SIMPLE GRADED RINGS OF SIEGEL MODULAR FORMS, DIFFERENTIAL OPERATORS AND BORCHERDS PRODUCTS

HIROKI AOKI
Department of Mathematics, Faculty of Science and Technology
Tokyo University of Science, Noda, Chiba, 278-8510, Japan
aoki_hiroki@ma.noda.tus.ac.jp

TOMOYOSHI IBUKIYAMA
Department of Mathematics, Graduate School of Science, Osaka University
Machikaneyama 1-16, Toyonaka, Osaka, 560-0043, Japan
ibukiym@math.wani.osaka-u.ac.jp

Received 30 July 2004
Revised 1 November 2004

In this paper, we show that the graded ring of Siegel modular forms of \( \text{Sp}(2, \mathbb{Z}) \) has a very simple unified structure for \( N = 1, 2, 3, 4 \), taking Neben-type case (the case with character) for \( N = 3 \) and 4. All are generated by 5 generators, and all the fifth generators are obtained by using the other four by means of differential operators, and it is also obtained as Borcherds products. As an appendix, examples of Euler factors of \( L \)-functions of Siegel modular forms of \( \text{Sp}(2, \mathbb{Z}) \) of odd weight are given.

Keywords: Siegel modular form; differential operator; Borcherds product.

Mathematics Subject Classification 2000: Primary 11F46; Secondary 11F60, 11F50

1. Introduction

In this paper, we give examples of congruence subgroups \( \Gamma \) of \( \text{Sp}(2, \mathbb{Z}) \) such that the graded ring \( A(\Gamma) \) of Siegel modular forms of \( \Gamma \) is generated by five modular forms, among which four are algebraically independent and the other is neatly described by using the first four. It is well-known as Igusa’s theorem that the ring of Siegel modular forms of degree two belonging to the full modular group \( \text{Sp}(2, \mathbb{Z}) \) are generated by four algebraically independent forms \( \phi_4, \phi_6, \chi_{10}, \chi_{12} \) and a modular form \( \chi_{35} \), where the suffices are the weights of the forms. It is also well known that \( \chi_{35} \) is expressed as a Borcherds product (cf. Gritsenko and Nikulin [9]). In Igusa’s paper [17], the modular form \( \chi_{35} \) was given by an average of theta constants over azygous triples and this is rather complicated. Here we give another very simple
way to construct $\chi_{35}$ from $\phi_4, \phi_6, \chi_{10}, \chi_{12}$ by using a differential operator. Now we explain the result for other groups. For any natural number $N$ and $n$, we put

$$\Gamma_0^{(n)}(N) = \left\{ g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, \mathbb{Z}) ; \ C \equiv 0 \mod N \right\}.$$  

When $n = 2$, we write $\Gamma_0(N) = \Gamma_0^{(2)}(N)$ for the sake of simplicity. We denote by $\psi_3$ the character of $\Gamma_0(3)$ defined by $\psi_3(\gamma) = \left( \frac{-3}{\det(D)} \right)$ and by $\psi_4$ the character of $\Gamma_0(4)$ defined by $\psi_4(\gamma) = \left( \frac{-1}{\det(D)} \right)$. We put $\Gamma_0^{\psi_3}(3) = \{ \gamma \in \Gamma_0(3) ; \psi_3(\gamma) = 1 \}$ and $\Gamma_0^{\psi_4}(4) = \{ \gamma \in \Gamma_0(4) ; \psi_4(\gamma) = 1 \}$.

**Theorem 1.1.** For each $\Gamma = Sp(2, \mathbb{Z}), \Gamma_0(2), \Gamma_0^{\psi_3}(3),$ or $\Gamma_0^{\psi_4}(4)$, the graded ring $A(\Gamma)$ of Siegel modular forms is generated by five Siegel modular forms consisting of four algebraically independent forms and another one constructed by the first four by using a differential operator. The fifth generator is also expressed as a Borcherds product.

As for precise generators and the differential operators, see Sec. 2. We note that the generators of the ring for $Sp(2, \mathbb{Z}), \Gamma_0(2)$ and $\Gamma_0^{\psi_4}(4)$ have been already known in [17, 12, 11]. So for these groups, a new point here is that the fifth generator is obtained by a simple differential operator and has a Borcherds product expression. The result on generators for $\Gamma_0^{\psi_3}(3)$ is newly obtained, and we can reprove our former result on $\Gamma_0(3)$ in [12] more easily and satisfactorily by our new result. Of course this kind of theorem is in a sense accidental, and there are some other discrete subgroups for which the same theorem holds even among subgroups containing the principal congruence subgroup $\Gamma(2)$, for example, the unique subgroup $\Gamma_\epsilon(1)$ of index two in $Sp(2, \mathbb{Z})$, but we omit further results here.

Since our expression of $\chi_{35}$ is so simple, it is now very easy to calculate many Fourier coefficients of $\chi_{35}$. Hence in Appendix A, we give a table of some Fourier coefficients. We also give Euler factors at 2 of spinor $L$ function of Siegel cusp forms $Sp(2, \mathbb{Z})$ of odd weight which have never appeared in any literature as far as the authors know.

Finally we would like to say a few words on our differential operators. A general theory of holomorphic differential operators on Siegel modular forms which produce new modular forms from given several modular forms was studied in Ibukiyama [13] in full generality. Our differential operator in the above theorem is one of these operators. In one variable case, this kind of operators are called Rankin–Cohen differential operators. Starting from modular forms of one variable of weight $k$ and $l$, we obtain a new modular form of weight $k + l + 2\nu$ by this operator where $\nu$ is a non-negative integer. We can also consider differential operators on more than two modular forms of one variable but we cannot get a new result, namely, this kind of operator is obtained by combination of those on two forms. As for Siegel modular forms, the situation is slightly different. For example, the Rankin–Cohen type differential operators on two Siegel modular forms of weight $k$ and $l$ are defined...
for degree two in [5] and for general degree in [7], and the weight of constructed
Siegel modular form is \( k + 1 + 2\nu \) where \( \nu \) is a non-negative integer. So, for example,
if we start from modular forms of even weight, then we get only a modular form
of even weight in this case. But as we see in the theorem, if we start from several
modular forms more than two, it happens that we get odd weight from even weights.
This is a very interesting point and a trick for our construction. The same sort of
construction is also valid for vector valued Siegel modular forms which will appear
elsewhere (cf. [14–16]).

2. Differential Operators

We review Siegel modular forms to fix notation. For any natural number \( n \), we
denote by \( H_n \) the Siegel upper half plane of degree \( n \)
\[
H_n = \{ Z \in M_n(\mathbb{C}); \quad ^{t}Z = Z, \quad Im(Z) > 0 \}. 
\]

For any commutative ring \( R \), we define the symplectic group over \( R \) by
\[
Sp(n, R) = \{ g \in M_{2n}(R); \quad ^{t}gJg = J \}
\]
where \( J = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix} \) and \( 1_n \) is the unit matrix. For a discrete subgroup \( \Gamma \subset Sp(n, \mathbb{R}) \) and a character \( \chi \) of \( \Gamma \), we say that a holomorphic function \( F(Z) \) on
\( H_n \) is a Siegel modular form of weight \( k \) with character \( \chi \) if
\[
F((AZ + B)(CZ + D)^{-1}) = \chi(\gamma) \det(CZ + D)^k F(Z) \quad \text{for any } \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma
\]
and bounded at cusps. We denote by \( A_k(\Gamma, \chi) \) the space of all Siegel modular forms
of weight \( k \) of \( \Gamma \) with character \( \chi \). When \( \chi \) is the identity character, we write
\( A_k(\Gamma) = A_k(\Gamma, \chi) \) and call \( f \in A_k(\Gamma) \) just a Siegel modular forms of weight \( k \). We
put \( A(\Gamma) = \bigoplus_{k=0}^{\infty} A_k(\Gamma) \). The space \( A(\Gamma) \) is a graded ring.

For \( Z \in H_n \), we write the \((i,j)\) components of \( Z \) by \( z_{ij} \). We put \( \langle n \rangle = \frac{n(n + 1)}{2} + 1 \). For \( \langle n \rangle \) numbers of Siegel modular forms \( f_i \in A_k(\Gamma) \) of weight \( k_i \) \((1 \leq i \leq \langle n \rangle)\), we define a new function \( \{f_1, \ldots, f_\langle n \rangle\}_{n+1} \) by

\[
\begin{bmatrix}
  k_1 f_1 & k_2 f_2 & \cdots & k_{\langle n \rangle-1} f_{\langle n \rangle-1} & k_{\langle n \rangle} f_{\langle n \rangle} \\
  \frac{\partial f_1}{\partial z_{11}} & \frac{\partial f_2}{\partial z_{11}} & \cdots & \frac{\partial f_{\langle n \rangle-1}}{\partial z_{11}} & \frac{\partial f_{\langle n \rangle}}{\partial z_{11}} \\
  \vdots & \vdots & \ddots & \vdots & \vdots \\
  \frac{\partial f_1}{\partial z_{nn}} & \frac{\partial f_2}{\partial z_{nn}} & \cdots & \frac{\partial f_{\langle n \rangle-1}}{\partial z_{nn}} & \frac{\partial f_{\langle n \rangle}}{\partial z_{nn}}
\end{bmatrix}
\]

Proposition 2.1. (1) The above function \( \{f_1, \ldots, f_\langle n \rangle\}_{n+1} \) is a Siegel cusp form
of weight \( k_1 + \cdots + k_{\langle n \rangle} + n + 1 \).

(2) If \( f_1, \ldots, f_\langle n \rangle \) are algebraically independent, then \( \{f_1, \ldots, f_\langle n \rangle\}_{n+1} \neq 0 \).
Proof. For \( \nu \geq 2 \), put \( F_\nu = f^{k_\nu}_\nu / f^{k_1}_1 \). Then \( F_\nu(\gamma Z) = F_\nu(Z) \), namely, these are automorphic functions on \( H_n \) invariant by \( \Gamma \). Define \( F(Z) \) by the Jacobian.

\[
F(Z) = \frac{\partial(F_2, \ldots, F_{(n)})}{\partial(z_{11}, \ldots, z_{nn})}
\]

Then we have

\[
F(Z) = \frac{\partial(F_2(\gamma Z), \ldots, F_{(n)}(\gamma Z))}{\partial(z_{11}, z_{12}, \ldots, z_{nn})}
= \frac{\partial(F_2(\gamma Z), \ldots, F_{(n)}(\gamma Z))}{\partial((\gamma Z)_{11}, (\gamma Z)_{12}, \ldots, (\gamma Z)_{nn})} \times \frac{\partial((\gamma Z)_{11}, \ldots, (\gamma Z)_{nn})}{\partial(z_{11}, z_{12}, \ldots, z_{nn})}
= F(\gamma Z) \text{det}(CZ + D)^{-n-1}.
\]

Hence \( F(Z) \) is a meromorphic Siegel modular form of weight \( n+1 \) of \( \Gamma \). We also have

\[
\frac{\partial}{\partial z_{ij}}(f^{k_\nu}_\nu / f^{k_1}_1) = k_1(f^{k_\nu}_\nu - f^{k_1}_1) \frac{\partial f_\nu}{\partial z_{ij}} - k_\nu(f^{k_\nu}_\nu / f^{k_1}_1 + 1) \frac{\partial f_1}{\partial z_{ij}}
= \left( \frac{k_1 f^{k_\nu}_\nu - 1}{f^{k_1}_1} \right) \left( \frac{\partial f_\nu}{\partial z_{ij}} - k_\nu f_\nu / f_1 \times \frac{\partial f_1}{\partial z_{ij}} \right).
\]

This implies that

\[
\{f_1, \ldots, f_{(n)}\}_{n+1} = \frac{f_1^{k_2 + \cdots + k_{(n)} + 1}}{k_1^{(n) - 1} f_2 f_3 \cdots f_{(n)} k_1 - 1} F(Z).
\]

This means that \( \{f_1, \ldots, f_{(n)}\}_{n+1} \) is a holomorphic Siegel modular form of weight \( k_1 + \cdots + k_{(n)} + n + 1 \). In particular, if \( f_\nu \) (\( 1 \leq \nu \leq (n) \)) are algebraically independent, then \( f^{k_\nu}_\nu / f^{k_1}_1 \) are local parameters of \( (n) - 1 \) dimensional variety \( \Gamma \backslash H_n \), so the functional determinant does not vanish identically. Hence we get (2).

