle (1.25) states that such an induced representation will lead to something automorphic, provided the character χ is itself automorphic, i.e., constructed from idele class characters. In our examples, these idele class characters were simply powers of the global absolute value.

Loosely speaking, automorphic forms constructed via induction always "come from a smaller subgroup", in our examples the subgroup $B(\mathbb{A})$. The question arises whether there are automorphic forms on $\mathrm{GL}(2,\mathbb{A})$ that do not arise via induction. These would be automorphic forms that genuinely live on $\mathrm{GL}(2,\mathbb{A})$, and do not come from a smaller subgroup. Evidently, such automorphic forms, if they exist at all, are more difficult to find, but maybe this makes them also more interesting. The question of existence of such automorphic forms is indeed subtle. Just to have a short name for them, let us call these more mysterious automorphic forms ***cusp forms***, without at this point trying to read any meaning into the word "cusp".

We would like to find a simple criterion for cuspidality. For this let us go back to the defining property (1.27) of the functions φ in the space $V(s)$. Recall that (1.27) was already a special case; in general, if χ is as in (1.24), the transformation property of a function φ in $\mathrm{ind}_{B(\mathbb{A})}^{G(\mathbb{A})}(\chi)$ is

$$\varphi(\begin{bmatrix} a & b \\ & d \end{bmatrix} g) = \chi_1(a)\chi_2(d)\varphi(g) \qquad \text{for all } g \in \mathrm{GL}(2,\mathbb{A}),\ a,d \in \mathbb{I},\ b \in \mathbb{A}. \tag{1.137}$$

In particular, φ satisfies

$$\varphi(\begin{bmatrix} 1 & b \\ & 1 \end{bmatrix} g) = \varphi(g) \qquad \text{for all } g \in \mathrm{GL}(2,\mathbb{A}),\ b \in \mathbb{A}. \tag{1.138}$$

This condition is independent of the inducing characters χ_1, χ_2. Specializing further to $g = 1$ (meaning the identity matrix – we refrain from writing 1_2 in order to ease notation), we get

$$\varphi(\begin{bmatrix} 1 & b \\ & 1 \end{bmatrix}) = \varphi(1) \qquad \text{for all } b \in \mathbb{A}. \tag{1.139}$$

If ρ denotes right translation, (1.139) may be rewritten as

$$(\rho(\begin{bmatrix} 1 & b \\ & 1 \end{bmatrix})\varphi)(1) = \varphi(1) \qquad \text{for all } b \in \mathbb{A}. \tag{1.140}$$

This equation says that the evaluation map $\varphi \mapsto \varphi(1)$ provides an $N(\mathbb{A})$-*invariant functional* on the space of $\mathrm{ind}_{B(\mathbb{A})}^{G(\mathbb{A})}(\chi)$, where $N(\mathbb{A})$ is the group defined in (1.23). In an irreducible space, there is always some φ with $\varphi(1) \neq 0$, since otherwise $\varphi(g) = (\rho(g)\varphi)(1)$ would be zero for all $g \in \mathrm{GL}(2,\mathbb{A})$. We proved that, if V is an irreducible subspace of $\mathrm{ind}_{B(\mathbb{A})}^{G(\mathbb{A})}(\chi)$, then the space

$$\mathrm{Hom}_{N(\mathbb{A})}(V,\mathbb{C}) \tag{1.141}$$

is non-zero. Here, we understand that $\mathbb{C}$ carries the trivial $N(\mathbb{A})$ action, so that this Hom space consists precisely of the $N(\mathbb{A})$-invariant functionals on V. With just a little bit more effort, one can show that the converse is also true: If, for a given representation (π, V) of $\mathrm{GL}(2,\mathbb{A})$, the space (1.141) is non-zero, then π is induced from some character χ of $B(\mathbb{A})$. Hence:

$$\pi \text{ is induced from } B(\mathbb{A}) \quad \Longleftrightarrow \quad \mathrm{Hom}_{N(\mathbb{A})}(V,\mathbb{C}) \neq 0 \tag{1.142}$$

This gives us a nice criterion to detect "inducedness". Note in particular that the condition on the right is *model-independent*, meaning that if you switch to an isomorphic representation V', the condition remains the same.

Now assume that (π, V) is an *automorphic* representation of $\mathrm{GL}(2,\mathbb{A})$. Recall that this means that π is irreducible, and that we can realize π as a space of automorphic forms on which $\mathrm{GL}(2,\mathbb{A})$ acts via right translation. *Then it is always possible to write down an element of the space* (1.141), namely,

$$V \ni \Phi \longmapsto \int\limits_{\mathbb{Q}\backslash\mathbb{A}} \Phi(\begin{bmatrix} 1 & x \\ & 1 \end{bmatrix})\, dx. \tag{1.143}$$

The function Φ, being an automorphic form, is left-invariant under rational points, so integrating over $\mathbb{Q}\backslash\mathbb{A}$ makes sense. Moreover, as noted in Sect. 1.1.1, the quotient $\mathbb{Q}\backslash\mathbb{A}$ is compact, so there are no convergence issues. By (1.142), if we want π not to be isomorphic to an induced representation, the functional (1.143) better be zero. Applying the functional to $\rho(g)\Phi$ instead of Φ, we see that then also

$$\int\limits_{\mathbb{Q}\backslash\mathbb{A}} \Phi(\begin{bmatrix} 1 & x \\ & 1 \end{bmatrix} g)\, dx = 0 \qquad \text{for all } g \in \mathrm{GL}(2,\mathbb{A}). \tag{1.144}$$

This will be our official definition of *cusp form*:

1.5.1 Definition. *i) An automorphic form Φ on $\mathrm{GL}(2,\mathbb{A})$ is called a **cusp form** if the condition (1.144) holds.*

*ii) An automorphic representation (π, V) is called a **cuspidal automorphic representation**, if every vector Φ in V is a cusp form.*

One can indeed show that if (π, V) is cuspidal, then it is not induced from $B(\mathbb{A})$. This makes the cuspidal automorphic representations the most interesting ones. Whether they even exist, though, is still not clear at this point!

1.5.2 Exploring the cuspidality condition

Assume that Φ is an automorphic form on $\mathrm{GL}(2,\mathbb{A})$, as defined on page 37. Suppose we would like to verify the cuspidality condition (1.144). Under some simplifying assumptions on Φ, let's see if we can get away with only testing certain elements $g \in \mathrm{GL}(2,\mathbb{A})$. One simplifying assumption which we have always made so far and will make again now, is that Φ is invariant under the center; see (1.132). Another simplifying assumption will be that Φ has weight $k \in \mathbb{Z}$, i.e.,

$$\Phi(gr(\theta)) = e^{ik\theta}\,\Phi(g) \qquad \text{for all } g \in \mathrm{GL}(2,\mathbb{A}),\ \theta \in \mathbb{R}. \tag{1.145}$$

Finally, we will assume that Φ is right invariant under all $\mathrm{GL}(2,\mathbb{Z}_p)$:

$$\Phi(gh) = \Phi(g) \qquad \text{for all } g \in \mathrm{GL}(2,\mathbb{A}),\ h \in \prod_{p<\infty} \mathrm{GL}(2,\mathbb{Z}_p). \tag{1.146}$$

By (1.145) and (1.146), using the Iwasawa decompositions (1.29) and (1.30), or right away the global Iwasawa decomposition (1.32), we see that it is enough to verify (1.144) for upper triangular matrices $g \in B(\mathbb{A})$. Writing g in the form

$$g = \begin{bmatrix} 1 & b \\ & 1 \end{bmatrix} \begin{bmatrix} a_1 & \\ & a_2 \end{bmatrix}, \qquad a_1, a_2 \in \mathbb{I},\ b \in \mathbb{A},$$

the b variable is absorbed into the x variable in (1.144), so that it is enough to verify (1.144) for diagonal matrices:

$$\int\limits_{\mathbb{Q}\backslash\mathbb{A}} \Phi(\begin{bmatrix} 1 & x \\ & 1 \end{bmatrix} \begin{bmatrix} a_1 & \\ & a_2 \end{bmatrix})\,dx = 0 \qquad \text{for all } a_1, a_2 \in \mathbb{I}. \tag{1.147}$$

Next we invoke the decomposition (1.9) and write

$$a_i = \gamma_i y_i \kappa_i, \qquad \gamma_i \in \mathbb{Q}^\times,\ y_i \in \mathbb{R}_{>0},\ \kappa_i \in \prod_{p<\infty} \mathbb{Z}_p^\times.$$

Then

$$\begin{aligned}
\int\limits_{\mathbb{Q}\backslash\mathbb{A}} \Phi(\begin{bmatrix} 1 & x \\ & 1 \end{bmatrix} \begin{bmatrix} a_1 & \\ & a_2 \end{bmatrix})\,dx &= \int\limits_{\mathbb{Q}\backslash\mathbb{A}} \Phi(\begin{bmatrix} 1 & x \\ & 1 \end{bmatrix} \begin{bmatrix} \gamma_1 & \\ & \gamma_2 \end{bmatrix} \begin{bmatrix} y_1 & \\ & y_2 \end{bmatrix} \begin{bmatrix} \kappa_1 & \\ & \kappa_2 \end{bmatrix})\,dx \\
&= \int\limits_{\mathbb{Q}\backslash\mathbb{A}} \Phi(\begin{bmatrix} 1 & x \\ & 1 \end{bmatrix} \begin{bmatrix} \gamma_1 & \\ & \gamma_2 \end{bmatrix} \begin{bmatrix} y_1 & \\ & y_2 \end{bmatrix})\,dx \qquad \text{(by (1.146))} \\
&= \int\limits_{\mathbb{Q}\backslash\mathbb{A}} \Phi(\begin{bmatrix} \gamma_1 & \\ & \gamma_2 \end{bmatrix} \begin{bmatrix} 1 & x\gamma_1^{-1}\gamma_2 \\ & 1 \end{bmatrix} \begin{bmatrix} y_1 & \\ & y_2 \end{bmatrix})\,dx \\
&= \int\limits_{\mathbb{Q}\backslash\mathbb{A}} \Phi(\begin{bmatrix} 1 & x\gamma_1^{-1}\gamma_2 \\ & 1 \end{bmatrix} \begin{bmatrix} y_1 & \\ & y_2 \end{bmatrix})\,dx \qquad \text{(by i) of Def. 1.4.4)} \\
&= \int\limits_{\mathbb{Q}\backslash\mathbb{A}} \Phi(\begin{bmatrix} 1 & x \\ & 1 \end{bmatrix} \begin{bmatrix} y_1 & \\ & y_2 \end{bmatrix})\,dx.
\end{aligned}$$

In the last step we performed a translation $x \mapsto x\gamma_1\gamma_2^{-1}$. (Whenever you integrate over $\mathbb{A}$ and perform a translation $x \mapsto xa$ with $a \in \mathbb{I}$, then you pick up a factor $|a|$. In our case, $|\gamma_1\gamma_2^{-1}| = 1$ by the product formula (1.12).) The upshot of this calculation is that, in order to verify the cuspidality condition (1.144), we only need to verify that

$$\int\limits_{\mathbb{Q}\backslash\mathbb{A}} \Phi(\begin{bmatrix} 1 & x \\ & 1 \end{bmatrix} \begin{bmatrix} y_1 & \\ & y_2 \end{bmatrix})\,dx = 0 \qquad \text{for all } y_1, y_2 \in \mathbb{R}_{>0}. \tag{1.148}$$

Next we invoke (1.6) to write

$$\int\limits_{\mathbb{Q}\backslash\mathbb{A}} \Phi(\begin{bmatrix} 1 & x \\ & 1 \end{bmatrix} \begin{bmatrix} y_1 & \\ & y_2 \end{bmatrix})\,dx = \int\limits_{\mathbb{Z}\backslash\mathbb{R}} \Phi(\begin{bmatrix} 1 & x \\ & 1 \end{bmatrix} \begin{bmatrix} y_1 & \\ & y_2 \end{bmatrix})\,dx. \tag{1.149}$$

Thus everything became reduced to an archimedean integral. Now write $y_1 = y^{1/2}z$ and $y_2 = y^{-1/2}z$ with $y, z > 0$. Observing (1.132), the cuspidality condition becomes

$$\int_{\mathbb{Z}\backslash\mathbb{R}} \Phi(\begin{bmatrix} 1 & x \\ & 1 \end{bmatrix}\begin{bmatrix} y^{1/2} & \\ & y^{-1/2} \end{bmatrix})\, dx = 0 \qquad \text{for all } y \in \mathbb{R}_{>0}. \tag{1.150}$$

We may expand the integrand, as a function of x, into a Fourier series:

$$\Phi(\begin{bmatrix} 1 & x \\ & 1 \end{bmatrix}\begin{bmatrix} y^{1/2} & \\ & y^{-1/2} \end{bmatrix}) = \sum_{n\in\mathbb{Z}} c_n(y)e^{2\pi i n x}, \tag{1.151}$$

with Fourier coefficients c_n that may depend on y. The cuspidality condition (1.150) is then equivalent to

$$c_0(y) = 0 \qquad \text{for all } y > 0 \tag{1.152}$$

(observe that $\int_{\mathbb{Z}\backslash\mathbb{R}} e^{2\pi inx}\, dx = 0$ if $n \neq 0$). Hence, at least under our simplifying assumptions, *cuspidality is equivalent to the vanishing of the* 0*-th Fourier coefficient of* Φ.

As in (1.73), let us define a function F on the upper half plane by

$$F(\tau) := y^{-k/2}\,\Phi(\begin{bmatrix} 1 & x \\ & 1 \end{bmatrix}\begin{bmatrix} y^{1/2} & \\ & y^{-1/2} \end{bmatrix}), \qquad \tau = x + iy,\ x, y \in \mathbb{R},\ y > 0. \tag{1.153}$$

(The factor $y^{-k/2}$ proved useful earlier for weight k functions, so we throw it in again.) As we elaborated in Chapter 1.4, the function F, being a function of a complex variable, may or may not be holomorphic (the holomorphy is related to Φ having the lowest weight property). Assume that F is holomorphic. Then the Cauchy-Riemann differential equations force $c_n(y) = c_n y^{k/2} e^{-2\pi ny}$ for a constant c_n. Substituting, we obtain the Fourier expansion for the function F,

$$F(\tau) = \sum_{n\in\mathbb{Z}} c_n e^{2\pi in\tau}. \tag{1.154}$$

But remember that, by the arguments in Sect. 1.3.4, the growth condition 1.3.3 does not allow for any negative terms in such an expansion. Hence

$$F(\tau) = \sum_{n=0}^{\infty} c_n e^{2\pi in\tau}. \tag{1.155}$$

By (1.152), we see that Φ is cuspidal if and only if $c_0 = 0$. We proved that, under the assumptions (1.132), (1.145) and (1.146), and the assumption that the function F defined in (1.153) is holomorphic, the cuspidality of Φ is equivalent to the vanishing of a single complex number! Of course, these assumptions are a bit restrictive, but they are at least satisfied if Φ corresponds to a classical holomorphic modular form (namely F) of weight k with respect to $\mathrm{SL}(2,\mathbb{Z})$. This will be the topic of the next section.

1.5.3 Classical modular forms

At this point the following definition is in the air, and it would be unwise to avoid it any longer.

1.5.2 Definition. *Let k be an integer. A function $F : \mathcal{H} \to \mathbb{C}$ is called a* ***modular form of weight*** *k if it satisfies the following conditions.*

i) F is holomorphic.

ii) F satisfies the transformation property

$$F\Big(\frac{a\tau+b}{c\tau+d}\Big)(c\tau+d)^{-k} = F(\tau) \qquad \text{for all } \gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}(2,\mathbb{Z}). \tag{1.156}$$

iii) F admits a Fourier expansion of the form

$$F(\tau) = \sum_{n=0}^{\infty} c_n e^{2\pi i n\tau}. \tag{1.157}$$

F is called a ***cusp form*** *if $c_0 = 0$ in the expansion (1.157).*

For example, by (1.77) and (1.93), the Eisenstein series $E_k(\tau)$ is a modular form of weight k; of course, it is not a cusp form. As in (1.78), property (1.156) is often written as

$$F\big|_k\gamma = F \qquad \text{for all } \gamma \in \mathrm{SL}(2,\mathbb{Z}), \tag{1.158}$$

where, for any function $f : \mathcal{H} \to \mathbb{C}$,

$$(f\big|_k h)(\tau) = (c\tau+d)^{-k} f\Big(\frac{a\tau+b}{c\tau+d}\Big) \qquad \text{for } h = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}(2,\mathbb{R}),\ \tau \in \mathcal{H}. \tag{1.159}$$

It will be advantageous to slightly generalize this *slash operation* by admitting matrices $h \in \mathrm{GL}(2,\mathbb{R})$ with positive determinant. Hence, we define

$$(f\big|_k h)(\tau) = \det(h)^{k/2}\,(c\tau+d)^{-k} f\Big(\frac{a\tau+b}{c\tau+d}\Big) \qquad \text{for } h = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}(2,\mathbb{R})^+,\ \tau \in \mathcal{H}. \tag{1.160}$$

It is straightforward to verify that (1.160) defines a *right action* of $\mathrm{GL}(2,\mathbb{R})^+$ on functions $f : \mathcal{H} \to \mathbb{C}$. The effect of the determinant factor is to make the center act trivially.

Recall that E_k originated from an automorphic form Φ_k on $\mathrm{GL}(2,\mathbb{A})$. In fact, we have the following

$$\textbf{General Principle: } \textit{Every modular form } F \textit{ originates from an automorphic form } \Phi. \tag{1.161}$$

To see this, let F be a given modular form of weight k. We will define $\Phi(g)$ for $g \in \mathrm{GL}(2,\mathbb{A})$. Use strong approximation (1.40) to write

$$g = \gamma h \kappa, \qquad \gamma \in \mathrm{GL}(2,\mathbb{Q}),\ h \in \mathrm{GL}(2,\mathbb{R})^+,\ \kappa \in \prod_{p<\infty} \mathrm{GL}(2,\mathbb{Z}_p). \tag{1.162}$$

Then set

$$\Phi(g) = (F\big|_k h)(i). \tag{1.163}$$

Let us verify that $\Phi(g)$ is well-defined. The issue is that the decomposition (1.162) is not unique. Assume we have an alternative decomposition $g = \gamma' h' \kappa'$. Then

$$\gamma'^{-1}\gamma = h'h^{-1}\kappa'\kappa^{-1} \in \mathrm{GL}(2,\mathbb{R})^+ \times \prod_{p<\infty} \mathrm{GL}(2,\mathbb{Z}_p). \tag{1.164}$$

Hence $\gamma'^{-1}\gamma$, which lies in $\mathrm{GL}(2,\mathbb{Q})$ (diagonally embedded into $\mathrm{GL}(2,\mathbb{A})$), lies in $\mathrm{GL}(2,\mathbb{Z}_p)$ for *all* primes p. This means the entries must be integers. Moreover, the determinant of $\gamma'^{-1}\gamma$, being a p-adic unit for all p, must be a unit in $\mathbb{Z}$, i.e., ± 1. But since $\gamma'^{-1}\gamma$ is in $\mathrm{GL}(2,\mathbb{R})^+$, the determinant must be $+1$. This shows $\tilde\gamma := \gamma'^{-1}\gamma \in \mathrm{SL}(2,\mathbb{Z})$. Looking only at the archimedean component in (1.164), we see $h' = \tilde\gamma h$. It follows that

$$F\big|_k h' = F\big|_k(\tilde\gamma h) = (F\big|_k\tilde\gamma)\big|_k h = F\big|_k h, \tag{1.165}$$

where the last step is justified by (1.158). This shows that the definition (1.163) is indeed unambiguous. In fact, Φ is well-defined precisely because F is a modular form!

It is easy to see that

$$F(\tau) = y^{-k/2}\,\Phi(\begin{bmatrix} 1 & x \\ & 1 \end{bmatrix}\begin{bmatrix} y^{1/2} & \\ & y^{-1/2} \end{bmatrix}), \qquad \tau = x + iy,\ y > 0, \tag{1.166}$$

just as in (1.73). This shows that F comes from an adelic function Φ, which is in fact uniquely determined by F.

Let us verify that Φ is an automorphic form in the sense of Definition 1.4.4. It is obvious from the construction that

$$\Phi(\gamma g \kappa) = \Phi(g) \qquad \text{for all } g \in \mathrm{GL}(2,\mathbb{A}),\ \gamma \in \mathrm{GL}(2,\mathbb{Q}),\ \kappa \in \prod_{p<\infty}\mathrm{GL}(2,\mathbb{Z}_p). \tag{1.167}$$

Hence Φ satisfies i) and iii) of Definition 1.4.4. The smoothness of Φ follows from the smoothness of F. A simple calculation shows that

$$\Phi(gr(\theta)) = e^{ik\theta}\,\Phi(g) \qquad \text{for all } g \in \mathrm{GL}(2,\mathbb{A}),\ \theta \in \mathbb{R}. \tag{1.168}$$

Hence Φ is not only K_∞-finite, but is in fact a function of weight k. For $\mathcal{Z}$-finiteness, observe that, since F is holomorphic,

$$\Omega_k F = \frac{k}{2}\Big(\frac{k}{2} - 1\Big)F; \tag{1.169}$$

here Ω_k is the operator defined in Lemma 1.4.3. By this lemma,

$$\Omega\Phi = \frac{k}{2}\Big(\frac{k}{2} - 1\Big)\Phi. \tag{1.170}$$

As we explained before Definition 1.4.4, being an eigenvector for Ω implies that Φ is $\mathcal{Z}$-finite.