When \( n = 2 \), for any \( Z \in H_2 \), we write

\[
Z = \begin{pmatrix} r & z \\ z & \omega \end{pmatrix}.
\]

For any Siegel modular form \( f \) of degree two, we denote by \( a(t_1, t_2, t; f) = a(t_1, t_2, t) \) the Fourier coefficient of \( f \) at

\[
\begin{pmatrix} t_1 & t/2 \\ t/2 & t_2 \end{pmatrix}
\]

Now first we apply this to level 1 case. We denote by \( \phi_4 \) or \( \phi_6 \) the unique modular form of \( Sp(2, \mathbb{Z}) \) of weight 4 or 6 such that \( a(0, 0, 0; \phi_4) = a(0, 0, 0; \phi_6) = 1 \).
We denote by $\chi_{10}$ or $\chi_{12}$ the unique cusp form of $Sp(2, \mathbb{Z})$ of weight 10 or 12 such that $a(1, 1, 1; \chi_{10}) = a(1, 1, 1; \chi_{12}) = 1$. Then the function

$$
\chi = \frac{1}{(2\pi i)^3} \{ \phi_4, \phi_6, \chi_{10}, \chi_{12} \} \frac{1}{3} = \frac{1}{(2\pi i)^3} \begin{vmatrix}
4\phi_4 & 6\phi_6 & 10\chi_{10} & 12\chi_{12} \\
\frac{\partial \phi_4}{\partial \tau} & \frac{\partial \phi_6}{\partial \tau} & \frac{\partial \chi_{10}}{\partial \tau} & \frac{\partial \chi_{12}}{\partial \tau} \\
\frac{\partial \phi_4}{\partial \omega} & \frac{\partial \phi_6}{\partial \omega} & \frac{\partial \chi_{10}}{\partial \omega} & \frac{\partial \chi_{12}}{\partial \omega} \\
\frac{\partial \phi_4}{\partial z} & \frac{\partial \phi_6}{\partial z} & \frac{\partial \chi_{10}}{\partial z} & \frac{\partial \chi_{12}}{\partial z}
\end{vmatrix}
$$

is a modular form of weight $4 + 6 + 10 + 12 + 3 = 35$. This does not vanish, since $\phi_4$, $\phi_6$, $\chi_{10}$, $\chi_{12}$ are algebraically independent. We denote by $E_4$ or $E_6$ the Eisenstein series of $SL_2(\mathbb{Z})$ of weight 4 or 6 with constant term 1. Let $\phi_{10,1}$ or $\phi_{12,1}$ be the Jacobi cusp form of weight 10 or 12 of index one given in Eichler-Zagier. The Fourier-Jacobi expansion of $\chi$ is given by

$$
(p^2)^{\frac{1}{2\pi i}} \left( 4E_4 \frac{\partial E_6}{\partial \tau} - 6E_6 \frac{\partial E_4}{\partial \tau} \right) \times \frac{1}{2\pi i} \left( \frac{\partial \phi_{10,1}}{\partial z} \phi_{12,1} - \phi_{10,1} \frac{\partial \phi_{12,1}}{\partial z} \right) + O(p^3)
$$

$$
= -3456\Delta_{12}\phi_{23,2}(p)^2 + O(p^3),
$$

where $q = \exp(2\pi i \tau)$, $\zeta = \exp(2\pi i z)$, $p = \exp(2\pi i \omega)$ and

$$
\phi_{23,2} = 12q(\zeta - \zeta^{-1}) + O(q^2)
$$

$$
= 12(4\pi i)(\Delta(\tau))^2 z + O(z^2)
$$

is a non-zero Jacobi form of weight 23 of index 2. We see that $a(2, 3, 1; \chi) = 2^9 \cdot 3^4$. We can see $\chi \neq 0$ also by this calculation. We put $\chi_{35} = \chi/(2^9 \cdot 3^4)$. Then $\chi_{35}$ is the unique Siegel modular form of weight 35 of degree 2 of $Sp(2, \mathbb{Z})$ such that $a(2, 3, 1; \chi_{35}) = 1$. Denote by $\Gamma_0(1)$ is the unique index two subgroup of $Sp(2, \mathbb{Z})$. We have a cusp form $\chi_5$ of $\Gamma_0(1)$ of weight 5 such that $\chi_5^5 = \chi_{10}$ [17]. Then $\{ \phi_4, \phi_6, \chi_5, \chi_{12}\}$ is the modular form of weight 30. We denote this by $\chi_{30}$. Then $\chi_{35}$ is equal to $\chi_5\chi_{30}$ up to constant. It is known that $A(Sp(2, \mathbb{Z})) = \mathbb{C}[\phi_4, \phi_6, \chi_{10}, \chi_{12}, \chi_{35}]$ and $A(\Gamma_0(1)) = \mathbb{C}[\phi_4, \phi_6, \chi_5, \chi_{12}, \chi_{30}]$.

We note that the Eisenstein series $\phi_4$, $\phi_6$ are the same function as those in Igusa [18, p. 848]. But our normalization of $\chi_{10}$, $\chi_{12}$, $\chi_{35}$ are different from Igusa’s notation. In fact, Igusa’s function $\chi_{10}$, $\chi_{12}$ or $\chi_{35}$ in [18, p. 848] is $-1/4$, $1/12$ or $i/4$ times of out $\chi_{10}$, $\chi_{12}$ or $\chi_{35}$.

**Remark.** $\chi_{35}$ is an odd function with respect to $z$. In the above $\frac{\partial F}{\partial z}$ for any Siegel modular forms $F$ of even weights are odd function with respect to $z$ and all the other derivatives are even. Hence this fits the fact. The Fourier coefficients $a(t_1, t_2, t)$ at $(t_1, i/2, t_2)$ satisfies $a(t_1, t_2, -1) = -a(t_1, t_2, t)$.

### 3. Siegel Modular Forms of Odd Weights of Level 2

The ring of modular forms of $\Gamma_0(2)$ was determined in Ibukiyama [12], and all the generators are given by theta constants. First we review this theorem here, and
then we show that in this case also the Siegel modular form of weight 19, which is the fundamental generator of odd weights forms, is obtained by differentiating even weights forms.

As usual, we define theta constants of characteristic $m = ^t(m', m'') \in \mathbb{Z}^4$ ($m', m'' \in \mathbb{Z}^2$) by

$$\theta_m(Z) = \sum_{p \in \mathbb{Z}^2} e \left( \frac{1}{2} \left( p + \frac{m'}{2} \right) \left( p + \frac{m''}{2} \right) \right),$$

where we put $e(x) = e^{2\pi i x}$ and $Z \in H_2$. We put

$$X = (\theta_{0000}^4 + \theta_{0010}^4 + \theta_{0100}^4 + \theta_{0110}^4)/4,$$

$$Y = (\theta_{0000} \theta_{0001} \theta_{0010} \theta_{0011})^2,$$

$$Z = (\theta_{0100}^4 - \theta_{0110}^4)^2/16384,$$

$$K = (\theta_{0100} \theta_{0110} \theta_{1000} \theta_{1001} \theta_{1010} \theta_{1110})^2/4096.$$

Each $X$, $Y$, $Z$, or $K$ is a modular form of $\Gamma_0(2)$ of weight 2, 4, 4, 6 respectively, and these four forms are algebraically independent. There also exists a cusp form $\chi_{19}$ of weight 19 belonging to $\Gamma_0(2)$, given explicitly by theta constants as follows.

$$\theta = \theta_{0000} \theta_{0001} \theta_{0010} \theta_{0011} \theta_{0100} \theta_{0110} \theta_{1000} \theta_{1001} \theta_{1010} \theta_{1110} \theta_{1011} \theta_{1111},$$

$$\theta' = (\theta_{1000}^4 - \theta_{1001}^4 - \theta_{1010}^4 + \theta_{1100}^4 + \theta_{1011}^4)/1536,$$

$$T = (\theta_{0100} \theta_{0110})^4/256,$$

$$\chi_{19} = \theta \theta' (8YZ - X^2T + YT + 1024ZT + 96T^2 - 8XK)/32.$$

Here $\alpha(2, 3, 1; \chi_{19}) = -1$.

We know that

**Proposition 3.1 [12, p. 34].**

$$A(\Gamma_0(2)) = \mathbb{C}[X, Y, Z, K] \oplus \chi_{19} \mathbb{C}[X, Y, Z, K].$$

where $\oplus$ means a direct sum as modules.

The only thing we would like to claim here is the following proposition.

**Proposition 3.2.**

$$\chi_{19} = \frac{1}{512(2\pi i)^2} \begin{vmatrix} 2X & 4Y & 4Z & 6K \\ \frac{\partial X}{\partial \tau} & \frac{\partial X}{\partial \zeta} & \frac{\partial X}{\partial \tau} & \frac{\partial X}{\partial \zeta} \\ \frac{\partial Y}{\partial \tau} & \frac{\partial Y}{\partial \zeta} & \frac{\partial Y}{\partial \tau} & \frac{\partial Y}{\partial \zeta} \\ \frac{\partial Z}{\partial \tau} & \frac{\partial Z}{\partial \zeta} & \frac{\partial Z}{\partial \tau} & \frac{\partial Z}{\partial \zeta} \\ \frac{\partial K}{\partial \tau} & \frac{\partial K}{\partial \zeta} & \frac{\partial K}{\partial \tau} & \frac{\partial K}{\partial \zeta} \end{vmatrix}.$$
The constant is obtained by calculation of the Fourier coefficient of this determinant at \((2, 3, 1)\). We see that \(\chi_{19}\) divides \(\chi_{35}\). This is obvious since \(\Gamma_0(2) \subset Sp(2, \mathbb{Z})\) but also clear from the relation

\[
\{\phi_4, \phi_6, \chi_{10}, \chi_{12}\}_3 = \{X, Y, Z, K\}_3 \times \frac{\partial(\phi_4, \phi_6, \chi_{10}, \chi_{12})}{\partial(X, Y, Z, K)},
\]

where we put

\[
\frac{\partial(\phi_4, \phi_6, \chi_{10}, \chi_{12})}{\partial(X, Y, Z, K)} = \begin{pmatrix}
\frac{\partial \phi_4}{\partial X} & \frac{\partial \phi_6}{\partial X} & \frac{\partial \chi_{10}}{\partial X} & \frac{\partial \chi_{12}}{\partial X} \\
\frac{\partial \phi_4}{\partial Y} & \frac{\partial \phi_6}{\partial Y} & \frac{\partial \chi_{10}}{\partial Y} & \frac{\partial \chi_{12}}{\partial Y} \\
\frac{\partial \phi_4}{\partial Z} & \frac{\partial \phi_6}{\partial Z} & \frac{\partial \chi_{10}}{\partial Z} & \frac{\partial \chi_{12}}{\partial Z} \\
\frac{\partial \phi_4}{\partial K} & \frac{\partial \phi_6}{\partial K} & \frac{\partial \chi_{10}}{\partial K} & \frac{\partial \chi_{12}}{\partial K}
\end{pmatrix}.
\]

The last functional determinant is calculated explicitly if we use the relations

\[
\phi_4 = 4X^2 - 3Y + 12288Z,
\]
\[
\phi_6 = -8X^3 + 9XY + 73728XZ - 27648K,
\]
\[
\chi_{10} = YK,
\]
\[
\chi_{12} = 3Y^2Z - 2XYK + 3072K^2.
\]

The result is

\[
81(-67108864X^2K^2 + 65536X^2Y^2Z - 16384XKY^2 - Y^4 - 16384Y^3Z + 68719476736ZK^2 - 67108864Y^2Z^2).
\]

4. Siegel Modular Forms of Level 3

We put

\[
\psi_3(\gamma) = \left( \begin{array}{c}
-3 \\
\det D
\end{array} \right) \quad \text{for} \quad \gamma = \left( \begin{array}{cc}
A & B \\
3C & D
\end{array} \right) \in \Gamma_0(3).
\]

We put \(\Gamma_0^{\psi_3}(3) = \{\gamma \in \Gamma_0(3); \psi_3(\gamma) = 1\}\). Then the group \(\Gamma_0^{\psi_3}(3)\) is index 2 in \(\Gamma_0(3)\). We have already given the structure of \(A^{(even)} = \bigoplus_{k=0}^{\infty} A_k(\Gamma_0(3))\) in [12], but in this section, we describe \(\bigoplus_{k=0}^{\infty} A_k(\Gamma_0^{\psi_3}(3))\) explicitly and reprove the result for \(A^{(even)}\) in [12] fairly easily.

For any even integral symmetric matrix \(S\) of size \(n\) and \(Z = \begin{pmatrix} \tau & z \\ z & \omega \end{pmatrix} \in H_2\), we define a theta function by

\[
\theta_S(Z) = \sum_{x,y \in \mathbb{Z}^n} e^{\pi i((xSx)\tau + 2(xSy)z + (ySy)\omega)}.
\]
We take the following three even symmetric matrices

\[
A_2 = \begin{pmatrix}
2 & 1 \\
1 & 2
\end{pmatrix},
\]

\[
E_6 = \begin{pmatrix}
2 & -1 & 0 & 0 & 0 & 0 \\
-1 & 2 & -1 & 0 & 0 & 0 \\
0 & -1 & 2 & -1 & 0 & -1 \\
0 & 0 & -1 & 2 & -1 & 0 \\
0 & 0 & 0 & -1 & 2 & 0 \\
0 & 0 & -1 & 0 & 0 & 2
\end{pmatrix}
\]

and

\[
E_6^* = 3E_6^{-1} = \begin{pmatrix}
4 & 5 & 6 & 4 & 2 & 3 \\
5 & 10 & 12 & 8 & 4 & 6 \\
6 & 12 & 18 & 12 & 6 & 9 \\
4 & 8 & 12 & 10 & 5 & 6 \\
2 & 4 & 6 & 5 & 4 & 3 \\
3 & 6 & 9 & 6 & 3 & 6
\end{pmatrix}
\]

We have \(\det(A_2) = 3\), \(\det(E_6) = 3\) and \(\det(E_6^*) = 243 = 3^5\) and these have level 3. By these conditions, we see \(\theta_{A_2}(Z) \in A_1(\Gamma_0(3), \psi_3)\) and \(\theta_{E_6}(Z), \theta_{E_6^*}(Z) \in A_3(\Gamma_0(3), \psi_3)\) (cf. [2]). We define another theta function with spherical function. We put

\[
S_4 = \begin{pmatrix}
1 & 0 & 3/2 & 0 \\
0 & 1 & 0 & 3/2 \\
3/2 & 0 & 3 & 0 \\
0 & 3/2 & 0 & 3
\end{pmatrix}
\]

and for \(x = (x_i), y = (y_i) \in \mathbb{R}^4\), we put

\[
Q(x, y) = \begin{pmatrix}
txS_4x & txS_4y \\
yS_4x & yS_4y
\end{pmatrix}.
\]

As in [12], we put

\[
\theta_4(Z) = \sum_{x, y \in \mathbb{Z}^4} (c^2 - d^2) \exp(2\pi i tr(Q(x, y)Z))
\]

where \(c = (x_1y_3 - x_3y_1) + (x_2y_4 - y_2x_4), d = (x_1y_4 - y_1x_4) + (x_3y_2 - y_3x_2) + (x_1y_2 - y_1x_2)\). We know that \(\theta_4 \in S_3(\Gamma_0(3))\) (cf. [12]).