Finally, moderate growth of Φ follows because there are no negative terms in the Fourier expansion (1.157); see our considerations following Definition 1.3.3. We verified that Φ is indeed an automorphic form.

The function Φ actually satisfies a stronger condition than being an eigenvector for Ω. Namely, with $L \in \mathfrak{sl}(2,\mathbb{C})$ being the element in (1.104), we have $L\Phi = 0$. This follows from Lemma 1.4.2 and is equivalent to the fact that F is holomorphic.

Our arguments have proven the following result.

1.5.3 Proposition. *The space of modular forms of weight k is isomorphic to the space of functions $\Phi:\ \mathrm{GL}(2,\mathbb{A}) \to \mathbb{C}$ satisfying the following conditions.*

i) $\Phi(\gamma g) = \Phi(g)$ *for all* $g \in \mathrm{GL}(2,\mathbb{A})$ *and* $\gamma \in \mathrm{GL}(2,\mathbb{Q})$.

ii) Φ *is smooth.*

iii) Φ *is right-invariant under* $\mathrm{GL}(2,\mathbb{Z}_p)$ *for all primes* p.

iv) $\Phi(gr(\theta)) = e^{ik\theta}\,\Phi(g)$ *for all* $g \in \mathrm{GL}(2,\mathbb{A})$ *and* $\theta \in \mathbb{R}$.

v) $L\Phi = 0$.

vi) Φ *has moderate growth in the sense of Definition 1.3.3.*

Given Φ, *the corresponding modular form* F *is given by* (1.166). *Given* F, *the corresponding* Φ *is given by* (1.163). *The function* Φ *satisfies the cuspidality condition*

$$\int\limits_{\mathbb{Q}\backslash\mathbb{A}} \Phi\Big(\begin{bmatrix} 1 & x \\ & 1 \end{bmatrix} g\Big)\, dx = 0 \qquad \text{for all } g \in \mathrm{GL}(2,\mathbb{A}) \tag{1.171}$$

if and only if F *is a cusp form.*

The principle (1.161) does not only hold for the type of modular forms defined in 1.5.2. It also holds for ***Maass forms*** (defined on $\mathcal{H}$), ***Hilbert modular forms*** (defined on several copies of $\mathcal{H}$), ***Siegel modular forms*** (defined on higher-dimensional generalizations of $\mathcal{H}$), and really every kind of modular form defined on some kind of "classical" domain. These all correspond to automorphic forms on various groups defined over various fields.

1.5.4 Ramanujan's Δ-function

We can finally prove that cusp forms, in the sense of Definition 1.5.1, do exist. The point is that *classical* cusp forms, as in Definition 1.5.2, are *easy* to construct from Eisenstein series. We can then use the correspondence expressed in Proposition 1.5.3 to obtain an adelic cusp form.

To construct cusp forms from the non-cuspidal Eisenstein series, first observe that modular forms can be *multiplied* to obtain other modular forms. More precisely, if F_1 has weight k_1 and F_2 has weight k_2, then F_1F_2 has weight $k_1 + k_2$. This is immediate from Definition 1.5.2.

1.5.4 Exercise. *Use Definition 1.4.4 to verify that automorphic forms can be multiplied to obtain other automorphic forms: The space of automorphic forms is in fact* a ring. *Prove also that* cusp forms *constitute an ideal in this ring.*

Now consider the Eisenstein series E_4 and E_6. These are modular forms of respective weights 4 and 6, with Fourier expansions given by (1.93). Take E_4^3 and E_6^2. These are two modular forms of weight 12. Whatever their Fourier expansions are, they both have a constant term 1. Hence, the function

$$\Delta := E_4^3 - E_6^2 \tag{1.172}$$

has no constant term. One quickly verifies from the Fourier expansions that it is not zero. It is therefore a cusp form of weight 12. We finally proved that cusp forms exist!

The Δ function is quite famous and was studied by Ramanujan [17]. Its Fourier expansion is usually written as

$$\Delta(z) = \sum_{n=1}^{\infty} \tau(n) e^{2\pi i n z}. \tag{1.173}$$

The Fourier coefficients $\tau(n)$ are known to be integers. The series $\tau(n)$ is often referred to as ***Ramanujan's τ-function*** and has been well studied.

One of the interesting questions about $\tau(n)$ is how fast this sequence grows. It is not difficult to prove that it grows at most like n^6. Ramanujan conjectured that it growths only like $n^{5.5}$. This turned out to be true, but was only proven in the 1970's as a consequence of work of Deligne. The Ramanujan conjecture, strange as it looks, has a beautiful interpretation in terms of automorphic representations; see the end of Sect. 1.6.5. A general version of this conjecture is believed to be true for all cuspidal automorphic representations, but this conjecture is still open, even for a group as small as GL(2).

1.5.5 Cuspidal automorphic representations

Now we've talked a lot about cuspidal automorphic *forms*, but less about cuspidal automorphic *representations*. Recall from Definition 1.5.1 that an automorphic representation (π, V) is called cuspidal if it consists entirely of cusp forms. One needs to verify the cuspidality only for *one* automorphic form in V:

1.5.5 Exercise. *Show that if (π, V) is an automorphic representation, and $\Phi_0 \in V$ is cuspidal, then every $\Phi \in V$ is cuspidal.*

As a hint, observe that, according to Definition 1.2.4, automorphic representations are always *irreducible*.

We proved in the previous section that cusp forms do exist. Let Φ be one of them; for example, Φ could be the cusp form corresponding to Ramanujan's Δ-function via Proposition 1.5.3. Can we construct from Φ a cuspidal automorphic representation? The answer is yes: We can let Φ generate a representation by taking the space V spanned by all right translates of Φ. Here, by "right translates" we mean all functions of the form

$$\rho(X)\rho(h)\rho(g)\Phi, \qquad X \in \mathfrak{g},\ h \in K_\infty,\ g \in \mathrm{GL}(2, \mathbb{A}_f), \tag{1.174}$$

where the Lie algebra $\mathfrak{g} = \mathfrak{gl}(2, \mathbb{R})$ acts on Φ via (1.134), and where $(\rho(g)\Phi)(x) = \Phi(xg)$, as usual. In other words, we let V be the smallest $(\mathfrak{g}, K_\infty)$-and $\mathrm{GL}(2, \mathbb{A}_f)$-module containing Φ. Note that V consists entirely of cuspidal automorphic forms.

We are very close to having constructed a cuspidal automorphic representation, BUT: Recall that, by definition, automorphic representations are supposed to be *irreducible*, and there is no guarantee that this is the case for V. One can show, however, that V breaks up into a *finite direct sum* of irreducible, invariant subspaces:

$$V = V_1 \oplus \ldots \oplus V_n. \tag{1.175}$$

Each of the V_i is indeed a cuspidal automorphic representation of $\mathrm{GL}(2, \mathbb{A})$. We stress that the breaking up as a direct sum of irreducibles is only true since Φ is a *cusp form*.

Recall from (1.67) that the globally induced representation $V(s)$ is a *restricted tensor product* of the locally induced representations $V_p(s)$ over all places p. There is an important result called the ***Tensor Product Theorem*** which states that *every* irreducible and admissible representation π of $\mathrm{GL}(2, \mathbb{A})$ can be written in such a way:

$$\pi \cong \bigotimes_{p \leq \infty} \pi_p, \tag{1.176}$$

where π_p is an irreducibe and admissible representation of the local group $\mathrm{GL}(2, \mathbb{Q}_p)$. Here, the precise meaning of "admissible", both in the global and the local case, need not concern us; the great majority of representations one encounters naturally are admissible. For a proof of the Tensor Product Theorem, see [5] or Theorem 3.3.3 of [3]. We will make a few more comments on this theorem in Sect. 1.6.1.

The Tensor Product Theorem applies in particular to the spaces V_i in (1.175). Let us denote by π_i the representation of $\mathrm{GL}(2, \mathbb{A})$ on V_i. Then

$$\pi_i \cong \bigotimes_{p \leq \infty} \pi_{i,p}, \tag{1.177}$$

where $\pi_{i,p}$ is an irreducibe, admissible representation of the local group $\mathrm{GL}(2, \mathbb{Q}_p)$. A natural question is, what *are* these local representations? For example, if the archimedean component $\pi_{i,\infty}$ is one of the three types listed in (1.127), which one is it?

Let's answer this question at least in one case, namely when Φ corresponds to the Ramanujan Δ-function. Recall that Δ is a holomorphic modular form of weight 12. By Lemma 1.4.2, the holomorphy of Δ is equivalent to $L\Phi = 0$ (this has also been stated as part of Proposition 1.5.3). It follows that $\pi_{i,\infty}$ contains a lowest weight vector of weight 12. There is only one representation in (1.127) with this property, namely $\mathcal{D}_{11}$. Hence $\pi_{i,\infty} \cong \mathcal{D}_{11}$. (See our remark at the end of Sect. 1.4.4.

The same argument shows that, whenever Φ corresponds to a holomorphic cusp form F of even weight $k > 0$, then $\pi_{i,\infty} \cong \mathcal{D}_{k-1}$ for all i.

In the case of Φ corresponding to Ramanujan's Δ-function, it turns out that V itself is irreducible. This is indeed the case whenever Φ corresponds to an *eigenform* under all *Hecke operators*. We have not introduced Hecke operators yet, but you might want to keep in mind for later that, under the correspondence between cuspidal Φ and F expressed in Proposition 1.5.3, *the irreducibility of the automorphic representation generated by Φ is equivalent to F being an eigenform for all Hecke operators.*

1.6 L-functions

Automorphic representations are complicated objects: Infinite tensor products of (almost always) infinite-dimensional representations. It turns out that one can associate to an automorphic representation, or in fact to *any* irreducible representation π of an adelic group like $\mathrm{GL}(2,\mathbb{A})$, a much simpler object, namely a function $L(s,\pi)$ of a complex variable s, which still captures some essential information about π.

If π is automorphic, this ***L-function*** satisfies a ***functional equation***. The prototype is the completed ***Riemann zeta function*** $Z(s)$, which satisfies the functional equation $Z(s) = Z(1-s)$. As we will see in (1.190), the underlying automorphic representation that has $Z(s)$ as its L-function is the *trivial* representation of the group $\mathrm{GL}(1,\mathbb{A})$.

1.6.1 The Tensor Product Theorem

In Sect. 1.5.5 we employed the *Tensor Product Theorem.* Recall its statement: If π is an irreducible, admissible representation of $\mathrm{GL}(2,\mathbb{A})$, then

$$\pi \cong \bigotimes_{p\leq\infty} \pi_p, \tag{1.178}$$

with irreducible, admissible representations π_p of the local groups $\mathrm{GL}(2,\mathbb{Q}_p)$. The precise definition of admissibility need not concern us at the moment. A few more remarks concerning the Tensor Product Theorem are in order.

- The theorem holds for *any* irreducible, admissible π and has nothing to do with π being automorphic.

- The isomorphism (1.178) is *abstract.* All you know is that it intertwines the $(\mathfrak{g},K_\infty)$-module and $\mathrm{GL}(2,\mathbb{A}_f)$-module structure. The right hand side of (1.178) knows nothing about the specific *model* of π.

- The π_p are automatically ***unramified*** (or ***spherical***) for almost all primes p. By definition, π_p being unramified means that there exists a non-zero vector v_0 in π_p invariant under $\mathrm{GL}(2,\mathbb{Z}_p)$. For example, the vector $\varphi_0 \in V_p(s)$ defined in (1.55) has this property.

- If you have a collection of local representations π_p, one for each $p \leq \infty$, and if almost all of them are unramified, then you can define a global representation π via (1.178).

- A random choice of local representations π_p will pretty much *never* lead to an *automorphic* representation π (in the sense of Definition 1.2.4). Automorphy is a very special coherence property among the π_p. If π as in (1.178) is automorphic, and if you switch out a *single* π_p for another representation, then it is very likely that you have destroyed the automorphic property.

The Tensor Product Theorem holds not only for $\mathrm{GL}(2)$, but also for $\mathrm{GL}(1)$, or $\mathrm{GL}(n)$, or in fact for much more general groups. Let's see what it says for $\mathrm{GL}(1)$. Recall that $\mathrm{GL}(1,\mathbb{A}) = \mathbb{I}$, the group of ideles. This group is abelian, so any irreducible representation is simply a character

$\mathbb{I} \to \mathbb{C}^\times$. As is customary, let us write χ instead of π for such a character. The Tensor Product Theorem then says that

$$\chi(x) = \prod_{p \le \infty} \chi_p(x_p), \qquad x = (x_p) \in \mathbb{I}, \tag{1.179}$$

with *local* characters $\chi_p : \mathbb{Q}_p^\times \to \mathbb{C}^\times$. Of course, in this case it is not a deep theorem: χ_p is simply the restriction of χ to the local group $\mathbb{Q}_p^\times$ (embedded into $\mathbb{I}$ at the p-th place).

But observe that almost all χ_p have to be *unramified.* In the case of GL(1) this means $\chi_p|_{\mathbb{Z}_p^\times} = 1$: The restriction of χ_p to the compact group $\mathrm{GL}(1, \mathbb{Z}_p) = \mathbb{Z}_p^\times$ has to be trivial. Recall from (1.10) that almost all x_p lie in $\mathbb{Z}_p^\times$. Hence, the product in (1.179) is *finite.* This is good; we don't want any subtle convergence issues here.

Conversely, given characters $\chi_p : \mathbb{Q}_p^\times \to \mathbb{C}^\times$ almost all of which are unramified, we may define $\chi : \mathbb{I} \to \mathbb{C}^\times$ by the formula (1.179). Very rarely though will we pick up an *automorphic* χ, meaning one that is trivial on $\mathbb{Q}^\times$. See Sect. 1.1.3 on how to obtain automorphic representations of $\mathrm{GL}(1, \mathbb{A})$ (also known as idele class characters) using strong approximation.

FAQ

- *I've heard about "strong multiplicity one"? What is it, and how is it related to the last bullet point above?*

  Strong Multiplicity One is a theorem for cuspidal automorphic representations of the group $\mathrm{GL}(n, \mathbb{A})$. It states that if

  $$\pi = \otimes \pi_p \qquad \text{and} \qquad \pi' = \otimes \pi'_p$$

  are two such representations, and if $\pi_p \cong \pi'_p$ for almost all p, then in fact $\pi_p \cong \pi'_p$ for *all* p. In other words, π and π' are globally isomorphic if they are locally isomorphic almost everywhere. Not only that, but it is even true that π and π', when realized as spaces of automorphic forms, are *identical.* In this sense cuspidal automorphic representations of $\mathrm{GL}(n, \mathbb{A})$ are rigid: In $\pi = \otimes \pi_p$, one cannot change finitely many π_p and still obtain another cuspidal automorphic representation.

  For groups other than GL(n), or for non-cuspidal representations, strong multiplicity one may fail. In these cases two automorphic representations may be locally almost everywhere equivalent without being globally equivalent. This is related to the phenomenon of *L-packets*: Local representations are grouped into finite packets, and exchanging a π_p by a representation in the same packet will conjecturally preserve the automorphic property. Thus, referring to the last bullet point on page 48, there are indeed situations where one can exchange a single π_p and retain the automorphic property.

  For GL(n) all L-packets have only one element, so this phenomenon does not occur. For more general groups, L-packets represent one of the complications in automorphic theory that are not yet fully understood.

1.6.2 L-functions for GL(1)

To each character χ of $\mathrm{GL}(1,\mathbb{A})$ we will associate a function $L(s,\chi)$ of a complex variable s. This ***global L-function*** will be a product of ***local L-factors***,

$$L(s,\chi)=\prod_{p\le\infty}L(s,\chi_p), \tag{1.180}$$

where the χ_p are the local characters appearing in the decomposition (1.179). The local factors $L(s,\chi_p)$ for $p<\infty$ look rather different from $L(s,\chi_\infty)$. We will first define all the local factors, also known as ***Euler factors***, and then think about convergence of the product (1.180).

Let's define the non-archimedean factors first. Recall that χ_p is a character of $\mathbb{Q}_p^\times$, and that we called χ_p unramified if it is trivial on $\mathbb{Z}_p^\times$. Now $\mathbb{Q}_p^\times$ is a direct product

$$\mathbb{Q}_p^\times=\langle p\rangle\times\mathbb{Z}_p^\times, \tag{1.181}$$

where $\langle p\rangle$ is the cyclic group generated by the prime p. Hence, if χ_p is unramified, then it is determined by the single number $\alpha:=\chi_p(p)$. This non-zero complex number is called the ***Satake parameter*** of χ_p. The definition of $L(s,\chi_p)$ is now as follows,

$$L(s,\chi_p)=\begin{cases}\dfrac{1}{1-\chi_p(p)p^{-s}} & \text{if } \chi_p \text{ is unramified,}\\ 1 & \text{if } \chi_p \text{ is ramified.}\end{cases} \tag{1.182}$$

Hence, in the unramified case, χ_p can be easily recovered from $L(s,\chi_p)$. Since almost all the χ_p in (1.179) are automatically unramified, we see that the product (1.180) contains a lot of information about π itself: Almost all the local components χ_p can be recovered from $L(s,\pi)$.

Next, let us define the archimedean factors $L(s,\chi_\infty)$. There are two kinds of characters χ_∞ of $\mathrm{GL}(1,\mathbb{R})=\mathbb{R}^\times$: Those with $\chi_\infty(-1)=1$, and those with $\chi_\infty(-1)=-1$. In the first case,

$$\chi_\infty(a)=|a|^t \qquad \text{for some } t\in\mathbb{C}. \tag{1.183}$$

In the second case,

$$\chi_\infty(a)=\mathrm{sgn}(a)|a|^t \qquad \text{for some } t\in\mathbb{C}. \tag{1.184}$$

We define

$$L(s,\chi_\infty)=\begin{cases}\pi^{-\frac{s+t}{2}}\,\Gamma\Big(\dfrac{s+t}{2}\Big) & \text{if } \chi_\infty \text{ is of the form (1.183),}\\ \pi^{-\frac{s+t+1}{2}}\,\Gamma\Big(\dfrac{s+t+1}{2}\Big) & \text{if } \chi_\infty \text{ is of the form (1.184).}\end{cases} \tag{1.185}$$

Here,

$$\Gamma(s)=\int_0^\infty y^{s-1}\,e^{-y}\,dy \tag{1.186}$$

is the usual Γ-function.

First example

Of course, the definitions (1.182) and (1.185) are rather unmotivated at this point. Let's consider some examples and see if they make sense. We will start with the simplest automorphic representation of all, the *trivial* representation of $\mathrm{GL}(1,\mathbb{A})$. Hence, $\chi : \mathbb{I} \to \mathbb{C}^\times$ is constantly 1. So are all its local components χ_p. According to (1.182),

$$L(s,\chi_p) = \frac{1}{1-p^{-s}} \qquad \text{for } p < \infty, \tag{1.187}$$

and according to (1.185),

$$L(s,\chi_\infty) = \pi^{-\frac{s}{2}}\,\Gamma\Big(\frac{s}{2}\Big). \tag{1.188}$$

Therefore, the global L-function is given by

$$L(s,\chi) = \pi^{-\frac{s}{2}}\,\Gamma\Big(\frac{s}{2}\Big) \prod_{p<\infty} \frac{1}{1-p^{-s}}. \tag{1.189}$$

Of course you recognize this function: The product

$$\zeta(s) = \prod_{p<\infty} \frac{1}{1-p^{-s}} \tag{1.190}$$

is the famous ***Riemann zeta function***; it can also be written as $\sum_{n=1}^\infty n^{-s}$. The product in (1.189) is known as the *completed* Riemann zeta function. Remarkably, the L-function of the simplest automorphic representation – the trivial representation – is one of the most interesting functions in all of mathematics!

As for convergence, it is easy to see that the product (1.190) is absolutely convergent for $\mathrm{Re}(s) > 1$. Riemann proved in his famous 1859 paper [18] that $\zeta(s)$ has a meromorphic continuation to the entire complex plane, and that the *completed* zeta function (1.189) – let's call it $Z(s)$ – satisfies the ***functional equation***

$$Z(s) = Z(1-s). \tag{1.191}$$

This will be a hallmark of those $L(s,\pi)$ for which π is *automorphic*: They all can be meromorphically continued to the entire complex plane and satisfy a functional equation similar to (1.191).

Second example

As our second example, let N be a positive integer and $\tilde\chi$ a primitive Dirichlet character mod N. Then $\tilde\chi$ determines an idele class character χ, as explained in Sect. 1.1.3. (In fact, up to multiplication by a power of the global absolute value, *all* χ are obtained in this way; see Proposition 1.1.2.) We would like to calculate $L(s,\chi)$, for which we need to factor χ in the form (1.179). See if you can prove the following facts:

- χ_p is unramified precisely for $p \nmid N$.
- For $p \nmid N$, the Satake parameter is given by $\chi_p(p) = \tilde\chi(p)^{-1}$.