To make symbols consistent with those in [12], we put

\[
\alpha_3 = \theta_{A_2},
\]

\[
\beta_3 = \theta_{E_6} - 10\theta_{A_2}^3 + 9\theta_{E_6},
\]

\[
\delta_3 = \theta_{E_6} - 9\theta_{E_6^*},
\]

\[
\gamma_4 = \theta_4.
\]
We put
\[ \chi_{14} = \frac{1}{2^{9} \cdot 3^{10}} \times \frac{1}{(2\pi i)^3} \begin{vmatrix} \alpha_1 & 3\beta_3 & 4\gamma_4 & 3\delta_4 \\ \frac{\partial \alpha_1}{\partial \tau} & \frac{\partial \beta_3}{\partial \tau} & \frac{\partial \gamma_4}{\partial \tau} & \frac{\partial \delta_4}{\partial \tau} \\ \frac{\partial \alpha_1}{\partial \omega} & \frac{\partial \beta_3}{\partial \omega} & \frac{\partial \gamma_4}{\partial \omega} & \frac{\partial \delta_4}{\partial \omega} \end{vmatrix}. \]

Then \( \chi_{14} \in A_{14}(\Gamma_0(3), \psi_3) \) with \( a(2, 3, 1; \chi_{14}) = 1 \).

We put
\[ B = \mathbb{C}[\alpha_1, \beta_3, \gamma_4, \delta_3], \]
\[ C = \mathbb{C}[\alpha_1^2, \beta_3^2, \delta_3^2, \gamma_4], \]
and denote by \( B^{(\text{odd})} \) or \( B^{(\text{even})} \) the submodule of the graded ring \( B \) consisting of odd degree or even degree elements, respectively. Namely, we have
\[ B^{(\text{even})} = \mathbb{C}[\alpha_1^2, \alpha_1 \beta_3, \alpha_1 \delta_3, \gamma_4, \beta_3^2, \delta_3^2, \beta_3 \delta_3]. \]

**Theorem 4.1.** The four modular forms \( \alpha_1, \beta_3, \gamma_4, \delta_3 \) are algebraically independent.

We have
\[ \bigoplus_{k=0}^{\infty} A_k(\Gamma_0^3(3)) = B \oplus B\chi_{14}, \]
\[ \bigoplus_{k=0}^{\infty} A_k(\Gamma_0(3)) = B^{(\text{even})} \oplus C\alpha_1 \chi_{14} \oplus C\beta_3 \chi_{14} \oplus C\delta_3 \chi_{14} \oplus C\alpha_1 \beta_3 \delta_3 \chi_{14}, \]
\[ \bigoplus_{k=0}^{\infty} A_k(\Gamma_0(3), \psi_3) = B^{(\text{odd})} \oplus B^{(\text{even})} \chi_{14}. \]

**Proof.** For any function \( F(Z) \) on \( H_2 \), the Witt operator \( W \) is defined by
\[ (W(F))(\tau, \omega) = F \begin{pmatrix} \tau & 0 \\ 0 & \omega \end{pmatrix}, \]
where \( W(F) \) is a function on \( H_1 \times H_1 \). In order to describe the images of generators under \( W \), we define functions on \( \tau \in H_1 \) by
\[ f_1(\tau) = \sum_{x \in \mathbb{Z}^2} e((t' x A_2 x) \tau) = 1 + 6q + 6q^3 + \cdots, \]
\[ f_2(\tau) = \sum_{x \in \mathbb{Z}^6} e((t' x E_6 x) \tau) = 1 + 72q + 270q^2 + 720q^3 + \cdots, \]
\[ f_3(\tau) = \sum_{x \in \mathbb{Z}^4} e((t' x E_8 x) \tau) = 1 + 54q^2 + 72q^3 + \cdots. \]

We have \( f_1 \in A_4(\Gamma_0^{(1)}(3), \psi) \), \( f_2, f_3 \in A_3(\Gamma_0^{(1)}(3), \psi) \), where we put \( \psi(\gamma) = \psi(d) = (a^{-3} \delta) \) for \( \gamma = \begin{pmatrix} a & \delta \\ 3c & d \end{pmatrix} \in \Gamma_0^{(1)}(3) \). We know that \( A_k(\Gamma_0^{(1)}(3)) = A_{k+1}(\Gamma_0^{(1)}(3), \psi) = 0 \).
for odd \( k \). We see by usual dimension formula that

\[
\sum_{k=0}^{\infty} (\dim A_k(\Gamma_0(3)) + \dim A_k(\Gamma_0'(3), \psi)) t^k = \frac{1}{(1-t)(1-t^3)}.
\]

Since \( f_1^3 - f_3 \) vanish at \( i\infty \) and \( f_1 \) does not, two forms \( f_1 \) and \( f_3 \) are algebraically independent. Comparing the dimensions we have

\[
\mathbb{C}[f_1, f_3] = \bigoplus_{k=0}^{\infty} (A_{2k}(\Gamma_0'(3)) \oplus A_{2k+1}(\Gamma_0'(3), \psi)).
\]

Also we have \( f_3 = (4f_1^3 - f_2)/3 \). We write \( f_i = f_i(\tau) \) and \( f_i' = f_i(\omega) \). Then since \( W(\alpha_1) = f_1 f_3, W(\theta E_4) = f_2 f_2 \) and \( W(\theta E_6) = f_3 f_3 \), we have

\[
\begin{align*}
W(\alpha_1) &= f_1 f_3, \\
W(\beta_3) &= 6(f_1 f_1')^2 + 2f_2 f_2' - 4(f_1^3 f_2' + (f_1')^3 f_2), \\
W(\delta_3) &= -16(f_1 f_1')^2 + 4(f_1^3 f_2 + (f_1')^3 f_2), \\
W(\gamma_4) &= 0.
\end{align*}
\]

Since it is obvious that \( W(\alpha_1), W(\beta_3), W(\delta_3) \) are algebraically independent, we see the four forms \( \alpha_1, \beta_3, \delta_3, \gamma_4 \) are algebraically independent by a standard induction as in Igusa [17, 1]. It is known that the dimension formula of modular forms belonging to \( \Gamma_0(3) \) is given by

\[
\sum_{k=0}^{\infty} \dim A_k(\Gamma_0(3)) t^k = \frac{1 + 2t^4 + t^6}{(1-t^2)(1-t^4)(1-t^6)}
\]

(cf. [12, p. 23]). Since \( B^{(even)} = C \oplus C\alpha_1 \beta_3 \oplus C\alpha_1 \delta_3 \oplus C\beta_3 \delta_3 \), we see \( B^{(even)} = \bigoplus_{k=0}^{\infty} A_{2k}(\Gamma_0(3)) \) by comparing the dimensions. We show that \( \alpha_1 \chi_{14}, \beta_3 \chi_{14}, \delta_3 \chi_{14}, \alpha_1 \beta_3 \delta_3 \chi_{14} \) are free generators of the module over \( C \). This is equivalent to say that \( \alpha_1, \beta_3, \delta_3, \alpha_1 \beta_3 \delta_3 \) are free over \( C \). Since \( C = \mathbb{C}[\alpha_1, \beta_3, \delta_3, \gamma_4] \), the proof is easily obtained by comparing parity of the degree of each variable. Hence we see that

\[
\bigoplus_{k=1}^{\infty} A_{2k-1}(\Gamma_0(3)) = C\alpha_1 \chi_{14} \oplus C\beta_3 \chi_{14} \oplus C\delta_3 \chi_{14} \oplus C\alpha_1 \beta_3 \delta_3 \chi_{14},
\]

comparing the dimensions. Next, we prove the assertion for \( \Gamma_0'(3) \). Assume that \( f \in A_k(\Gamma_0(3), \psi_3) \). Then we see \( \alpha_1 f \in A_{k+1}(\Gamma_0(3)) \). If \( k \) is even, then \( \alpha_1 f \) is divisible by \( \chi_{14} \). If \( \chi_{14}/\alpha_1 \) is holomorphic, then \( \chi_{14}/\alpha_1 \in A_{13}(\Gamma_0(3)) \). But we have \( A_{13}(\Gamma_0(3)) = \{0\} \) by dimension formula and this is a contradiction. Put \( g = \alpha_1 f/\chi_{14} \). Then \( g \in B \). Then we have \( \alpha_1^2 f^2 = g^2 \chi_{14}^2 \). We have \( \alpha_1, f^2, g, \chi_{14} \in B \) and \( B \) is a weighted polynomial ring. Since \( \chi_{14}/\alpha_1 \) is not holomorphic, \( \chi_{14}^2 \) is not divisible by \( \alpha_1^2 \) as an element of the weight polynomial ring. So \( \alpha_1 \) divide \( g^2 \). But since \( g \in B \), this means that \( g \) is divisible by \( \alpha_1 \) and \( g/\alpha_1 \in A_{k-14}(\Gamma_0(3)) \subset B \). Hence \( f \in B_{\chi_{14}} \). If \( k \) is odd, then \( \alpha_1 f \in A_{k+1}(\Gamma_0(3)) \subset B^{(even)} \). Again, we see \( \alpha_1^2 f^2 \in B^{(even)}, f^2 \in B^{(even)} \). So if we put \( \alpha_1 f = P(\alpha_1, \beta_3, \delta_3, \gamma_4) \) where \( P \) is a
polynomial of even (weighted) total degree, then \( P^2 \) is divided by \( \alpha_1^2 \). This means that \( \alpha_1 \) divides \( P \) in \( B \) and hence we have \( f \in B \). The assertion for \( \Gamma_0^\infty(3) \) is obvious from these.

In [12], we needed seven generators for \( \bigoplus_{k=0}^{\infty} A_{2k}(\Gamma_0(3)) \). Notation being as in [12, p. 21], (but see correction of several misprints in [12] in the last page of this paper,) we can show that \( t_2 = \alpha_1^2, u_4 = \gamma_4, v_6 = \delta_3^2, w_6 = \beta_3^2, x_4 = \alpha_1 \delta_3, y_4 = \alpha_1 \delta_3, \) and \( z_6 = \beta_3 \delta_3 \). Also, by comparing the images under the Witt operator and Fourier coefficients of both sides, we can show that

\[
\begin{align*}
\phi_4 &= 8x_4 + 41t_2^2 - 162u_4 + 5y_4, \\
\phi_6 &= 277t_2^3 - 2187t_2u_4 + 80t_2x_4 + \frac{11}{8}v_6 + w_6 + \frac{7}{2}z_6 + \frac{91}{2}t_2y_4, \\
\chi_{10} &= (64t_2^3u_4 + 32x_4t_2u_4 - 16y_4t_2u_4 - 4z_6u_4 + 4u_4v_6 + 4u_4w_6)/6144, \\
\chi_{12} &= (-124416t_4^2u_4 - 192t_2y_4w_6 - 768t_2^3u_4 + 16t_2y_4v_6 + 256t_2x_4w_6 \\
&+ 4096t_2x_4 - 1536t_2^2w_6 - 1024t_2y_4 + 16w_6^2 - 16u_4 w_6 - 6z_6 + 4v_6 z_6 + 4096t_2^6 + 7776t_2z_6u_4 + 10077696t_2^2u_4^2 \\
&- 62208t_2^2x_4u_4 + 31104t_2^2y_4u_4 + 2519424x_4u_4^2 - 1944t_2y_4u_4 \\
&- 7776t_2w_6u_4 - 68024448u_4^3 - 1259712y_4u_4^5)/3981312.
\end{align*}
\]

In particular, we have

\[
\chi_{10} = \gamma_4(8\alpha_1^3 + 2\beta_3 - \delta_3)^2/6144.
\]

For the sake of simplicity, from now on we put \( a = \alpha_1, b = \beta_3, c = \gamma_4, d = \delta_3 \). To write down the formula for \( \chi_{14} \), we define two modular forms \( f_{18} \) of weight 18 and \( f_{24} \) of weight 24 of \( \Gamma_0(3) \) as follows.

\[
f_{18} = 63489484a^3b^2 + 201216a^6d^3c - 5178816a^5d^3c - 60949905408a^7c^2b \\
+ 27264a^3b^3d^2 - 5441955840a^7c^2d - 79501824a^5c^3d - 2304a^6c^2d^2 \\
- 124032a^3b^4d - 1115609472a^4c^2b^2 - 7938048a^{12}bd \\
- 5462016a^9b^2d + 279936a^2d^4c + 2516904576a^4c^2d^2 - 5598720a^2b^4c \\
- 949542912a^{11}cb + 152285184a^{11}cd - 416544768a^8b^2c + 16796160a^8d^2c \\
+ 1344435190272a^3c^3b - 44079842304c^3bd - 1365504a^6d^3d \\
+ 28065792a^{12}bd - 4096b^6 - 143259487488a^3c^3d \\
+ 15073280a^{18} + 125d^6 + 174680064a^8d^6 + 6158592a^2b^3dc \\
- 979776a^2d^3bc - 839808a^2b^5d^2c - 24670199808a^{10}c^2 \\
- 34012224ad^2bc^2 - 32134205039616a^2c^4 + 476171136ab^2d^2c^2 \\
+ 58786560a^5b^2dc + 839808a^3d^2bc + 7890835968a^4c^2bd
\]