- χ_∞ is trivial on $\mathbb{R}_{>0}$.
- $\chi_\infty(-1) = \tilde{\chi}(-1)$.

This is enough to determine the L-function: If $\tilde{\chi}$ is an *even* Dirichlet character, i.e., if $\tilde{\chi}(-1) = 1$, then

$$L(s,\chi) = \pi^{-\frac{s}{2}}\,\Gamma\Big(\frac{s}{2}\Big)\prod_{p<\infty}\frac{1}{1-\tilde{\chi}(p)^{-1}p^{-s}}. \tag{1.192}$$

If $\tilde{\chi}$ is an *odd* Dirichlet character, i.e., if $\tilde{\chi}(-1) = -1$, then

$$L(s,\chi) = \pi^{-\frac{s+1}{2}}\,\Gamma\Big(\frac{s+1}{2}\Big)\prod_{p<\infty}\frac{1}{1-\tilde{\chi}(p)^{-1}p^{-s}}. \tag{1.193}$$

The non-archimedean part

$$L(s,\tilde{\chi}^{-1}) := \prod_{p<\infty}\frac{1}{1-\tilde{\chi}(p)^{-1}p^{-s}} \tag{1.194}$$

is known as a ***Dirichlet*** L***-function***, and (1.192), (1.193) are *completed* Dirichlet L-functions[7]. Just as for the Riemann zeta function, the infinite product is known to converge for $\mathrm{Re}(s) > 1$, has meromorphic continuation to all of $\mathbb{C}$, and satisfies a functional equation. The latter now takes the slightly more complicated form

$$L(s,\chi^{-1}) = i^{-\delta}\tau(\tilde{\chi})N^{-s}L(1-s,\chi), \tag{1.195}$$

where $\delta \in \{0,1\}$ is defined by $\tilde{\chi}(-1) = (-1)^\delta$, and where $\tau(\tilde{\chi})$ denotes the *Gauss sum*

$$\tau(\tilde{\chi}) = \sum_{j\in(\mathbb{Z}/N\mathbb{Z})^\times}\tilde{\chi}(j)e^{2\pi i j/N}. \tag{1.196}$$

In view of Proposition 1.1.2 we can say that *the* L*-functions of automorphic representations of* $\mathrm{GL}(1,\mathbb{A})$ *are precisely the completed Dirichlet* L*-functions* (and shifts of such, coming from the power of the absolute value). The latter, which are generalizations of the Riemann zeta function, are of great importance in number theory. Seems we are on to something!

Taking averages

The fact that the local factors (1.182) and (1.185) lead precisely to the Dirichlet L-functions with the "correct" archimedean completion factors provides a certain justification for their definition. To motivate these factors even further, let's take an "average" of an idele class character χ. After all, taking averages of an otherwise complicated function can give valuable information.[8] In fact, we may elevate this observation to a

> **General Principle:** *Taking averages of automorphic objects reveals important information.* (1.197)

[7] We apologize for the somewhat confusing notation: The L-function in (1.194), which is the standard notation for a Dirichlet L-function, is an Euler product over finite primes only. Those in (1.192) and (1.193) are Euler products over the finite primes *and* the archimedean place. In general, if you see something like $L(s,\pi)$ in the literature, you should make sure what the definitions are: Sometimes this symbol stands for a complete, sometimes for an incomplete Euler product.

[8] On a road trip, you are probably more interested in your average speed rather than the detailed time-velocity graph of your car.

Average of course means integration, so what we have in mind is something like

$$\int_{\mathbb{I}} \chi(a)\, d^\times a, \tag{1.198}$$

where $d^\times a$ denotes a Haar measure on the ideles. Now this doesn't make sense for various reasons, one of them being convergence. Also, even if it were convergent, integration over a character gives zero unless the character is trivial. So what we are going to do is to throw in a ***Schwartz function*** f:

$$\int_{\mathbb{I}} \chi(a) f(a)\, d^\times a. \tag{1.199}$$

The effect of f will be to destroy the character property, so that we won't automatically get zero, and also to make the integral convergent. We will construct f as a pure tensor of *local* Schwartz functions:

$$f(a) = \prod_{p \le \infty} f_p(a_p), \qquad a = (a_p) \in \mathbb{A}. \tag{1.200}$$

Note that f is a function on $\mathbb{A}$; the integral (1.199) will only see its restriction to $\mathbb{I}$. Now local Schwartz functions are *rapidly decreasing* for $p = \infty$, and *locally constant with compact support* for $p < \infty$. The prototypical examples are

$$f_\infty(a) = e^{-\pi a^2} \qquad \text{for } a \in \mathbb{R}, \tag{1.201}$$

and

$$f_p(a) = \begin{cases} 1 & \text{if } a \in \mathbb{Z}_p, \\ 0 & \text{if } a \notin \mathbb{Z}_p, \end{cases} \tag{1.202}$$

for $p < \infty$. If $\chi = \otimes \chi_p$, then the integral (1.199) factors as

$$\int_{\mathbb{I}} \chi(a) f(a)\, d^\times a = \prod_{p \le \infty} \int_{\mathbb{Q}_p^\times} \chi_p(a) f_p(a)\, d^\times a. \tag{1.203}$$

Let us calculate each local integral assuming that $\chi = |\cdot|^s = \otimes |\cdot|_p^s$ and f_p as in (1.201) and (1.202). For $p < \infty$,

$$\begin{aligned}
\int_{\mathbb{Q}_p^\times} \chi_p(a) f_p(a)\, d^\times a &= \int_{\mathbb{Z}_p \setminus \{0\}} |a|_p^s\, d^\times a && \text{(by definition of } f_p) \\
&= \sum_{n=0}^{\infty} \int_{p^n \mathbb{Z}_p^\times} |a|_p^s\, d^\times a && \Big(\text{since } \mathbb{Z}_p = \bigsqcup_{n=0}^{\infty} p^n \mathbb{Z}_p^\times\Big) \\
&= \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p^\times} |p^n a|_p^s\, d^\times a && \text{(property of Haar measure)} \\
&= \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p^\times} |p^n|_p^s\, d^\times a
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty} |p^n|_p^s && \text{(normalization of Haar measure)} \\
&= \sum_{n=0}^{\infty} p^{-ns} && \text{(definition of } |\cdot|_p) \\
&= \frac{1}{1-p^{-s}}. && \text{(geometric series)}
\end{aligned}
$$

For $p = \infty$,

$$
\begin{aligned}
\int\limits_{\mathbb{Q}_p^\times} \chi_p(a) f_p(a)\, d^\times a &= \int\limits_{\mathbb{R}^\times} |a|_\infty^s \, e^{-\pi a^2} \, d^\times a && \text{(definition of } f_\infty) \\
&= 2 \int\limits_0^\infty a^s \, e^{-\pi a^2} \, d^\times a \\
&= \int\limits_0^\infty y^{s/2} \, e^{-\pi y} \, d^\times y \\
&= \pi^{-s/2} \int\limits_0^\infty y^{s/2} \, e^{-y} \, d^\times y && \text{(property of Haar measure)} \\
&= \pi^{-s/2} \int\limits_0^\infty y^{s/2-1} \, e^{-y} \, dy && (d^\times y = \frac{dy}{y}) \\
&= \pi^{-s/2} \, \Gamma\Big(\frac{s}{2}\Big). && \text{(see (1.186))}
\end{aligned}
$$

We see that we get *precisely*

$$
\int\limits_{\mathbb{I}} |a|^s \, f(a) \, d^\times a = Z(s), \tag{1.204}
$$

the completed Riemann zeta function. One could do a more general calculation and see that the Dirichlet L-functions also appear as averages of automorphic forms χ (weighted by appropriate Schwartz functions). This may provide another motivation for the definitions (1.182) and (1.185).

Remarks:

i) Integrals as in (1.199) are called *zeta integrals*, the origin of the name propably being the connection with the Riemann zeta function. They were studied by Tate in his famous 1950 thesis [19]. Using Fourier analysis, he showed that they can be used to prove the functional equations (1.191) and (1.195).

ii) In a formula like (1.204) one would say that the zeta integral on the left hand side *represents* the L-function on the right hand side. Representing L-functions as averages over automorphic forms is one of the main methods to prove statements about L-functions like analytic continuation and functional equation.

1.6.3 L-functions for GL(2)

In this section we consider L-functions for irreducible, admissible representations of $\mathrm{GL}(2,\mathbb{A})$. By the Tensor Product Theorem, every such representation factors as

$$\pi = \bigotimes_{p\leq\infty} \pi_p, \tag{1.205}$$

where π_p is an irreducible, admissible representation of the local group $\mathrm{GL}(2,\mathbb{Q}_p)$. The L-function of π is defined as an infinite product

$$L(s,\pi) = \prod_{p\leq\infty} L(s,\pi_p). \tag{1.206}$$

Our job is to describe the local L-factors $L(s,\pi_p)$. To give a quick impression, we will not do this in generality, but only for some of the representations we have encountered. In the p-adic case,

$$L(s,\pi_p) = \begin{cases} \dfrac{1}{(1-p^{-s+t/2})(1-p^{-s-t/2})} & \text{if } \pi_p = V_p(t), \\ \dfrac{1}{1-p^{-s-1/2}} & \text{if } \pi_p = \mathrm{St}_{\mathrm{GL}(2)}. \end{cases} \tag{1.207}$$

(Recall the Steinberg representation $\mathrm{St}_{\mathrm{GL}(2)}$ introduced on page 20.) For the archimedean case, it will be convenient to introduce the abbreviations

$$\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\,\Gamma\Big(\frac{s}{2}\Big), \qquad \Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\,\Gamma(s). \tag{1.208}$$

Note that $\Gamma_{\mathbb{R}}(s)$ is the archimedean factor for the Riemann zeta function; see (1.188). With these notations, the L-factors for the representations occuring in Table (1.127) are

$$L(s,\pi_\infty) = \begin{cases} \Gamma_{\mathbb{R}}\Big(s+\dfrac{t}{2}\Big)\Gamma_{\mathbb{R}}\Big(s-\dfrac{t}{2}\Big) & \text{if } \pi_\infty = V_\infty(t), \\ \Gamma_{\mathbb{C}}\Big(s+\dfrac{\ell}{2}\Big) & \text{if } \pi_\infty = \mathcal{D}_\ell, \\ \Gamma_{\mathbb{R}}\Big(s+\dfrac{\ell}{2}\Big)\Gamma_{\mathbb{R}}\Big(s-\dfrac{\ell}{2}\Big) & \text{if } \pi_\infty = \mathcal{F}_\ell. \end{cases} \tag{1.209}$$

These formulas are enough to write down the L-functions of the automorphic representations in our Eisenstein series examples. Let's work this out.

L-functions for real analytic Eisenstein series

Recall from Sect. 1.2.3 that the real analytic Eisenstein series $E(\tau,s)$ originated from a distinguished vector Φ_0 in the automorphic representation $V(s)$. In this section we will write t instead of s, so as not to get in conflict with the variable for the L-function. We saw in Sect. 1.3.2 that

$$V(t) = \prod_{p\leq\infty} V_p(t) \tag{1.210}$$

with local induced representations $V_p(t)$. Note that the standard model for $V(t)$ is not a space of automorphic forms, but functions on $\mathrm{GL}(2,\mathbb{A})$ with the typical transformation property of an induced representation. However, the summation process (1.39) leads to a model for $V(t)$ consisting of automorphic forms, and according to Definition 1.2.4 this is enough to call $V(t)$ an automorphic representation. For the calculation of the L-function, *any* model for $\pi := V(t)$ is fine, since $L(s,\pi)$ only depends on the isomorphism class of the local representations $\pi_p := V_p(t)$.

From (1.207) and (1.209) we get, with $\pi = V(t)$ and $\pi_p = V_p(t)$,

$$\begin{aligned} L(s,\pi) &= \prod_{p\le\infty} L(s,\pi_p) \\ &= \Gamma_{\mathbb{R}}\Big(s+\frac{t}{2}\Big)\Gamma_{\mathbb{R}}\Big(s-\frac{t}{2}\Big)\prod_{p<\infty}\frac{1}{(1-p^{-s+t/2})(1-p^{-s-t/2})}. \end{aligned}$$

Using, as before, the notation $Z(s)$ for the completed Riemann zeta function (1.189), we see that

$$L(s,\pi) = Z\Big(s+\frac{t}{2}\Big)Z\Big(s-\frac{t}{2}\Big). \tag{1.211}$$

Hence $L(s,\pi)$ has meromorphic continuation to all of $\mathbb{C}$, and from (1.191) we get the ***functional equation***

$$L(s,\pi) = L(1-s,\pi). \tag{1.212}$$

We will see that L-functions for automorphic representations of $\mathrm{GL}(2,\mathbb{A})$ always satisfy a functional equation of this kind.

L-functions for holomorphic Eisenstein series

Now let π be the automorphic representation (1.70). This representation contains a distinguished vector leading to the holomorphic Eisenstein series $E_k(\tau)$. From (1.207) and (1.209), we get

$$L(s,\pi) = \Gamma_{\mathbb{C}}\Big(s+\frac{k-1}{2}\Big)\prod_{p<\infty}\frac{1}{(1-p^{-s+(k-1)/2})(1-p^{-s-(k-1)/2})}. \tag{1.213}$$

The infinite product looks again like two translates of the Riemann zeta function, but the archimedean factor doesn't. Note however, by ***Legendre's formula*** for the Γ-function, that

$$\Gamma_{\mathbb{C}}(s) = \Gamma_{\mathbb{R}}(s)\Gamma_{\mathbb{R}}(s+1). \tag{1.214}$$

Hence,

$$\begin{aligned} L(s,\pi) &= \Gamma_{\mathbb{R}}\Big(s+\frac{k-1}{2}\Big)\Gamma_{\mathbb{R}}\Big(s+\frac{k+1}{2}\Big)\prod_{p<\infty}\frac{1}{(1-p^{-s+(k-1)/2})(1-p^{-s-(k-1)/2})} \\ &= Z\Big(s+\frac{k-1}{2}\Big)\Gamma_{\mathbb{R}}\Big(s+\frac{k+1}{2}\Big)\prod_{p<\infty}\frac{1}{1-p^{-s+(k-1)/2}} \\ &= Z\Big(s+\frac{k-1}{2}\Big)Z\Big(s-\frac{k-1}{2}\Big)\frac{\Gamma_{\mathbb{R}}\big(s+\frac{k+1}{2}\big)}{\Gamma_{\mathbb{R}}\big(s-\frac{k-1}{2}\big)}. \end{aligned} \tag{1.215}$$

This implies that $L(s,\pi)$ has meromorphic continuation to all of $\mathbb{C}$. Using $s\Gamma(s)=\Gamma(s+1)$, it is easy to verify that

$$\frac{\Gamma_{\mathbb{R}}\big(s+\frac{k+1}{2}\big)}{\Gamma_{\mathbb{R}}\big(s-\frac{k-1}{2}\big)}=(-1)^{k/2}\,\frac{\Gamma_{\mathbb{R}}\big(1-s+\frac{k+1}{2}\big)}{\Gamma_{\mathbb{R}}\big(1-s-\frac{k-1}{2}\big)} \tag{1.216}$$

(recall that k is an *even* integer). It follows that $L(s,\pi)$ satisfies the functional equation

$$L(s,\pi)=(-1)^{k/2}\,L(1-s,\pi). \tag{1.217}$$

In some sense, the L-functions in (1.211) and (1.215) are "easy": They are essentially products of two Riemann zeta functions. Accordingly, their analytic properties (like meromorphic continuation and functional equation) follow from the corresponding properties of the zeta function. This phenomenon happens whenever our automorphic representation is globally induced, like $V_\infty(t)$. In this case, the L-function is a product of L-functions of the inducing data, a reflection of the fact that local L-factors are ***inductive***; we will talk more about this in the next section.

The more interesting L-functions arise from *cuspidal* automorphic representations. Recall that "cuspidal" means "not induced". In this case it is still true, but much less obvious, that $L(s,\pi)$ has meromorphic continuation and satisfies a functional equation. Before we discuss L-functions of cusp forms, we will present some purely local theory in the next section.

1.6.4 L-factors for unramified representations

In this section we will assume that p is a prime number. Recall that an irreducible, admissible representation π_p of $\mathrm{GL}(2,\mathbb{Q}_p)$ is called *unramified* or *spherical* if there exists a non-zero vector v_0 in the space of π_p with the property

$$\pi_p(g)v_0=v_0 \qquad \text{for all } g\in\mathrm{GL}(2,\mathbb{Z}_p). \tag{1.218}$$

(We already mentioned this terminology on page 48.) The group $\mathrm{GL}(2,\mathbb{Z}_p)$ is compact, and in fact is a ***maximal compact subgroup*** of $\mathrm{GL}(2,\mathbb{Q}_p)$. The invariance property (1.218) is the best that can be expected of a vector v_0: If v_0 is, in addition, invariant under *any* other element, then v_0 is invariant under *all* of $\mathrm{GL}(2,\mathbb{Q}_p)$, and so π_p is necessarily the trivial representation.
It turns out that the unramified representations are easy to describe. Since doing this for $\mathrm{GL}(n)$ is not much more difficult than for $\mathrm{GL}(2)$, we will treat the general case. Of course, we will call an irreducible, admissible representation π_p of $\mathrm{GL}(n,\mathbb{Q}_p)$ ***unramified*** or ***spherical*** if there exists a non-zero vector v_0 in the space of π_p with the property

$$\pi_p(g)v_0=v_0 \qquad \text{for all } g\in\mathrm{GL}(n,\mathbb{Z}_p). \tag{1.219}$$

A vector v_0 with the property (1.219) is also called a ***spherical vector***. Again, $\mathrm{GL}(n,\mathbb{Z}_p)$ is a *maximal* compact subgroup of $\mathrm{GL}(n,\mathbb{Q}_p)$. Think of a spherical vector having the best possible invariance-under-compact-subgroups property; one cannot ask for more.

Spherical representations can be constructed by ***induction*** from the ***Borel subgroup*** B. By definition, B consists of all upper triangular matrices. Hence,

$$B(\mathbb{Q}_p)=\{\begin{bmatrix}a_1&*&*\\&\ddots&*\\&&a_n\end{bmatrix}:\ a_i\in\mathbb{Q}_p^\times,\ *\in\mathbb{Q}_p\}. \tag{1.220}$$

Characters $\chi : B(\mathbb{Q}_p) \to \mathbb{C}^\times$ are of the form

$$\chi(\begin{bmatrix} a_1 & * & * \\ & \ddots & * \\ & & a_n \end{bmatrix}) = \chi_1(a_1) \cdot \ldots \cdot \chi_n(a_n), \tag{1.221}$$

where $\chi_1, \ldots, \chi_n$ are characters of $\mathbb{Q}_p^\times$. Given any such χ, we may form the induced representation

$$\mathrm{ind}_{B(\mathbb{Q}_p)}^{\mathrm{GL}(n,\mathbb{Q}_p)}(\chi). \tag{1.222}$$

By definition, the space of this representation consists of all locally constant functions $\varphi : \mathrm{GL}(n, \mathbb{Q}_p) \to \mathbb{C}$ with the transformation property

$$\varphi(bg) = \chi_1(a_1) \cdot \ldots \cdot \chi_n(a_n)\varphi(g) \qquad \text{for all } b = \begin{bmatrix} a_1 & * & * \\ & \ddots & * \\ & & a_n \end{bmatrix} \in B(\mathbb{Q}_p),\, g \in \mathrm{GL}(n, \mathbb{Q}_p), \tag{1.223}$$

on which $\mathrm{GL}(n, \mathbb{Q}_p)$ acts by right translation. However, this is ***unnormalized*** induction, and we prefer ***normalized*** induction. Recall how in (1.27) we threw in a factor $|a/d|^{1/2}$. We will do something similar now. Instead of (1.223), we consider functions with the transformation property

$$\varphi(bg) = \chi_1(a_1) \cdot \ldots \cdot \chi_n(a_n)\delta(b)\varphi(g) \qquad \text{for all } b = \begin{bmatrix} a_1 & * & * \\ & \ddots & * \\ & & a_n \end{bmatrix} \in B(\mathbb{Q}_p) \tag{1.224}$$

and $g \in \mathrm{GL}(n, \mathbb{Q}_p)$, where

$$\delta(b) = \prod_{1 \le i < j \le n} |a_i/a_j|^{1/2}. \tag{1.225}$$

The space of all such functions, which in addition are required to be locally constant, is the standard model of the induced representation (1.222). Following a modern notation going back to [20], we write

$$\chi_1 \times \ldots \times \chi_n \tag{1.226}$$

instead of $\mathrm{ind}_{B(\mathbb{Q}_p)}^{\mathrm{GL}(n,\mathbb{Q}_p)}(\chi)$.