Graded Rings of Siegel Modular Forms, Differential Operators and Borcherds Products 259
\[ f_{24} = -1970749440 a^7 b^5 c^2 - 165888 a^2 b^6 c - 43008 a^8 b^4 d^2 \\
- 16926659444736 a b c^5 + 4478976 ac^2 b^5 + 8411086946304 a^5 c^4 b \\
+ 27895062528 a^6 b^5 d^2 + 15881803923456 a^8 c^4 + 201553920 c^2 b^2 d^2 \\
+ 3604480 a^{12} b^4 + d^8 + 16777216 a^{24} + 103195607040 a^{16} c^2 \\
+ 28311552 a^{18} b^2 - 1048576 a^{16} d^2 + 589824 a^9 b^5 \\
- 2208301056 a^{20} c + 8324176896 a^3 c^3 bd^2 - 340402176 a^4 c^2 b^2 d^2 \\
+ 101559956668416 e^8 - 256 a^6 d^6 - 40310784 c^6 b^4 + 2048 a^3 b^7 \\
+ 13107200 a^{15} b^3 - 64532889133056 a^4 c^5 - 1867840847424 a^{12} c^3 \\
+ 24576 a^{12} d^4 - 12317184 ac^2 b^2 d^2 - 64 b^6 d^2 + 48 d^4 b^4 \\
- 77760 a^2 cb^2 d^4 + 20155392 a^3 c^4 d^6 - 12 d^6 b^2 + 217728 a^4 c^2 d^2 \\
+ 33554432 a^{11} b + 53248 a^6 b^6 + 103514112 a^{11} d^2 bc - 3072 a^{3} b^5 d^2 \\
- 1617408 a^8 d^4 c - 594542592 a^{11} b^3 c + 35831808 a^8 b^2 d^2 c \\
- 93560832 a^8 b^4 c + 1110158991360 a^2 c^6 b^2 - 71833817088 a^2 d^2 c^4 \\
+ 8424 a^2 d^6 c - 808704 a^5 d^4 bc - 404397785088 a^6 c^3 b^2 - 3312451584 a^{17} cb \\
+ 8448 a^6 d^4 b^2 - 6967296 a^5 b^5 c + 4976640 a^{5} b^3 d^2 c + 2799360 a^6 d b^2 c \\
- 1572864 a^9 d^2 b - 933888 a^{12} b^2 d^2 + 1152 a^3 d^3 b^3 + 636014592 a^4 b^4 c^2 \\
- 128 a^3 d^4 b - 278528 a^9 b^3 d^2 + 24576 a^9 d^4 b + 2519424 a^4 d^4 c^2 \\
- 36199084032 a^3 c^3 b^3 + 114661785600 a^{13} c^2 b + 103514112 a^{14} d^2 c \\
- 1507945807872 a^9 c^3 b + 8814624768 a^7 b^3 c^2 + 47584641024 a^{10} c^2 b^2 \\
- 3224862720 a^{10} c^2 d^2 - 1974730752 a^{14} b^2 c. \]

By Igusa [18, p. 849], we get

\[ (\chi_{35})^2 = -\chi_{10} f_{18} f_{24} / (2^{70} : 3^{24}). \]

(Note that our notations are slightly different from those in Igusa [18].)

To obtain a relation between \(\chi_{35}\) and \(\chi_{14}\), we calculate the following functional determinant

\[ \text{jac}_{35} = \frac{\partial(\phi_4, \phi_6, \chi_{10}, \chi_{12})}{\partial(a, b, c, d)}. \]
First we review the results in [11]. We define a character 
\[ \chi_{35} = c \cdot \frac{\text{Jac}_{35}}{f_{18}/(2^{25} \cdot 3^6)}. \]

Since \(2^{34} \chi_{35} = \text{Jac}_{35}(2^{310} \chi_{14})\), we get
\[ \chi_{35} = \chi_{14} \times f_{18}(8a^3 + 2b - d)/2^{25}.\]

Incidentally, we have \(a(0, 0, 0; f_{18}) = 2^{21}\), \(a(0, 0, 0; 8a^3 + 2b - d) = 2^4\), and the product is \(2^{25}\). We have \(a(2, 3, 1; \chi_{35}) = a(2, 3, 1; \chi_{14}) = 1\) and \(a(t_1, t_2, t_{12}; \chi_{35}) = a(t_1, t_2, t_{12}; \chi_{14}) = 0\) if \(t_1 < 2\) and \(t_2 \leq 3\) or \(t_1 \leq 2\) and \(t_2 < 3\). This fits the above relation.

By virtue of the relation \(\chi_{10} = c(8a + 2b - d)^2/(2^{11} \cdot 3)\) and the above, we get
\[ \chi_{14}^2 = -c \cdot f_{24}/(2^{31} \cdot 3^{25}) \]
where \(c\) is the cusp form of weight 4.

5. Siegel Modular Forms of Level 4

First we review the results in [11]. We define a character \(\psi_4\) of \(\Gamma_0(4)\) by \(\psi_4(\gamma) = (\frac{A}{D})\) for any \(\gamma = (\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}) \in \Gamma_0(4)\). We define a subgroup of \(\Gamma_0(4)\) with index two by
\[ \Gamma^\psi_0(4) = \{ \gamma \in \Gamma_0(4); \psi_4(\gamma) = 1 \}. \]

We put
\[ f_{1/2} = \theta_{0000}(2Z), \]
\[ f_1 = f_{1/2}^2, \]
\[ g_2 = \theta_{0000}(2Z)^4 + \theta_{0100}(2Z)^4 + \theta_{1000}(2Z)^4 + \theta_{1100}(2Z)^4, \]
\[ h_2 = \theta_{0000}(2Z)^4 + \theta_{0001}(2Z)^4 + \theta_{0010}(2Z)^4 + \theta_{0011}(2Z)^4, \]
\[ f_3 = (\theta_{0001}(2Z)\theta_{0010}(2Z)\theta_{0011}(2Z))^2, \]
\[ \chi_{11} = \chi_5(2Z)(\theta_{0000}(2Z)^4 - \theta_{0010}(2Z)^4) \times (\theta_{0001}(2Z)^4 - \theta_{0011}(2Z)^4)(\theta_{0010}(2Z)^4 - \theta_{0011}(2Z)^4)/4096. \]

Then we have \(f_1 \in A_1(\Gamma_0(4), \psi_4), g_2, h_2 \in A_2(\Gamma_0(4)), f_3 \in A_3(\Gamma_0(4), \psi_4)\) and \(\chi_{11} \in A_{11}(\Gamma_0(4))\). We know that \(\dim A_{11}(\Gamma_0(4)) = 1\) and \(a(3, 2, 1; \chi_{11}) = -1\) (cf. [11]).

**Theorem 5.1 (Hayashida–Ibukiyama [11]).** Four forms \(f_1, g_2, h_2, f_3\) are algebraically independent and we have
\[ \bigoplus_{k=0}^{\infty} A_k(\Gamma^\psi_0(4)) = \mathbb{C}[f_1, g_2, h_2, f_3; \chi_{11}]. \]
Again we see that
\[
\chi_{11} = -\frac{1}{2^{18} \cdot 3(2\pi i)^3} \times \begin{vmatrix} f_1 & 1 & g_2 & 2h_2 & 3f_3 \\ \frac{\partial f_1}{\partial \tau} & \frac{\partial g_2}{\partial \tau} & \frac{\partial h_2}{\partial \tau} & \frac{\partial 3f_3}{\partial \tau} \\ \frac{\partial f_1}{\partial \omega} & \frac{\partial g_2}{\partial \omega} & \frac{\partial h_2}{\partial \omega} & \frac{\partial 3f_3}{\partial \omega} \\ \frac{\partial f_1}{\partial \alpha} & \frac{\partial g_2}{\partial \alpha} & \frac{\partial h_2}{\partial \alpha} & \frac{\partial 3f_3}{\partial \alpha} \end{vmatrix}.
\]

Indeed, since \(f_1\) and \(f_3\) are Siegel modular forms with character and \(g_2\) and \(h_2\) are without character, we see that \(\{f_1, g_2, h_2, f_3\}_3\) is a Siegel modular form without character of weight 11. Comparing a Fourier coefficient, we get the result.

6. Borcherds Product

In this section, we give a construction of \(\chi_{19}, \chi_{14}\) and \(\chi_{11}\) by Borcherds product. As for Siegel paramodular groups, a concrete theory of Borcherds product is treated in Gritsenko and Nikulin [9]. But the Borcherds product for congruence subgroups \(\Gamma_0(N)\) is not well-known, since \(\Gamma_0(N)\) is not an automorphism group of a lattice in general. So first we explain the essence of our way to construct Borcherds product shortly. Since we use a weaker version of Jacobi forms for construction of Borcherds product, we explain this first.

For any function \(\phi\) on \(H_1 \times \mathbb{C}\) and \(g \in SL_2(\mathbb{R})\) and \(X = (\lambda, \mu) \in \mathbb{R}^2, \kappa \in \mathbb{R}\), we put
\[
\phi_{[k,m]}(g) = (c\tau + d)^{-k} e^{m\left(-cz^2 \over c\tau + d\right)} \phi \left( a\tau + b \over c\tau + d \right),
\]
\[
\phi_{[m]}(X, \kappa) = e^{m(\lambda^2 \tau + 2\lambda z + \lambda \mu + \kappa)} \phi(\tau, z + \lambda \tau + \mu),
\]
where we put \(e^m(x) = e^{2imx}\) for any \(x \in \mathbb{C}\). For simplicity, we write \(\phi_{[m]}(X, 0) = \phi_{[m]}(X)\). Let \(\Gamma\) be a subgroup of \(SL_2(\mathbb{Z})\) of finite index. We denote by \(\Gamma^J\) the semi direct product \(\Gamma \times \mathbb{Z}^2\) where the product is defined as in Eichler–Zagier [6, p. 9]. Then the above operations of \(\gamma \in \Gamma\) and \(X \in \mathbb{Z}^2\) define an action of \(\Gamma^J\). For any integers \(k\) and \(m\) with \(m \geq 0\), a holomorphic function \(\phi\) on \(H_1 \times \mathbb{C}\) is called a weak Jacobi form of weight \(k\) and index \(m\) of \(\Gamma^J\) when it satisfies the following conditions:

1. \(\phi_{[k,m]}(\gamma) = \phi\) for any \(\gamma \in \Gamma\).
2. \(\phi_{[m]}(X) = \phi\) for any \(X \in \mathbb{Z}^2\).
3. \(\phi\) has the Fourier expansion
\[
\phi(\tau, z)_{[k,m]} = \sum_{n,l} c_\gamma(n,l) e(n\tau + lz)
\]
for all \(\gamma \in SL_2(\mathbb{Z})\) and besides \(c_\gamma(n,l) = 0\) unless \(n \geq 0\) (cf. [6]). (In the usual definition of Jacobi forms we assume that \(c_\gamma(n,l) = 0\) unless \(4nm - l^2 \geq 0\).) Instead of the condition (3), we sometimes use weaker condition.

(3') There is \(n_0\) such that \(c_\gamma(n,l) = 0\) for all \(n < n_0\) and all \(\gamma \in SL_2(\mathbb{Z})\).
If \( \phi \) satisfies (1), (2) and (3'), we say that \( \phi \) is a very weak Jacobi form. We denote by \( J_{k,m}^{\text{weak}}(\Gamma^J) \) or \( J_{k,m}^{w}(\Gamma^J) \) the space of weak, or very weak Jacobi forms, respectively. Sometimes we consider Jacobi forms with character \( \chi \) of \( \Gamma \), taking the condition \( \phi|_{k,m}[\gamma] = \chi(\gamma)\phi \) instead of the condition (1). The space of such Jacobi forms is denoted by \( J_{k,m}(\Gamma^J, \chi) \) and so on. It is obvious that if \( \phi(\tau, z) \) is very weak then \( \alpha(\tau)\phi(\tau, z) \) is a weak Jacobi form for some modular form \( \alpha(\tau) \) of \( \Gamma \). (For example, take \( \alpha \) to be some power of \( \Delta \) where \( \Delta \) is the Ramanujan delta function.)

Also we denote by \( A(\Gamma) \) the graded ring of all holomorphic modular forms on \( H_1 \) belonging to \( \Gamma \) and by \( A_k(\Gamma, \chi) \) the space of holomorphic modular forms of \( \Gamma \) of weight \( k \) with character \( \chi \). When \( \chi \) is trivial, we write \( A_k(\Gamma) = A_k(\Gamma, \chi) \). We define \( \phi_{0,1} \in J_{0,1}^{\text{weak}}(SL_2(\mathbb{Z})) \), \( \phi_{-2,1} \in J_{-2,1}^{\text{weak}}(SL_2(\mathbb{Z})) \) and \( \phi_{-1,2} \in J_{-1,2}^{\text{weak}}(SL_2(\mathbb{Z})) \) as in Eichler–Zagier [6, pp. 108 and 110]. Namely we put

\[
\phi_{-2,1} = \frac{\phi_{10,1}}{\Delta} = (-2 + \zeta + \zeta^{-1}) + q(-12 + 8\zeta + 8\zeta^{-1} - 2\zeta^2 - 2\zeta^{-2}) + \cdots,
\]
\[
\phi_{0,1} = \frac{\phi_{12,1}}{\Delta} = (10 + \zeta + \zeta^{-1}) + q(108 - 64\zeta - 64\zeta^{-1} + 10\zeta^2 + 10\zeta^{-2}) + \cdots,
\]
\[
\phi_{-1,2} = \frac{\phi_{11,2}}{\Delta} = (\zeta - \zeta^{-1}) + q(3(\zeta - \zeta^{-1}) + \zeta^3 - \zeta^{-3}) + \cdots
\]
where \( \phi_{10,1}, \phi_{12,1} \) or \( \phi_{11,2} \) is a unique element in \( J_{10,1}(SL_2(\mathbb{Z})), J_{12,1}(SL_2(\mathbb{Z})), J_{11,2}(SL_2(\mathbb{Z})) \) up to constant.

By the way, we have the following product expansion (cf. [3, p. 184, Theorem 6.5]).

\[
\phi_{-2,1}(\tau, z) = \zeta^{-1}(1 - \zeta^2) \prod_{n=1}^{\infty} \frac{(1 - q^n\zeta^2)(1 - q^n\zeta^{-1})^2}{(1 - q^n)^4},
\]
\[
\phi_{-1,2}(\tau, z) = \zeta^{-1}(1 - \zeta^2) \prod_{n=1}^{\infty} \frac{(1 - q^n\zeta^2)(1 - q^n\zeta^{-2})}{(1 - q^n)^2}.
\]

We denote by \( J_{\text{even},*}^{\text{weak}}(\Gamma^J) \) the graded ring of weak Jacobi forms of even weight of arbitrary index. We also denote by \( J_{\text{odd},*}^{\text{weak}}(\Gamma^J) \) the vector space of weak Jacobi forms of arbitrary odd weight, and by \( J_{*,*}^{\text{weak}}(\Gamma^J) \) the space of arbitrary weight and index.