Now assume that all the χ_i are unramified; this means that $\chi_i(x) = 1$ for all $x \in \mathbb{Z}_p^\times$. *In this case* $\chi_1 \times \ldots \times \chi_n$ *contains a* $\mathrm{GL}(n, \mathbb{Z}_p)$*-invariant vector* φ_0, which is in fact unique up to multiples. This follows from the Iwasawa decomposition

$$\mathrm{GL}(n, \mathbb{Q}_p) = B(\mathbb{Q}_p) \cdot K_p, \qquad \text{where } K_p = \mathrm{GL}(n, \mathbb{Z}_p), \tag{1.227}$$

a generalization of (1.29). Using (1.227), we *define* φ_0 by

$$\varphi_0(bg) = \chi_1(a_1) \cdot \ldots \cdot \chi_n(a_n)\delta(b) \qquad \text{for } b = \begin{bmatrix} a_1 & * & * \\ & \ddots & * \\ & & a_n \end{bmatrix} \in B(\mathbb{Q}_p),\ g \in \mathrm{GL}(n, \mathbb{Z}_p). \tag{1.228}$$

You should make sure to understand the importance of the unramifiedness of the χ_i while doing the following

1.6.1 Exercise. *Verify that φ_0 is well-defined.*

Evidently, φ_0 is constantly one on $\mathrm{GL}(n, \mathbb{Z}_p)$, and every vector that is constantly one on $\mathrm{GL}(n, \mathbb{Z}_p)$ coincides with φ_0.

Most of the time $\chi_1 \times \ldots \times \chi_n$ turns out to be irreducible[9]. Hence, in this case, we have constructed a spherical representation. Even if $\chi_1 \times \ldots \times \chi_n$ is not irreducible, it contains a unique spherical ***constituent*** (a quotient of a subrepresentation). So, in any case, the χ_i determine a spherical representation. One can show that the order of $\chi_1, \ldots, \chi_n$ is irrelevant; each permutation of the χ_i leads to the same spherical representation.

Even more is true: *Any spherical representation of* $\mathrm{GL}(n, \mathbb{Z}_p)$ *is obtained in this way.* This follows from work of Borel; see [1]. Hence, there is a bijection between the spherical representations of $\mathrm{GL}(n, \mathbb{Q}_p)$ and unramified characters $\chi_1, \ldots, \chi_n$ of $\mathbb{Q}_p^\times$ up to permutation.

To simplify matters further, note that, by (1.181), each χ_i is determined by the single complex number $\chi_i(p)$. The numbers $\chi_1(p), \ldots, \chi_n(p)$ are called the ***Satake parameters*** of the corresponding spherical representation.

We summarize: Each spherical representation π_p of $\mathrm{GL}(n, \mathbb{Q}_p)$ is obtained as the spherical constituent of an induced representation $\chi_1 \times \ldots \times \chi_n$, where the χ_i are unramified characters of $\mathbb{Q}_p^\times$. The numbers $\chi_1(p), \ldots, \chi_n(p)$ are the Satake parameters of π_p. They are unique up to permutation[10].

Now we are able to write down the L-factor $L(s, \pi_p)$ of an unramified representation π_p of $\mathrm{GL}(n, \mathbb{Q}_p)$. It is

$$L(s, \pi_p) = \frac{1}{(1 - \alpha_1 p^{-s}) \cdot \ldots \cdot (1 - \alpha_n p^{-s})}, \tag{1.229}$$

where $\alpha_1, \ldots, \alpha_n$ are the Satake parameters of π_p. Note this is compatible with (1.182). It is also compatible with (1.207), since

$$V_p(t) = |\cdot|^{t/2} \times |\cdot|^{-t/2}. \tag{1.230}$$

Formula (1.229) is an example for a general fact called the ***inductivity*** of L-factors. By this we mean that if π_p is induced, then $L(s, \pi_p)$ equals the product of the L-factors of the inducing data. For example, if $\pi_p = \chi_1 \times \ldots \times \chi_n$, then

$$L(s, \pi_p) = L(s, \chi_1) \cdot \ldots \cdot L(s, \chi_n), \tag{1.231}$$

where $L(s, \chi_i)$ are the L-factors for GL(1) defined in (1.182). This formula generalizes (1.229).

If π is an automorphic representation of $\mathrm{GL}(n, \mathbb{A})$ (or in fact any irreducible, admissible representation of this group), then, according to the Tensor Product Theorem, we can write $\pi = \otimes \pi_p$ with irreducible, admissible representations π_p of $\mathrm{GL}(n, \mathbb{Q}_p)$. Recall that π_p is automatically spherical for almost all p. Hence, in the L-function

$$L(s, \pi) = \prod_{p<\infty} L(s, \pi_p), \tag{1.232}$$

[9] It may seem counterintuitive to denote an irreducible object by a symbol like "$\chi_1 \times \ldots \times \chi_n$", but that's just the way it is.

[10] What we see here is the appearance of the ***Weyl group***. For GL(n), the Weyl group is just the symmetric group S_n.

almost all the local factors are of the general form (1.229).

We are now in a position to reformulate the Ramanujan conjecture. We mentioned on page 46 that the original Ramanujan conjecture was a statement about the growth of the Fourier coefficients of Ramanujan's Δ-function. In representation theoretic terms, a more general version of Ramanujan's conjecture is the following statement. *Let $\pi = \otimes\pi_p$ be a cuspidal, automorphic representation of* $\mathrm{GL}(n, \mathbb{A})$. *Let p be a prime for which π_p is unramified. Then all the Satake parameters of π_p have absolute value* 1. If all the Satake parameters have absolute value 1, we also say that π_p is ***tempered***. Observe that the Ramanujan conjecture is only for *cusp* forms; it is wrong for non-cusp forms. As of this writing, this version of the Ramanujan conjecture is unproven, even for GL(2), except in special cases.

1.6.5 L-functions for cusp forms

In this section we will get to know some features of L-functions of *cuspidal* automorphic representations of $\mathrm{GL}(2, \mathbb{A})$. They still have meromorphic (in fact, analytic) continuation to all of $\mathbb{C}$ and satisfy a functional equation, but not for the obvious reason that they factor into two Riemann zeta functions: L-functions for cuspidal automorphic representations are *irreducible.*

Taking averages

At the end of Sect. 1.6.2, the Euler factors for the Riemann zeta function more or less naturally dropped out when we took certain averages of automorphic forms as in (1.199). This is nice, so maybe we can apply the principle (1.197) to do something similar for automorphic forms on GL(2). Let's think about a good way to take an average of an automorphic form Φ on $\mathrm{GL}(2, \mathbb{A})$. The simple-minded attempt

$$\int\limits_{\mathrm{GL}(2,\mathbb{A})} \Phi(g)\, dg \tag{1.233}$$

doesn't make a lot of sense, for reasons of convergence. Better is

$$\int\limits_{\mathrm{GL}(2,\mathbb{Q})\backslash \mathrm{GL}(2,\mathbb{A})} \Phi(g)\, dg, \tag{1.234}$$

taking the automorphy of Φ into account. Even if this converges, a single number won't tell us much about Φ, since after all we may always scale Φ. Better than a single number is a function of s, so we consider

$$\int\limits_{\mathrm{GL}(2,\mathbb{Q})\backslash \mathrm{GL}(2,\mathbb{A})} \Phi(g)\, |\det(g)|^s\, dg. \tag{1.235}$$

Note that the factor $|\det(g)|^s$ is well-defined on the quotient by the product formula (1.12). For different values of s the integral (1.235) produces different kinds of averages of Φ, and one may hope to obtain some useful information that way. To simplify this integral, observe that compact groups don't matter much when taking averages; anyway the function Φ is right invariant under almost all $\mathrm{GL}(2, \mathbb{Z}_p)$. Therefore, in view of the global Iwasawa decomposition (1.32), we get almost the same information if we only integrate over the Borel subgroup,

$$\int_{B(\mathbb{Q})\backslash B(\mathbb{A})} \Phi(g)\,|\det(g)|^s\,dg. \tag{1.236}$$

In a typical element $\begin{bmatrix} a_1 & b \\ & a_2 \end{bmatrix}$ of $B(\mathbb{A})$, the b-variable gets integrated over $\mathbb{Q}\backslash\mathbb{A}$. This also being compact, we will simply omit it:

$$\int_{\mathbb{Q}^\times\backslash\mathbb{I}} \int_{\mathbb{Q}^\times\backslash\mathbb{I}} \Phi(\begin{bmatrix} a_1 & \\ & a_2 \end{bmatrix})\,|a_1a_2|^s\,d^\times a_1\,d^\times a_2. \tag{1.237}$$

We see that there are still convergence issues in this integral; just try an inner integral over the center, consisting of all scalar matrics $\begin{bmatrix} a & \\ & a \end{bmatrix}$, and assume that Φ is center-invariant. So let's just not integrate over the center (we could have seen right away, in any of the above integrals, that integrating over the center is silly). Then we are down to

$$\int_{\mathbb{Q}^\times\backslash\mathbb{I}} \Phi(\begin{bmatrix} a & \\ & 1 \end{bmatrix})\,|a|^s\,d^\times a. \tag{1.238}$$

Now *this* integral turns out to be convergent, for *all* values of s, provided that Φ is a *cusp form*. It turns out to be a very useful cross-section of the automorphic form Φ. In fact, for suitable Φ, this integral *represents* the L-function $L(s,\pi)$, similar to the way that the left hand side of (1.204) represents the Riemann zeta function:

1.6.2 Theorem. *Let π be a cuspidal, automorphic representation of* $\mathrm{GL}(2,\mathbb{A})$. *Then there exists a cusp form Φ in the space of π such that*

$$\int_{\mathbb{Q}^\times\backslash\mathbb{I}} \Phi(\begin{bmatrix} a & \\ & 1 \end{bmatrix})\,|a|^{s-1/2}\,d^\times a = L(s,\pi). \tag{1.239}$$

From the outset, equation (1.239) is only valid for $\mathrm{Re}(s)$ large enough, since the Euler product defining $L(s,\pi)$ converges only in such a region. But the integral on the left-hand side of (1.239) converges for *all* values of s, and in fact represents an *entire* function of s. Hence, equation (1.239) proves the ***analytic continuation*** of $L(s,\pi)$ to all of $\mathbb{C}$. After $L(s,\pi)$ has been continued, evidently, the equation is valid for all $s \in \mathbb{C}$.

Note that the L-functions (1.211) and (1.215) have poles coming from the poles of the Riemann zeta function. In contrast, L-functions for cuspidal automorphic representations are always *entire*.[11]

If $\pi = \otimes\pi_p$, then the cusp form Φ in Theorem 1.6.2 corresponds to a ***pure tensor*** $\otimes v_p$, where v_p is a distinguished vector in the local representation π_p. The proof of Theorem 1.6.2 involves the notion of ***Whittaker model***, and is part of Jacquet-Langlands' theory developed in [8]. See §6 of [6] for an overview, and [3] for a more recent reference.

[11]There are some exotic exceptions to this statement for certain higher-rank groups, but at least for $\mathrm{GL}(n)$ the statement is true.

Example: Holomorphic cusp forms

Let F be a holomorphic cusp form of weight k, as in Definition 1.5.2. Let $\Phi : \mathrm{GL}(2,\mathbb{A}) \to \mathbb{C}$ be the cuspidal automorphic form corresponding to F via Proposition 1.5.3. Recall from (1.166) that the relationship between F and Φ is given by

$$F(\tau) = y^{-k/2}\,\Phi(\begin{bmatrix} 1 & x \\ & 1 \end{bmatrix}\begin{bmatrix} y^{1/2} & \\ & y^{-1/2} \end{bmatrix}), \qquad \tau = x + iy,\ y > 0. \tag{1.240}$$

We will assume that Φ generates an *irreducible* representation π; as explained in Sect. 1.5.5, this may be assumed without much harm. Hence π is a cuspidal automorphic representation, and we may talk about $L(s,\pi)$.

It turns out that Φ is the distinguished cusp form in π whose existence is guaranteed by Theorem 1.6.2. Hence,

$$\int\limits_{\mathbb{Q}^\times\backslash\mathbb{I}} \Phi(\begin{bmatrix} a & \\ & 1 \end{bmatrix})\,|a|^{s-1/2}\,d^\times a = L(s,\pi). \tag{1.241}$$

We may use this equation to deduce the functional equation of $L(s,\pi)$. Let

$$w = \begin{bmatrix} & 1 \\ -1 & \end{bmatrix} \in \mathrm{GL}(2,\mathbb{Q}), \tag{1.242}$$

diagonally embedded into $\mathrm{GL}(2,\mathbb{A})$ as usual. We will write $w = w_\infty w_f$, where w_∞ has trivial non-archimedean components, and w_f has trivial archimedean component. Note that $w_\infty = r(\pi/2)$, using the notation (1.31). We calculate

$$\begin{aligned}
\int\limits_{\mathbb{Q}^\times\backslash\mathbb{I}} \Phi(\begin{bmatrix} a & \\ & 1 \end{bmatrix})\,|a|^{s-1/2}\,d^\times a &= \int\limits_{\mathbb{Q}^\times\backslash\mathbb{I}} \Phi(w\begin{bmatrix} a & \\ & 1 \end{bmatrix})\,|a|^{s-1/2}\,d^\times a && (\Phi \text{ is an automorphic form})\\
&= \int\limits_{\mathbb{Q}^\times\backslash\mathbb{I}} \Phi(\begin{bmatrix} 1 & \\ & a \end{bmatrix}w)\,|a|^{s-1/2}\,d^\times a\\
&= \int\limits_{\mathbb{Q}^\times\backslash\mathbb{I}} \Phi(\begin{bmatrix} 1 & \\ & a \end{bmatrix}r(\pi/2))\,|a|^{s-1/2}\,d^\times a && (\Phi \text{ is right } \prod_{p<\infty} K_p\text{-invariant})\\
&= e^{ik\pi/2}\int\limits_{\mathbb{Q}^\times\backslash\mathbb{I}} \Phi(\begin{bmatrix} 1 & \\ & a \end{bmatrix})\,|a|^{s-1/2}\,d^\times a && (\Phi \text{ has weight } k)\\
&= (-1)^{\frac{k}{2}}\int\limits_{\mathbb{Q}^\times\backslash\mathbb{I}} \Phi(\begin{bmatrix} a^{-1} & \\ & 1 \end{bmatrix})\,|a|^{s-1/2}\,d^\times a && (\Phi \text{ is center-invariant})\\
&= (-1)^{\frac{k}{2}}\int\limits_{\mathbb{Q}^\times\backslash\mathbb{I}} \Phi(\begin{bmatrix} a & \\ & 1 \end{bmatrix})\,|a^{-1}|^{s-1/2}\,d^\times a && (\text{property of Haar measure})\\
&= (-1)^{\frac{k}{2}}\int\limits_{\mathbb{Q}^\times\backslash\mathbb{I}} \Phi(\begin{bmatrix} a & \\ & 1 \end{bmatrix})\,|a|^{(1-s)-1/2}\,d^\times a.
\end{aligned}$$

From (1.241) we thus get the functional equation

$$L(s,\pi) = (-1)^{k/2} L(1-s,\pi). \tag{1.243}$$

Note that this is the same functional equation as (1.217), the one for the holomorphic Eisenstein series E_k. In fact, (1.243) holds for all (automorphic representations generated by the adelizations of) holomorphic modular forms of weight k.

Using (1.240), let us calculate the left hand side of (1.241) in terms of F. By (1.9) and the invariance properties of Φ,

$$\begin{aligned}
\int\limits_{\mathbb{Q}^\times\backslash\mathbb{I}} \Phi(\begin{bmatrix} a & \\ & 1\end{bmatrix})\,|a|^{s-1/2}\,d^\times a &= \int\limits_{\mathbb{R}_{>0}} \Phi(\begin{bmatrix} a & \\ & 1\end{bmatrix})\,|a|_\infty^{s-1/2}\,d^\times a \\
&= \int\limits_{\mathbb{R}_{>0}} \Phi(\begin{bmatrix} a^{1/2} & \\ & a^{-1/2}\end{bmatrix})\,a^{s-1/2}\,d^\times a \\
&= \int\limits_{\mathbb{R}_{>0}} a^{k/2} F(ia)\,a^{s-1/2}\,d^\times a \\
&= \int\limits_0^\infty F(ia)\,a^{s+k/2-1/2}\,d^\times a.
\end{aligned}$$

Therefore,

$$\int\limits_0^\infty F(ia)\,a^{s+k/2-1/2}\,d^\times a = L(s,\pi). \tag{1.244}$$

The integral on the left hand side of (1.244) is known as a ***Mellin transform*** of F. Let us substitute the classical Fourier expansion

$$F(\tau) = \sum_{n=1}^\infty c_n e^{2\pi i n\tau}, \tag{1.245}$$

which we already encountered in (1.155). Note that there is no constant term since Φ is a *cusp* form. Then

$$\begin{aligned}
L(s,\pi) &= \int\limits_0^\infty F(ia)\,a^{s+k/2-1/2}\,d^\times a \\
&= \int\limits_0^\infty \sum_{n=1}^\infty c_n e^{-2\pi n a}\,a^{s+k/2-1/2}\,d^\times a \\
&= \sum_{n=1}^\infty c_n \int\limits_0^\infty e^{-2\pi n a}\,a^{s+k/2-1/2}\,d^\times a \\
&= \sum_{n=1}^\infty c_n (2\pi n)^{-(s+\frac{k-1}{2})} \int\limits_0^\infty e^{-a}\,a^{s+k/2-1/2}\,d^\times a
\end{aligned}$$

$$= (2\pi)^{-(s+\frac{k-1}{2})}\,\Gamma\Big(s+\frac{k-1}{2}\Big)\sum_{n=1}^{\infty} c_n n^{-(s+\frac{k-1}{2})}. \tag{1.246}$$

The shifted L-function

$$L(s,F) := L\Big(s-\frac{k-1}{2},\pi\Big) = (2\pi)^{-s}\,\Gamma(s)\sum_{n=1}^{\infty} c_n n^{-s} \tag{1.247}$$

is known in the literature as the completed ***L-function of the modular form*** F. It satisfies the functional equation

$$L(s,F) = (-1)^{k/2} L(k-s,F), \tag{1.248}$$

which follows from (1.243).

Finally, we will look at the Euler factors of $L(s,\pi)$. We explained in Sect. 1.5.5 that, since F is holomorphic of weight k, the archimedean component π_∞ of π is the discrete series representation $\mathcal{D}_{k-1}$. By (1.209),

$$L(s,\pi_\infty) = \Gamma_{\mathbb{C}}(s). \tag{1.249}$$

We saw this factor naturally come out of the calculation (1.246) (at least up to a factor 1/2; the factor doesn't matter – we could have normalized F and Φ more carefully).

As for the non-archimedean factors, observe that the local component π_p is unramified for *all* primes p. Therefore, by (1.229),

$$L(s,\pi_p) = \frac{1}{(1-\alpha_p p^{-s})(1-\beta_p p^{-s})}, \tag{1.250}$$

where $\alpha_p, \beta_p \in \mathbb{C}^\times$ are the Satake parameters of π_p. But Φ is invariant under the center, which implies that in the representation π_p the center acts trivially. This is equivalent to saying that the characters χ_1 and χ_2 in the induced representation $\chi_1 \times \chi_2$ (see (1.226)) are inverses of each other: $\pi_p = \chi_1 \times \chi_1^{-1}$. Therefore $\beta_p = \alpha_p^{-1}$: there is really only *one* complex number characterizing the local component π_p. Combining everything, we see that

$$L(s,\pi) = \Gamma_{\mathbb{C}}(s)\prod_{p<\infty}\frac{1}{(1-\alpha_p p^{-s})(1-\alpha_p^{-1}p^{-s})}. \tag{1.251}$$

We stress that the Euler factors look the same for each prime p only since we are working with modular forms for $\mathrm{SL}(2,\mathbb{Z})$. Had we worked with modular forms with ***level*** - the subject of a later chapter - then there would be finitely many "bad" factors in the product (1.251) that looked different.

If we expand the factors $(1-\alpha_p^{\pm 1}p^{-s})^{-1}$ using the geometric series, then we recognize the infinite product as a Dirichlet series $\sum d_n n^{-s}$. Comparing the Dirichlet series in (1.246) and (1.251), we see that

$$c_p = p^{\frac{k-1}{2}}(\alpha_p + \alpha_p^{-1}). \tag{1.252}$$

Hence we get a direct relationship between the Fourier coefficients of the modular form F and the Satake parameters of the local representation π_p: The coefficient c_p completely determines π_p. Moreover, you can now easily see the connection between the classical version and the representation theory version of the ***Ramanujan conjecture***:

- *Classically*, the Ramanujan conjecture is the statement that the Fourier coefficients c_n of a holomorphic cusp form of weight k grow no faster than $n^{(k-1)/2}$; see page 46.

- *Representation theoretically*, the Ramanujan conjecture is the statement that the Satake parameters of π_p have absolute value 1, for all p; see page 60.

The relationship (1.252) makes plausible the statement that these two conjectures are equivalent. Maybe you can prove the equivalence by more carefully comparing the Dirichlet series in (1.246) and (1.251).

Holomorphic cusp forms for GL(2) are some of the few instances where the Ramanujan conjecture has actually been proven, namely as a consequence of Deligne's proof [4] of the ***Weil conjectures***.

Chapter 2

Refinements

2.1 More general levels

[[To do]]

2.2 More general central characters

[[To do]]

2.3 More general fields

[[To do]]

2.4 More general groups

[[To do]]

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