**Proposition 6.1.** We have

\[
J_{*,*}^{\text{weak}}(\Gamma^J) = A(\Gamma)[\phi_{-2,1}, \phi_{0,1}] \oplus \phi_{-1,2} A(\Gamma)[\phi_{-2,1}, \phi_{0,1}].
\]

When \(-12 \in \Gamma\), more precisely we have

\[
J_{\text{even},*}^{\text{weak}}(\Gamma^J) = A(\Gamma)[\phi_{-2,1}, \phi_{0,1}],
\]
\[
J_{\text{odd},*}^{\text{weak}}(\Gamma^J) = \phi_{-1,2} J_{\text{even},*}^{\text{weak}}(\Gamma).
\]
Proof. Since our proof is almost the same as in [6], we sketch an outline shortly here. First we assume that $-1_2 \in \Gamma$. Assume that $k$ is even. For any $\phi \in J_{k,m}^{weak}(\Gamma^J)$, we see $\phi(\tau, z) = \phi(\tau, z)$ so take the Taylor expansion $\phi = \sum_{t=0}^{\infty} f_t(\tau) z^{2t}$. We denote by $\hat{\phi}$ the group generated by $\phi$ and $A$. Since $\hat{\phi}$ is defined by the natural projection to $\hat{\Gamma}$, we see the following exact sequence as before.

$$0 \rightarrow J_{k,m}^{weak}(\Gamma^J)(r+1) \rightarrow J_{k,m}^{weak}(\Gamma^J)(r) \rightarrow A_{k+2r}(\Gamma).$$

Also we can show that $\phi(\tau, z) = 0$ has exactly $2m$ zeros counting multiplicity inside the fundamental parallelotope and we get $J_{k,m}^{weak}(\Gamma^J)(m+1) = \{0\}$. Hence we get

$$\dim J_{k,m}^{weak}(\Gamma^J) \leq \sum_{r=0}^{m} \dim A_{k+2r}(\Gamma).$$

Now $\phi_{0,1}, \phi_{-2,1} \in J_{even,\Gamma}(\Gamma^J)$ and since $\phi_{0,1} = 12 + O(z^2)$ and $\phi_{-2,1} = (2\pi i z)^2 + O(z^4)$, the functions $\phi_{0,1}^{\theta}, \phi_{-2,1}^{\theta} (0 \leq a, 0 \leq b)$ are linearly independent over $A(\Gamma)$. Hence the maximum of the above inequality is attained.

The case for odd $k$ is similarly proved. We see $\phi(\tau, z) = -\phi(\tau, -z)$, so take the Taylor expansion $\phi = \sum_{t=0}^{\infty} f_t(\tau) z^{2t+1}$. We denote by $J_{k,m}^{weak}(\Gamma^J)(r)$ the space of those $\phi \in J_{k,m}^{weak}(\Gamma^J)$ such that $f_t = 0$ unless $t \geq r$. By the mapping from $\phi$ to $f_r$, we get the exact sequence

$$0 \rightarrow J_{k,m}^{weak}(\Gamma^J)(r+1) \rightarrow J_{k,m}^{weak}(\Gamma^J)(r) \rightarrow A_{k+2r+1}(\Gamma).$$

Also we can show $J_{k,m}^{weak}(\Gamma^J)(m-1) = \{0\}$. Hence we get

$$\dim J_{k,m}^{weak}(\Gamma^J) \leq \sum_{r=0}^{m-2} \dim A_{k+2r+1}(\Gamma)$$

and

$$J_{odd,\Gamma}^{weak}(\Gamma^J) = \phi_{-1,2} A(\Gamma)[\phi_{-2,1}, \phi_{0,1}].$$

Now we assume that $-1_2 \notin \Gamma$. For any Jacobi weak form $\phi(\tau, z) \in J_{k,m}^{weak}(\Gamma^J)$ we consider two Jacobi forms $f$ and $g$ defined by

$$f = \phi(\tau, z) + \phi(\tau, z)_{|k,m[-1_2]},$$

$$g = \phi(\tau, z) - \phi(\tau, z)_{|k,m[-1_2]}.$$

We denote by $\hat{\Gamma}$ the group generated by $\Gamma$ and $-1_2$ and by $\chi$ the character of $\hat{\Gamma}$ which is defined by the natural projection to $\hat{\Gamma}/\Gamma \cong \{\pm 1\}$. Then $f \in J_{k,m}(\hat{\Gamma})$. Since $A(\hat{\Gamma}) = A_{even}(\Gamma) = \bigoplus_{k=0}^{\infty} A_{2k}(\Gamma)$, we see that $f \in A_{even}(\Gamma)[\phi_{-2,1}, \phi_{0,1}]+ \phi_{-2,1} A_{even}(\Gamma)[\phi_{-2,1}, \phi_{0,1}]$. On the other hand, we have $g \in J_{k,m}(\hat{\Gamma}, \chi)$. We have the following exact sequence as before.

$$0 \rightarrow J_{k,m}^{weak}(\hat{\Gamma}, \chi)(s+1) \rightarrow J_{k,m}^{weak}(\hat{\Gamma}, \chi)(s) \rightarrow A_{k+2s+1}(\hat{\Gamma}, \chi).$$

For each $k$ we get

$$\dim J_{k,m}^{weak}(\hat{\Gamma}, \chi) \leq \sum_{s=0}^{m-2} \dim A_{k+2s+1}(\hat{\Gamma}, \chi).$$
and for odd $k$ we get
\[ \dim J_{k,m}^{\text{weak}}(\tilde{\Gamma}, \chi) \leq \sum_{s=0}^{m} \dim A_{k+2s}(\tilde{\Gamma}, \chi). \]

Since $A_k(\tilde{\Gamma}, \chi) = A_k(\Gamma)$ if $k$ is odd, we see that
\[ g \in A_{\text{odd}}(\Gamma)[\phi_{-2,1}, \phi_{0,1}] + A_{\text{odd}}(\Gamma)[\phi_{-2,1}, \phi_{0,1}]. \]

By these we get the conclusion of the proposition. \hfill \Box

In a sense the Borcherds product is obtained as a log version of Saito–Kurokawa lifting, so we need the result that the discrete group we consider is generated by Jacobi group and an elementary change of variables. We shall show this now.

We consider four kinds of elements of $Sp(2, \mathbb{Z})$.

\[ T = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad u(x) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ x & 1 & 0 & 0 \\ 0 & 0 & 1 & -x \\ 0 & 0 & 0 & 1 \end{pmatrix}, \]
\[ C(a, b, c, d) = \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ cN & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad u(S) = \begin{pmatrix} 1_2 & S \\ 0 & 1_2 \end{pmatrix}, \]

where $x \in \mathbb{Z}$, \((a_\mathcal{N}, b_\mathcal{N}) \in \Gamma_0^{(1)}(N), tS = S \in M_2(\mathbb{Z})$.

**Lemma 6.2.** For any natural number $N$, the group $\Gamma_0(N)$ is generated by the above four kinds of matrices.

**Proof.** Take $\gamma = (\begin{pmatrix} a_\mathcal{N} & b_\mathcal{N} \\ c_\mathcal{N} & d_\mathcal{N} \end{pmatrix}) \in \Gamma_0(N)$ and put $A = (a_{ij}), C = (c_{ij})$. We can assume $a_{11}, a_{21} \geq 0$ by multiplying $C(-1, 0, 0, -1)$ or $TC(-1, 0, 0, -1)T$ if necessary. Multiplication of $T$ and $u(x)$ gives the Euclid algorithm to the pair $(a_{11}, a_{21})$, so we can assume that $a_{21} = 0$. Then, since $\gamma \in SL_4(\mathbb{Z})$, $a_{11}$ and $N$ are coprime. Denote by $m$ the g.c.d. of $a_{11}$ and $c_{11}$. Then if we put $c_0 = c_{11}/m$ and $d_0 = -a_{11}/m$, then $Nc_0$ and $d_0$ are coprime. Hence there exists $a_0, b_0$ such that $a_0d_0 - b_0c_0 = N$. Multiplying $C(a_0, b_0, c_0, d_0)$ from left for a suitable choice of $a_0, b_0$, we may assume that the first column of $\gamma$ is $(m, 0, 0, Nc_{22})$. Here $m$ and $Nc_{22}$ are coprime. Multiplying $u(1)$ from left, this is changed into $(m, m, Nc_{22}, NC_{22})$. We take a pair of coprime integers $c_1, d_1$ such that $c_1m + d_1c_0 = 0$ and a pair $a_1, b_1$ such that $a_1d_1 - Nb_1c_1 = 1$. Then multiplying $C(a_1, b_1, c_1, d_1)$ and $TC(a_1, -b_1, -c_1, d_1)T$ from left, the first column becomes $(a_2, a_2, 0, 0)$ for some $a_2 \in \mathbb{Z}$. Since $\gamma \in SL_4(\mathbb{Z})$, we have $a_2 = \pm 1$. By multiplying $u(-1)$ from left, we get $(1, 0, 0, 0)$. Since $\gamma \in Sp(2, \mathbb{R})$, we have $c_{12} = d_{12} = 0$ and $d_{11} = 1$. Then $\left(\begin{smallmatrix} a_{22} \\ b_{22} \\ c_{22} \\ d_{22} \end{smallmatrix}\right) \in \Gamma_0^{(1)}(N)$. Hence by taking $TC(d_{22}, -b_{22}, -c_{22}, N, a_{22})T\gamma$, we can assume that $C = 0$, $d_{12} = 0$, and $a_{11} = a_{22} = d_{11} = d_{22} = 1$. This is a product of $u(x)$ and $u(S)$ for suitable $x$ and $S$. \hfill \Box
To construct a kind of Saito–Kurokawa lifting of weight 0, we define a Hecke operator shifting indices as in [6]. We put

\[ \Delta_N(t) = \left\{ g = \begin{pmatrix} a & b \\ cN & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, \det(g) = t \right\}. \]

For any \( \phi \in J_{k,m}^\infty(\Gamma_0(1)(N)) \), we define

\[ (\phi|_{k,m} V_l)(\tau, z) = t^{k-1} \sum_{(a \ b \ cN \ d) \in \Gamma_0(1)(N) \setminus \Delta_N(t)} (c\tau + d)^{-k} e^{\frac{-cz^2}{c\tau + d}} \times \phi \left( \frac{a\tau + b}{c\tau + d}, \frac{tz}{c\tau + d} \right). \]

This operator is a mapping from \( J_{k,m}^\infty(\Gamma_0(1)(N)) \) to \( J_{k,m}^\infty(\Gamma_0(1)(N)) \).

Now to calculate everything more concretely, we should describe representatives of \( \Gamma_0(1)(N) \setminus \Delta_N(t) \). We take a complete set of cusps of \( \Gamma_0(1)(N) \). Namely, we take a complete set of representatives \( \{g_s\} \) of double coset \( \Gamma_0(1)(N) \setminus SL_2(\mathbb{Q})/P(\mathbb{Q}) \), where \( P(\mathbb{Q}) = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a, b \in \mathbb{Q} \right\} \). Since \( SL_2(\mathbb{Q}) = SL_2(\mathbb{Z})P(\mathbb{Q}) \), we may assume \( g_s \in SL_2(\mathbb{Z}) \). Now take \( g \in \Delta_N(t) \). Since \( g \left( \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right) \in SL_2(\mathbb{Q}) \), we have

\[ g \in \Gamma_0(1)(N)g_s \left( \begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix} \right) \]

for some \( s, a > 0, b, d > 0 \) with \( ad = t \). If

\[ \Gamma_0(1)(N)g_1 \left( \begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix} \right) = \Gamma_0(1)(N)g_2 \left( \begin{smallmatrix} a_2 & b_2 \\ 0 & d_2 \end{smallmatrix} \right) \]

for \( a_2 > 0, d_2 > 0 \), then obviously \( g_1 = g_2, a = a_2, d = d_2 \). Define a natural number \( h_s \) by

\[ g_s^{-1}\Gamma_0(1)(N)g_s \cap P(\mathbb{Z}) = \left\{ \pm \left( \begin{smallmatrix} 1 & h_s n \\ 0 & 1 \end{smallmatrix} \right) : n \in \mathbb{Z} \right\}, \]

where we put \( P(\mathbb{Z}) = P(\mathbb{Q}) \cap SL_2(\mathbb{Z}) \). Then \( b \equiv b_2 \mod h_s d \). We put \( g_s = \left( \begin{smallmatrix} x & y \\ z & w \end{smallmatrix} \right) \). Then the condition for \( g_s \left( \begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix} \right) \in \Delta_N(t) \) is that \( az_s \equiv 0 \mod N \) and \( ad = t \). As a whole we see that

\[ \Gamma_0(1)(N) \setminus \Delta_N(t) = \bigcup_s \left\{ g_s \left( \begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix} \right) : a, b, d \in \mathbb{Z}, ad = t, az_s \equiv 0 \mod N, b = 0, \ldots, h_s d - 1 \right\}. \]
Now we take \( \phi(\tau, z) \in \mathcal{M}^{(1)}(\Gamma_0^1(N)) \). We put
\[
\phi(\tau, z)_{[0,1]}[c_s] = \sum_{n,l} c_s(n,l)e(n\tau + lz).
\]
Here we note that \( \phi(\tau, z)_{[0,1]}[c_s] = \phi(\tau, z)_{[0,1]}[\lambda, \mu] \) for any integers \( \lambda, \mu \in \mathbb{Z} \).

So we get \( c_s(4n - l^2) = c_s(n,l) \). Here \( n \in h_s^{-1}Z \) might not be an integer. But if \( 4 \not| \ h_s \) then \( l \) mod 2 is determined only by \( 4n - l^2 \) and in this case we write \( c_s(4n - l^2) = c_{s,t}(4n - l^2) \) for simplicity.

We define an operator \( L \) by
\[
L\phi = \sum_{t=1}^{\infty} (\phi_{[0,1]}V_t)(\tau, z)e^{2\pi it\omega}.
\]
For each cusp, we put \( \phi_s(\tau, z) = \phi(\tau, z)_{[0,1]}[c_s] \) and \( n_s = N/(z_s, N) \). Then we have
\[
L\phi = \sum_{s} \frac{1}{l_{s}} \sum_{d|l_{s}, a_{s}, \equiv 0 (N)} \sum_{h_s,d-1} c_{s,l}(4an + l^2)e((n\tau + l\omega)z)
\]
\[
= \sum_{s} h_s \sum_{a=1}^{\infty} \frac{1}{an_s} \sum_{m=1}^{\infty} c_{s,l}(4mn - l^2)e(n\tau + l\omega)z
\]
\[
= \sum_{s} \frac{h_s}{n_s} \log \left( \prod_{l,m,n \in \mathbb{Z}, m \geq 1} (1 - e(n\tau + l\omega + m\omega))^{c_{s,l}(4mn - l^2)} \right).
\]
We note that \( L\phi \) converges only for those \( Z \in H_2 \) such that \( \text{Im}(Z) \) is big enough, namely \( \text{Im}(Z) > Y_0 \) is positive definite for some fixed \( Y_0 \). But by analytic continuation, \( \exp(L\phi) \) becomes multi-valued meromorphic function on \( H_2 \) according to the general theory of Borcherds product (cf. [3, p. 177, Theorem 5.1] or [4, p. 88, Theorem 3.22]). Its singularities and zeros are on so called rational quadratic divisors. The behaviour on rational quadratic divisors is determined by exponents, and in our case if \( n^{-1}h_s c_{s,l}(N) \) are integers for \( N < 0 \), then the product is single valued (here if \( 4mn - l^2 \geq 0 \) we see \( |e(n\tau + l\omega + m\omega)| < 1 \) on \( H_2 \) and we can take the principal value as a branch of \( (1 - e(n\tau + l\omega + m\omega))^{c_{s,l}(4mn - l^2)} \), and if \( c_{s,l}(N) > 0 \) for \( N < 0 \) besides, then it is holomorphic).

Now we see that \( L\phi \) is invariant by the action of \( \Gamma_0(1)(N)^J \). Since \( \phi \) is of weight 0, this is also true for \( \exp(L\phi) \) which is actually given by the following infinite product
\[
\prod_{l,m,n \in \mathbb{Z}, m \geq 0} (1 - e(n\tau + l\omega + m\omega))^{c_{s,l}(4mn - l^2)} h_s/n_s,
\]
where we put \( p = e(\omega) \). If this is invariant by exchange of \( \tau \) and \( \omega \), then by Lemma 6.2, we see that this is a Siegel modular form belonging to \( \Gamma_0(N) \). But
actually this is false since we have terms with \( n \leq 0 \) while all \( m \geq 1 \) in the above product. So to get invariance for this exchange, we must multiply something to \( \exp(L\phi) \). To give such a candidate, we prepare notations. For integers \( m, l, n \in \mathbb{Z} \), we write \((n, l, m) > 0 \) if (1) \( m > 0 \), \( n, l \in \mathbb{Z} \), or (2) \( m = 0 \), \( n > 0 \), \( l \in \mathbb{Z} \), or (3) \( m = n = l > 0 \). For \( \phi(\tau, z) \in J^m_{1,0} \), we denote by \( c_{s, l}(N) \) the Fourier coefficients at each cusp as before. We put \( d(\phi) = \sum_{s} \sum_{m > 0, n < 0, m, n, l \in \mathbb{Z}} \frac{n^{-1}}{s^1} h_{s, c_s, l}(4mn - l^2) \).

For some integer \( b \) and positive integers \( a, c \), we put

\[
F(\tau, z, \omega) = q^{a} \zeta^{b} \prod_{(n, l, m) > 0} (1 - (q^n \zeta^l p^m)^{n_s})_s^{-1} h_{s, c_s, l}(4mn - l^2),
\]

\[
f(\tau, z) = q^{a} \zeta^{b} \prod_{s} \prod_{n, l \geq 1} ((1 - (q^n \zeta^l)^{n_s})(1 - (q^n \zeta^l)^{n_s})_s^{-1} h_{s, c_s, l}(-l^2))
\times \prod_{n = 1}^{\infty} (1 - q^{n n_s})_s^{-1} h_{s, c_s, l}(0) \prod_{l \geq 1} (1 - \zeta^{l n_s})_s^{-1} h_{s, c_s, l}(-l^2).
\]

We note here that if \( N < 0 \), we have \( c_{s, l}(N) \neq 0 \) only for finitely many \( N \), so in the above product expression of \( f(\tau, z) \), only finitely many different \( l \) appears. Indeed if \( \phi \) is very weak, there is \( n_0 \) such that \( c_s(n, r) = 0 \) if \( n < n_0 \). So, if \( N < 4n_0 - 1 \) for \( N \equiv -r^2 \mod 4 \) \( (r = 0 \text{ or } 1) \), then \( c_{s, l}(N) = c_s((N + r^2)/4, r) = 0 \). Also for a fixed \( N < 0 \), the number of triples \((n, m, r)\) with \( m \geq 1 \) and \( n < 0 \) such that \( 4mn - l^2 = N < 0 \) is finite.

We denote by \( \chi \) a Dirichlet character modulo \( N \) and define a character of \( \Gamma_0(N) \) or \( \Gamma_0^{(1)}(N) \) by \( \chi_2(\gamma) = \chi(\det(D)) \) or \( \chi_1(\gamma) = \chi(d) \) for \( \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \) or \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \), respectively.

**Proposition 6.3.**

1. If the relation
   \[
a - c = \sum_{m,n>0} \sum_{l,m,n \in \mathbb{Z}} mh_{s, c_s, l}(4mn - l^2)
   \]
   is satisfied, then we have \( F(\omega, z, \tau) = (-1)^{d(\phi)} F(\tau, z, \omega) \).

2. If \( f(\tau, z) \in J_{k,e}(\Gamma_0^{(1)}(N), \chi_1) \) and \( (-1)^{d(\phi)} = (-1)^k \chi(-1) \), then we have \( F(\tau, z, \omega) \in A_k(\Gamma_0(N), \chi_2) \).

**Proof.** The product \( F(\tau, z, \omega) \) was defined by multiplying something to \( \exp(L\phi) \) so that it is relatively symmetric for \( \tau \) and \( \omega \). The idea of this definition is explained as follows. The part of the product in \( F \) for \( m \geq 1 \), \( n, l \in \mathbb{Z} \) appears already in \( \exp(L\phi) \). There if \( m \geq 1 \), \( n \geq 1 \), then \((1 - (q^n \zeta^l p^m)^{n_s})(1 - (q^n \zeta^l)^{n_s})_s^{-1} h_{s, c_s, l}(4mn - l^2) \) have the same exponent \( n^{-1} h_{s, c_s, l}(4mn - l^2) \). So this part is symmetric by exchange of \( \tau \) and \( \omega \) without any change. When \( m \geq 1 \) and \( n = 0 \), to make the product symmetric, the part \( \prod_{s,l} \prod_{m \geq 1} (1 - (q^n \zeta^l)^{n_s})_s^{-1} h_{s, c_s, l}(-l^2) \) is multiplied to \( \exp(L\phi) \) in \( F(\tau, z, \omega) \). So this part becomes also symmetric. Finally when \( m \geq 1 \) and \( n \leq -1 \), instead
of multiplying a power of \((1 - (q^m \zeta^l p^n)^{n_1})\), we multiply \(q^n p^c\) to avoid a negative power of \(p\). Since \((1 - (q^m \zeta^l p^n)^{n_1}) = (-q^{-m} \zeta^{-l} p^{-n})^{n_1}(1 - (q^m \zeta^l p^n)^{n_1})\), the part coming from \((-q^m \zeta^l p^n)\) and \(q^n p^c\) remains. Namely, we have

\[
\frac{F(\tau, z; \omega)}{F(\omega, z; \tau)} = q^{a - c} p^{c - a} (-1)^d(\phi)(pq^{-1})^{A_s B},
\]

where

\[
A = \sum_{s} \sum_{m > 0, n < 0, m, n \in \mathbb{Z}} mh_s c_{s,l}(4mn - l^2),
\]

\[
B = \sum_{s} \sum_{m > 0, n < 0, m, n \in \mathbb{Z}} lh_s c_{s,l}(4mn - l^2).
\]

We have \(B = 0\) since \(c_{s,l}(4mn - l^2) = c_{s,-l}(4mn - l^2)\). By our condition in (1), we also have \(a - c = A\). Hence (1) is proved. Since \(F(\tau, z; \omega) = f(\tau, z) \exp(L \phi)\) and \(\exp(L \phi)\) is invariant by the action of \(\Gamma_0^{(1)}(N)\), we see that under the action of \(\Gamma_0^{(1)}(N)\) the function \(F(\tau, z; \omega)\) behaves like a Siegel modular form of weight \(k\) with character \(\chi_2\). By (1) and Lemma 6.2, we get \(F(\tau, z; \omega) \in A_k(\Gamma_0(N), \chi_2)\).

In the above, we explained how to construct Borcherds product from an element of \(J^u_{1,0}\). Now conversely, if we are given a Siegel modular form \(F \in S_k(\Gamma_0(N), \chi_2)\) and want to express \(F\) by Borcherds product, how can one find \(\phi \in J^u_{1,0}\) which we start from? About this we explain rough idea which is experimentally valid. We take the Fourier–Jacobi coefficient \(f_m(\tau, z)\) of \(p^m\) of \(F\) for the smallest \(m\) such that \(f_m \neq 0\). Then if \(f_m(\tau, z)\) is expressed as an infinite product and if \(F\) can be written as a Borcherds product, then \(f_m(\tau, z)\) should appear as a factor of the product. Since there is no \(p\) in this product, this part should be the part for \(m \leq 0\) which should be multiplied to \(\exp(L \phi)\) to create symmetry. So \(f_m(\tau, z)\) should be equal to \(f(\tau, z)\) in the above proposition. Although we do not know which products in \(f_m(\tau, z)\) should correspond to which cusps, we just distribute some parts of \((1 - (q^m \zeta^l p^n)^{r})\) to a cusp with \(n_s = r\) and hope that there exists \(\phi \in J^u_{0,1}(\Gamma_0^{(1)}(N))\) such that \(f(\tau, z) = f_m(\tau, z) p^m\) for some \(a, b, c\) satisfying the condition of (2). Concrete calculations that this idea works will be given in the examples below.

Now we shall show that \(\chi_{19} \in S_{19}(\Gamma_0(2))\), \(\chi_{14} \in S_{14}(\Gamma_0(3), \psi_3)\), \(\chi_{11} \in S_{11}(\Gamma_0(4))\) are all obtained by Borcherds products.

For any prime \(p\), \(\Gamma_0^{(1)}(p)\) has two cusps represented by \(g_1 = 1_2^1\) and \(g_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\) and we have \(n_1 = 1, n_2 = p\) and \(h_1 = 1, h_2 = p\). We have

\[
\Gamma_0^{(1)}(p) \setminus \Delta_p(t) = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in M_2(\mathbb{Z}); ad = t, b = 0, \ldots , d - 1 \right\} \cup \left\{ g_2 \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in M_2(\mathbb{Z}); ad = t, a \equiv 0 \pmod{p}, b = 0, \ldots , pd - 1 \right\}.
\]

First, we consider Siegel modular forms of level 2. Since

\[
\chi_{19} = \eta(\tau)^8 \eta(2\tau)^{32} \phi_{-1,2} p^2 + \cdots
\]
and

\[ \phi_{-1,2} = \zeta^{-1}(1 - \zeta^2) \prod_{n=1}^{\infty} (1 - q^n \zeta^2)(1 - q^n \zeta^{-2})(1 - q^n)^{-2}, \]

and \( \dim S_{19}(\Gamma_0(2)) = 1 \), if we can find a very weak Jacobi form \( \phi(\tau, z) \in J_{0,1}\Gamma_0^{(1)}(2) \) such that

\[ c_1(-4) = 1, \quad c_1(0) = 6, \quad c_1(n) = 0 \text{ if } n \leq 0, \quad n \neq 0, -4, n \in \mathbb{Z}, \]
\[ c_2(0) = 32, \quad c_2(n) = 0 \text{ if } n \leq 0, \quad n \neq 0, n \in \mathbb{Z}, \]

its Borcherds lift should be \( \chi_{19} \), where \( c_1(n) \) or \( c_2(n) \) are the Fourier coefficients of \( \phi \) or \( \phi_2 = \phi_{[0,1][g_2]} \).

Now we construct this \( \phi \). Define modular forms of \( \Gamma_0^{(1)}(2) \) by

\[ \alpha(\tau) = \theta_{00}(2\tau)^4 + \theta_{10}(2\tau)^4 \in A_2(\Gamma_0^{(1)}(2)), \]
\[ \beta(\tau) = \frac{1}{16} \theta_{00}(2\tau)^4 \theta_{10}(2\tau)^4 \in A_4(\Gamma_0^{(1)}(2)). \]

Their Fourier expansion at cusps \( \infty \) and \( 0 \) are given by

\[ \alpha(\tau) = 1 + 24q + 24q^2 + \cdots, \]
\[ \beta(\tau) = 0 + q + 8q^2 + \cdots, \]

\[ \left( \alpha \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right)(\tau) = -\frac{1}{2} \alpha(\tau/2) = -\frac{1}{2} - 12q^{\frac{1}{2}} - 12q + \cdots, \]
\[ \left( \beta \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right)(\tau) = \frac{1}{256} \left( \alpha(\tau/2)^2 - 64\beta(\tau/2) \right) = \frac{1}{256} - \frac{1}{16} q^{\frac{1}{2}} + \frac{7}{16} q + \cdots. \]

We note that \( \beta(\tau) = \eta(2\tau)^{16}/\eta(\tau)^{8} \). Hence \( \beta \) has no zero on \( H_1 \). Now \( \beta \phi \) should be a weak Jacobi form of weight 4 and index 1. Then by virtue of Proposition 6.1, \( \beta \phi \) should be written as

\[ \beta(\tau) \phi(\tau, z) = (a\alpha(\tau)^3 + b\alpha(\tau)\beta(\tau)) \phi_{-2,1}(\tau, z) + (c\alpha(\tau)^2 + d\beta(\tau)) \phi_{0,1}(\tau, z) \]

for some \( a, b, c, d \in \mathbb{R} \). Imposing the condition on the Fourier coefficients we demanded, now we find

\[ \phi(\tau, z) = \frac{\alpha(\tau)}{\beta(\tau)} \left( \frac{\varphi_{0,1}(\tau, z) - \alpha(\tau)\varphi_{-2,1}(\tau, z)}{12} \right) + \frac{32}{3} \alpha(\tau) \varphi_{-2,1}(\tau, z) - \frac{8}{3} \varphi_{0,1}(\tau, z). \]

Incidentally we can show that this has integral coefficients. Indeed we easily see \( \alpha - 1 \in 24q\mathbb{Z}[[q]] \). Also we see \( \phi_{0,1} - \phi_{-2,1} \in 12\mathbb{Z}[[q, \zeta, \zeta^{-1}]] \) since \( (E_4\phi_{0,1} - E_6\phi_{-2,1})/12 \)
is the Eisenstein Jacobi form of weight 4 and of index 1 having integral coefficients where $E_4$ or $E_6$ are Eisenstein series of $SL_2(\mathbb{Z})$ of weight 4 or 6, each belonging to $1 + 24\mathbb{Z}[[q]]$. Since we have

$$\phi(\tau, z) = \left( \frac{\alpha(\tau)^2}{\beta(\tau)} - 32 \right) \left( \varphi_{0,1}(\tau, z) - \alpha(\tau) \varphi_{-2,1}(\tau, z) \right) + 8\alpha(\tau) \varphi_{-2,1}(\tau, z)$$

$$= q^{-1} + (6 + \zeta + \zeta^{-1}) + (8 + 128(\zeta + \zeta^{-1}) + 6(\zeta^2 + \zeta^{-2}))q + \cdots,$$

we see that all Fourier coefficients of $\phi(\tau, z)$ are integers. Also we have

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{16\alpha(\tau)}{\beta(\tau)^2} \\ \alpha(\tau)^2 - 64\beta(\tau)^2 \end{bmatrix} - 8 \begin{bmatrix} \varphi_{0,1}(\tau, z) - \alpha(\tau) \varphi_{-2,1}(\tau, z) \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \varphi_{0,1}(\tau, z) - \alpha(\tau) \varphi_{-2,1}(\tau, z) \\ -2 \end{bmatrix},$$

and all Fourier coefficients of $\phi[1 \quad 0 \quad 1 \quad 0](\tau, z)$ are integers.

**Conclusion.** Borcherds lift of

$$\phi = \frac{\alpha^2}{\beta} \left( \frac{\varphi_{0,1} - \alpha \varphi_{-2,1}}{12} \right) + \frac{32}{3} \frac{\alpha \varphi_{-2,1} - 8}{3} \varphi_{0,1}$$

is $\chi_{19}$. Namely we have

$$\chi_{19} = q^3 \zeta^{-1} \eta^2 \prod_{(n,l,m)>0} \left( 1 - q^n \zeta^l \eta^m \right)^{c_1(4nm-l^2)} \left( 1 - q^{2n} \zeta^{2l} \eta^{2m} \right)^{c_2(4nm-l^2)}.$$

Secondly, we consider a Siegel modular form of level 3. Since

$$\chi_{14} = \eta(3\tau)^9 \eta(3\tau)^{21} \varphi_{-1,2}(\tau, z)p^2 + \cdots,$$

and $\dim S_{14}(\Gamma_0(3), \psi_3) = 1$, if we can find a very weak Jacobi form $\phi(\tau, z) \in J_{14,1}(\Gamma_0(3))$ such that

$$c_1(-4) = 1, c_1(0) = 7, \quad c_1(n) = 0 \text{ if } n \leq 0, \quad n \neq 0, -4, \ n \in \mathbb{Z},$$

$$c_2(0) = 21, \quad c_2(n) = 0 \text{ if } n < 0, \ n \in \mathbb{Z},$$

its Borcherds lift should be $\chi_{14}$.

Now we construct this $\phi$. Put

$$\alpha(\tau) = \sum_{x, y \in \mathbb{Z}} q^{x^2+3xy+3y^2} \in A_1(\Gamma_0^{(1)}(3), \psi),$$

and

$$\beta(\tau) = \alpha(\tau)^3 - \frac{54\eta(3\tau)^9}{(\eta(\tau))^3} \in A_3(\Gamma_0^{(1)}(3), \psi),$$
where \( \psi_{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} = (\frac{3}{a}) \) is a character of \( \Gamma_0^{(1)}(3) \). Their Fourier coefficients are

\[
\alpha(\tau) = 1 + 6q + 6q^2 + \cdots, \\
\beta(\tau) = 1 - 6q - 54q^2 - 252q^3 + \cdots,
\]

\[
\left( \alpha \left| \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right. \right)(\tau) = -\frac{i}{\sqrt{3}} \alpha \left( \frac{\tau}{3} \right)
\]

\[
= -\frac{i}{\sqrt{3}} \left( 1 + 6q + 6q^2 + \cdots \right),
\]

\[
\left( \beta \left| \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right. \right)(\tau) = -\frac{i}{3\sqrt{3}} \beta \left( \frac{\tau}{3} \right)
\]

\[
= -\frac{i}{3\sqrt{3}} \left( 1 - 36q^2 - 54q^3 - 252q^4 + \cdots \right).
\]

We note that \( \alpha^3 - \beta \) has no zero on \( H_1 \) since it is expressed as an infinite product. Because \((\alpha^3 - \beta)\phi\) should be a weak Jacobi form of weight 3 and index 1 with character \( \chi \), \((\alpha^3 - \beta)\phi\) should satisfy the condition

\[
(\alpha(\tau)^3 - \beta(\tau))\phi(\tau, z) = (a\alpha(\tau)^5 + b\alpha(\tau)^2\beta(\tau))\phi_{-2,1}(\tau, z)
\]

\[
+ (c\alpha(\tau)^3 + d\beta(\tau))\phi_{0,1}(\tau, z)
\]

for some \( a, b, c, d \in \mathbb{R} \). Comparing its Fourier coefficients, now we find

\[
\phi(\tau, z) = \frac{54\alpha(\tau)^3}{\alpha(\tau)^3 - \beta(\tau)} \left( \frac{\varphi_{0,1}(\tau, z) - \alpha(\tau)^2\varphi_{-2,1}(\tau, z)}{12} \right)
\]

\[
+ \frac{15}{2} \alpha(\tau)^2\varphi_{-2,1}(\tau, z) - \frac{1}{2} \varphi_{0,1}(\tau, z).
\]

Since we have \( \alpha(\tau)^2 - 1 \in 12\mathbb{Z}[\eta][\xi] \), and \( \alpha(\tau)^3 - \beta(\tau) \in 54\mathbb{Z}[\eta][\xi] \), all the Fourier coefficients of \( \phi \) are integers by using \( \phi_{0,1} \equiv \phi_{-2,1} \mod 12 \). Also

\[
\left( \phi \left| \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right. \right)(\tau, z)
\]

\[
= \left( \frac{54\alpha(\tau)^3}{\alpha(\tau)^3 + \beta(\tau)} - 6 \right) \left( \frac{\varphi_{0,1}(\tau, z) - \alpha(\tau)^2\varphi_{-2,1}(\tau, z)}{12} \right)
\]

\[
+ 3 \left( \frac{\alpha(\tau)^3 - \beta(\tau)}{\alpha(\tau)^3 + \beta(\tau)} \right) \alpha(\tau)^2 \phi_{-2,1}(\tau, z),
\]

Hence all Fourier coefficients of \( \left( \phi \left| \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right. \right)(\tau, z) \) are integers.

**Conclusion.** Borcherds lift of

\[
\phi = \frac{54\alpha^3}{\alpha^3 - \beta} \left( \frac{\varphi_{0,1} - \alpha^2\varphi_{-2,1}}{12} \right) + \frac{15}{2} \alpha^2\varphi_{-2,1} - \frac{1}{2} \varphi_{0,1}
\]
is $\chi_{14}$. Namely
\[ \chi_{14} = q^3\zeta^{-1}p^2 \prod_{(n,l,m)>0} (1 - q^n\zeta^l p^m)^{e_1(4nm-l^2)} (1 - q^{3n}\zeta^{3l} p^{3m})^{e_2(4nm-l^2)}, \]
where the meaning of $(n,l,m) > 0$ is as before.

Thirdly, we consider Siegel modular forms of level 4. Cusps of $\Gamma_0^{(1)}(4)$ are represented by $g_1 = 1_2, g_2 = (1,0) \ 2 \ 1$ and $g_3 = (0,1) \ 4 \ 1_0$. Representatives of $\Delta_4(t)$ are given by
\[ \Gamma_0^{(1)}(4) \setminus \Delta_4(t) = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : ad = t, \ b = 0, \ldots, d - 1 \right\} \cup \left\{ g_2 \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : ad = t, \ a \in 2\mathbb{Z}, \ b = 0, \ldots, d - 1 \right\} \cup \left\{ g_3 \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : ad = t, \ a \in 4\mathbb{Z}, \ b = 0, \ldots, 4d - 1 \right\}. \]

Because
\[ \chi_{11} = \eta(\tau)^8 \eta(4\tau)^{16} \phi_{-1,2}p^2 + \cdots, \]
and $\dim S_{11}(\Gamma_0(4)) = 1$, if we can find a very weak Jacobi form $\phi(\tau, z) \in J_{0,1}^{(1)}(\Gamma_0(4))$ such that
\[ c_1(-4) = 1, \quad c_1(0) = 6, \quad c_1(n) = 0 \quad \text{if } n \leq 0, \quad n \neq 0, -4, \quad n \in \mathbb{Z}, \]
\[ c_2(n) = 0 \quad \text{if } n \leq 0, \quad n \in \mathbb{Z}, \]
\[ c_3(0,0) = 16, \quad c_3(n,l) = 0 \quad \text{if } n < 0, \quad n \in \mathbb{Z}, \]
its Borcherds lift should be $\chi_{11}$.

Now we construct this $\phi$. Define modular forms of $\Gamma_0^{(1)}(4)$ by
\[ \alpha(\tau) = \theta_{00}(2\tau)^4 \in A_2(\Gamma_0^{(1)}(4)), \]
\[ \beta(\tau) = \theta_{10}(2\tau)^4 \in A_2(\Gamma_0^{(1)}(4)). \]
Their Fourier coefficients are
\[ \alpha(\tau) = 1 + 8q + 24q^2 + \cdots, \]
\[ \beta(\tau) = 0 + 16q + 0q^2 + \cdots, \]
\[ \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right) \right) \right) (\tau) = -\frac{1}{4} \alpha(\tau/4) \]
\[ = -\frac{1}{4} - 2q^{1/2} - 6q^{3/2} + \cdots, \]
\[ \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right) \right) \right) (\tau) = \frac{1}{4} (\beta(\tau/4) - \alpha(\tau/4)) \]
\[ = -\frac{1}{4} + 2q^{1/2} - 6q^{3/2} + \cdots. \]
and
\[
\begin{pmatrix}
1 & 0 \\
2 & 1
\end{pmatrix}
\begin{pmatrix}
\alpha \\
\beta
\end{pmatrix}(\tau) = \beta(\tau),
\]
\[
\begin{pmatrix}
1 & 0 \\
2 & 1
\end{pmatrix}
\begin{pmatrix}
\beta \\
\alpha
\end{pmatrix}(\tau) = \alpha(\tau).
\]

We remark that \(16\eta(\tau)^{24} = \alpha\beta(\alpha - \beta)^4\). Hence \(\beta\) has no zero on \(H_1\). Because \(\beta\phi\) should be a weak Jacobi form of weight 2 and index 1, \(\beta\phi\) should satisfy the condition
\[
\beta(\tau)\phi(\tau, z) = (a\alpha(\tau)^2 + b\alpha(\tau)\beta(\tau) + c\beta(\tau)^2)\phi_{-2,1}(\tau, z)
+ (d\alpha(\tau) + e\beta(\tau))\phi_{0,1}(\tau, z)
\]
for some \(a, b, c, d, e \in \mathbb{R}\). Comparing its Fourier coefficients, now we find
\[
\phi(\tau, z) = \frac{4\alpha(\tau)}{\beta(\tau)} \left( \frac{\phi_{0,1}(\tau, z) - \alpha(\tau)\varphi_{-2,1}(\tau, z)}{3} \right)
+ \frac{20}{3}\alpha(\tau)\varphi_{-2,1}(\tau, z).
\]

Since \(\alpha(\tau) + \beta(\tau) - 1 \in 24\mathbb{Z}[q]^{\infty}\). It is shown as before that the Fourier coefficients of the following three forms are integers.
\[
\phi(\tau, z) = \frac{16\alpha(\tau)}{\beta(\tau)} \left( \frac{\phi_{0,1}(\tau, z) - (\alpha(\tau) + \beta(\tau))\varphi_{-2,1}(\tau, z)}{12} \right)
+ 8\alpha(\tau)\varphi_{-2,1}(\tau, z),
\]
\[
\frac{1}{2} \begin{pmatrix}
1 & 0 \\
2 & 1
\end{pmatrix}
\begin{pmatrix}
\tau \\
\zeta
\end{pmatrix}(\tau, z) = \frac{8\beta(\tau)}{\alpha(\tau)} \left( \frac{\phi_{0,1}(\tau, z) - (\alpha(\tau) + \beta(\tau))\varphi_{-2,1}(\tau, z)}{12} \right)
+ 4\beta(\tau)\varphi_{-2,1}(\tau, z),
\]
\[
\begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix}
\begin{pmatrix}
\tau \\
\zeta
\end{pmatrix}(\tau, z) = \frac{16\alpha(\frac{\tau}{4})}{\alpha(\frac{\tau}{4}) - \beta(\frac{\tau}{4})} \left( \frac{\phi_{0,1}(\tau, z) - (\alpha(\frac{\tau}{4}) + \beta(\frac{\tau}{4}))\varphi_{-2,1}(\tau, z)}{12} \right)
+ 3 \left( \frac{\alpha(\frac{\tau}{4}) - \beta(\frac{\tau}{4})}{\alpha(\frac{\tau}{4}) - \beta(\frac{\tau}{4})} \right)\varphi_{-2,1}(\tau, z).
\]

**Conclusion.** Borcherds lift of
\[
\phi = \frac{4\alpha}{\beta} \left( \frac{\varphi_{0,1} - \alpha\varphi_{-2,1}}{3} \right) + \frac{20}{3}\alpha\varphi_{-2,1}
\]
is \(\chi_{11}\). Namely, we have
\[
\chi_{11} = q^{3}\zeta^{-1}p^{2} \prod_{(n,m) > 0} (1 - q^{n}\zeta^{2}p^{m})^{c_{1}(4nm - \ell^{2})}
\times (1 - q^{2n}\zeta^{2}p^{2m})^{c_{2}(4nm - \ell^{2})/2} (1 - q^{4n}\zeta^{4}p^{4m})^{c_{3}(4nm - \ell^{2})}.
\]
Appendix A. Euler Factors

The simple expression of \( \chi_{35} \) makes it easy to calculate the Fourier coefficients of Siegel modular forms of odd weights of \( Sp(2, \mathbb{Z}) \). So, we give here some examples of Euler 2 factors of spinor \( L \) functions with some other eigen values.

The Euler \( p \)-factor of the spinor \( L \)-function is given by

\[
(1 - \lambda(p)p^{-s} + (\lambda(p)^2 - \lambda(p^2) - p^{2k-4})p^{-s} - \lambda(p)p^{2k-3-3s} + p^{4k-6-4s})^{-1}
\]

where \( \lambda(p^a) \) is the eigenvalue of the Hecke operator \( T(p^a) \) (cf. Andrianov [1, p. 88]). We define the Hecke polynomial of a Siegel modular form \( F \) at \( p \) by

\[
H_p(u, F) = u^4 - \lambda(p)u^3 + (\lambda(p)^2 - \lambda(p^2) - p^{2k-4})u^2 - \lambda(p)p^{2k-3}u + p^{4k-6}.
\]

If we denote \( a(t_1, t_2, t) \) the Fourier coefficient of Siegel modular form \( F \) of odd weight \( k \) at \( \left( \frac{t_1}{t_2} \right) \) and \( a(t_1, t_2, t; p^a) \) the coefficient of \( T(p^a)F \) at the same matrix, we have

\[
\begin{align*}
   a(2, 3, 1; 2) &= a(4, 6, 2) + 2^{k-2}(-a(2, 3, 1) + a(1, 6, 1)), \\
   a(2, 4, 1; 2) &= a(4, 8, 2) + 2^{k-2}(-a(2, 4, 1) + a(1, 8, 1)), \\
   a(2, 3, 1; 4) &= a(8, 12, 4) + 2^{k-2}(-a(4, 6, 2) + a(2, 12, 2)) \\
   &- 2^{2k-4}(a(2, 3, 1) + a(1, 6, 1)), \\
   a(2, 4, 1; 4) &= a(8, 16, 4) + 2^{k-2}(a(2, 16, 2) - a(4, 8, 2)) \\
   &- 2^{2k-4}(a(1, 8, 1) + a(2, 4, 1)), \\
   a(2, 3, 1; 3) &= a(6, 9, 3) - 3^{k-2}(a(2, 3, 1) + a(1, 6, 1)), \\
   a(2, 4, 1; 3) &= a(6, 12, 3), \\
   a(2, 3, 1; 5) &= a(10, 15, 5), \\
   a(2, 4, 1; 5) &= a(10, 20, 5) - 5^{k-2}a(2, 4, 1), \\
   a(3, 4, 2; 2) &= a(6, 8, 4) - 2^{k-2}a(2, 6, 2).
\end{align*}
\]

By using these relations, we can calculate Hecke polynomials at 2. We put \( \chi_{39} = \phi_4 \chi_{35}, \chi_{41} = \chi_{35} \phi_6, \chi_{43} = \chi_{35} \phi_4^2, \chi_{45,e} = \chi_{35} \phi_4 \phi_6, \chi_{45,c} = \chi_{35} \chi_{10}. \)

Since we get

\[
\begin{align*}
   T(2)\chi_{45,e} &= -10766446755840\chi_{45,e} + 6671813516187402240\chi_{45,c}, \\
   T(2)\chi_{45,c} &= 2052096\chi_{45,e} - 11638111469568\chi_{45,c}.
\end{align*}
\]

We get eigenforms

\[
\chi_{45, \pm} = \chi_{45,e} + (-212384 \pm 32\sqrt{3219068329})\chi_{45,c}.
\]

The eigenvalues and Hecke polynomials \( H_2(u, F) \) at two for the above \( F \) are given as follows.

\[
\begin{align*}
   \lambda(2, \chi_{35}) &= -25073418240 = -2^{17} \cdot 3^3 \cdot 5 \cdot 13 \cdot 109, \\
   \lambda(4, \chi_{35}) &= 138590166352717152256 = 2^{34} \cdot 313 \cdot 25773193,
\end{align*}
\]
\begin{align*}
\lambda(2, \chi_{39}) &= -283917680640 = -2^{20} \cdot 3^2 \cdot 5 \cdot 11 \cdot 547, \\
\lambda(4, \chi_{39}) &= -3619320962049810366464 = -2^{40} \cdot 109 \cdot 1249 \cdot 24179, \\
\lambda(2, \chi_{41}) &= -1478110740480 = -2^{18} \cdot 3^3 \cdot 5 \cdot 11 \cdot 3797, \\
\lambda(4, \chi_{41}) &= 205535162772980871725056 = 2^{36} \cdot 4127 \cdot 72472263, \\
\lambda(2, \chi_{43}) &= -4069732515840 = -2^{17} \cdot 3^3 \cdot 5 \cdot 127 \cdot 1181, \\
\lambda(4, \chi_{43}) &= -70252138637293975936384, \\
\lambda(2, \chi_{45, \pm}) &= -2^{34} \cdot 137 \cdot 461 \cdot 32587 \cdot 198689, \\
\lambda(4, \chi_{45, \pm}) &= 2^{45} (-3563629466346751 \mp 3^2 \cdot 1249236899 \sqrt{3219068329}),
\end{align*}

Here we used the following Fourier coefficients.

\[ H_2(\chi_{35}, u) = u^4 + 25073418240u^3 + 41629915959040938880u^2 \\
+ 267 \cdot 25073418240u + 2^{134}, \]

\[ = (u^3 + 196608(63765 + \sqrt{931783609})u + 2^{67}) \times (u^2 + 196608(63765 - \sqrt{931783609})u + 2^{67}), \]

\[ H_2(\chi_{39}, u) = u^4 + 283917680640u^3 + 65339104410568260321280u^2 \\
+ 283917680640 \cdot 2^{75}u + 2^{150}, \]

\[ = (u^2 + (14958840320 + 1572864\sqrt{12276590561})u + 2^{75}) \times (u^2 + (14958840320 - 1572864\sqrt{12276590561})u + 2^{75}), \]

\[ H_2(\chi_{41}, u) = u^4 + 1478110740480u^3 + 167704474344569574522880u^2 \\
+ 1478110740480 \cdot 2^{70}u + 2^{158}, \]

\[ = (u^2 - (-739055370240 + 393216\sqrt{505009125721})u + 2^{79}) \times (u^2 - (-739055370240 - 393216\sqrt{505009125721})u + 2^{79}), \]

\[ H_2(\chi_{43}, u) = u^4 + 4069732515840u^3 + 18752233335756256738017280u^2 \\
+ 4069732515840 \cdot 2^{83}u + 2^{166}, \]

\[ = (u^2 - (-20348666257920 + 196608\sqrt{122398046613649})u + 2^{83}) \times (u^2 - (-20348666257920 - 196608\sqrt{122398046613649})u + 2^{83}), \]

\[ H_2(\chi_{45, \pm}, u) = u^4 - 2^{17} \cdot 3(-67 \cdot 425207 \pm 167 \sqrt{3219068329})u^3 \\
+ 2^{52} \cdot 13(31 \cdot 37 \cdot 2746651 \mp 3^2 \cdot 29 \cdot 71\sqrt{3219068329})u^2 \\
- 2^{104} \cdot 3(-67 \cdot 425207 \pm 167 \sqrt{3219068329})u + 2^{174}. \]
\begin{align*}
\text{det}(2T) & \quad \text{matrix} & \quad \chi_{35} & \quad \chi_{39} \\
23 & (1, 6, 1) & 0 & 0 \\
31 & (1, 8, 1) & 0 & 0 \\
23 & (2, 3, 1) & 1 & 1 \\
31 & (2, 4, 1) & -69 & 171 \\
92 & (2, 12, 2) & 0 & 0 \\
124 & (2, 16, 2) & 0 & 0 \\
92 & (4, 6, 2) & -16483483648 & -14647827168 \\
124 & (4, 8, 2) & 1137360371712 & -25047862345728 \\
207 & (6, 9, 3) & -6265491005023317 & -82431566868072467 \\
279 & (6, 12, 3) & 81589405848930960 & -21795652752174648840 \\
368 & (8, 12, 4) & 70785606262933807104 & -486137399845176383476 \\
496 & (8, 16, 4) & 4884171642142432690176 & -83135719773525161646496 \\
775 & (10, 15, 5) & 9470081642319930937500 & 8882040686918464920237500 \\
775 & (10, 20, 5) & -8686092830378577431953125 & 13906716477716715796158046875 \\
\end{align*}
Some more eigenvalues at other primes are given by
\[
\begin{align*}
\lambda(3, \chi_{35}) &= -112824551571578840 = -2^3 \cdot 3^{11} \cdot 5 \cdot 4817 \cdot 346429, \\
\lambda(5, \chi_{35}) &= 9470081642319930937500 = 2^2 \cdot 3^3 \cdot 5^7 \cdot 13 \cdot 23 \cdot 41 \cdot 137 \cdot 66829019, \\
\lambda(3, \chi_{39}) &= -12745957457172040 = 2^3 \cdot 3^{12} \cdot 5 \cdot 5995959921, \\
\lambda(5, \chi_{39}) &= 8882040686918464920937500 = 2^2 \cdot 3^2 \cdot 5^7 \cdot 11 \cdot 47 \cdot 61 \cdot 73 \cdot 46141 \cdot 29729663, \\
\lambda(3, \chi_{41}) &= -1283634468791983080 = -2^3 \cdot 3^{12} \cdot 5 \cdot 122321 \cdot 493657, \\
\lambda(5, \chi_{41}) &= -5831121557266319789062500 = 2^2 \cdot 3^3 \cdot 5^9 \cdot 11 \cdot 20413117 \cdot 123110838817, \\
\lambda(3, \chi_{43}) &= -65782425978552959640 = -2^3 \cdot 3^{14} \cdot 5 \cdot 343836777839, \\
\lambda(5, \chi_{43}) &= -448901104534445302863489062500 = 2^2 \cdot 3^3 \cdot 5^8 \cdot 41 \cdot 773 \cdot 17657 \cdot 1901457470879, \\
\lambda(3, \chi_{45}, \pm) &= -940124365399227162888 \mp 6607391123670336 \sqrt{3219068329} \\
&= -2^3 \cdot 3^{15} \cdot 1552657 \cdot 5274739 \pm 2^4 \cdot 31^2 \cdot 7487 \sqrt{3219068329}, \\
\lambda(5, \chi_{45}, \pm) &= -915248281886926157933039062500 \\
&= 2^6 \cdot 3^9 \cdot 13(-11 \cdot 83 \cdot 3290133403025431747) \\
&\pm 2^6 \cdot 29^2 \cdot 751 \cdot 3733 \cdot 511939 \sqrt{3219068329},
\end{align*}
\]

Correction of the paper [12]

(1) p. 20, in the definition of \( f_6 \), the coefficients of \( \theta_2 \theta_4 \) should be \(-2271/3328\) instead of \( 2271/3328 \).

(2) p. 21, in \( v_6 \) and \( w_6 \), the coefficients of \( \theta_4 \theta_6 \) is \(-3321/4\) instead of \( 3321/4 \).

(3) p. 24, the coefficient of \( g_2 \chi_6 \) is \(-1728\) instead of \(-1729\).

(4) p. 27, \( \uparrow 8 \), the right-hand side of \( W(E_4 - 81E'_4) - 100 \) instead of \( 100 \) and +10 instead of –10.

(5) p. 29, l.17, the coefficient of \( \theta_2 \theta_6 \) should be 110592 instead of 432 and that of \( \theta_2 \theta_6 \) is \(-124416\) instead of \( 124416 \).

(6) p. 29, \( \uparrow 4 \), \( E_4 + E'_4 \) should be replaced by \( E_4 + 81E'_4 \). The coefficient of \( \theta_2 f_6 \) is 172800 instead of 675/256.

(7) p. 30, in Eq. (1), the coefficient of \( (E_4 + 81E'_4)f_6 \) is \(-108\) instead of 180 and that of \( \theta_2^2 f_6 \) is \(-20520\) instead of \(-34200\).

References

Graded Rings of Siegel Modular Forms, Differential Operators and Borcherds Products 279