On representations of finite groups in the space of Siegel modular forms and theta series

Dedicated to Professor Masayoshi Nagata
on his sixtieth birthday

By

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Let $p$ be a prime. A representation $\pi_1^{(m)}$ of the symplectic group $Sp(m)$ over the finite field $\mathbb{Z}/p\mathbb{Z}$ is realized in the space of Siegel modular forms of genus $m$, of level $p$, and of weight $k$. When $m=1$, Hecke discovered that the difference of multiplicities of two specific irreducible representations in $\pi_1^{(m)}$ is equal to the class number of $\mathbb{Q}(\sqrt{-p})$, if $p \equiv 3 \mod 4$, $p > 3$, $k \geq 2$; he also found a beautiful explanation of this fact by the special modular forms, called "eingliedrig forms", which correspond to $L$-functions with Grössencharacters of $\mathbb{Q}(\sqrt{-p})$ by the Mellin transformation. The basic philosophy suggested by this classical work is, in its raw form, that the existence of special (or "lifted") modular forms would produce a difference of multiplicities of certain representations in $\pi_1^{(m)}$.

This paper, in essence, is a document on experiments which are made to examine this picture in the case of Siegel modular forms of genus 2 and of level $p$. Main results obtained through the course of investigations are Theorems 2.6, 2.8 and 3.2, and examples in § 4.

We shall explain the contents of each section. In § 1, Hecke's work quoted above shall be briefly reviewed. In § 2, we shall first generalize Hecke's notion "eingliedrig" and "zweigliedrig" to Siegel modular forms of genus 2; for certain representations $\theta_9$ and $\theta_{11}$ of $Sp(2)$ over $\mathbb{Z}/p\mathbb{Z}$, we shall introduce the notion of $\theta_9$ and $\theta_{11}$-eingliedrig forms. Then we shall prove the relation between the difference of multiplicities of $\theta_9$ and $\theta_{11}$ in $\pi_1^{(p)}$ and the existence of eingliedrig forms (Theorem 2.8). We shall define the Hecke operator $T(p)$ for the level $p$ of modular forms and determine the absolute value of its eigenvalues (Theorem 2.6); the eigenvalues of $T(p)$ are not real in general and the real eigenvalues correspond exactly to eingliedrig forms. This phenomenon is similar to Hecke's "Nebentypus" case, although the corresponding statement for the classical case does not seem to be rigorously proved.

In § 3, we shall prove that Siegel modular forms constructed from a pair of elliptic modular forms in our previous paper [15] are $\theta_{11}$-eingliedrig in the case

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of prime level. In §4, we shall decompose $S_4(Γ^{(3)}(p))$ (see §0, for the notation) into eigen spaces of Hecke operators for $p=3$, $k=2, 4, 6, 8$, $p=11$, $k=4$, and $p=7$, $k=6$. Though the theta series studied in §3 are “special”, we can construct a major part of $S_4(Γ^{(3)}(p))$ by taking products of them of lower weights; and the theory developed in §2 can be applied efficiently to the explicit decomposition. We have calculated eigenvalues of $T(p)$ in these cases, which may have interesting arithmetical meanings as in the classical case.

In §5, we shall formulate Conjectures about eingliedrig forms suggested by these examples. The $θ_{11}$-eingliedrig forms would be precisely those constructed in §3 (Conjecture 5.1). For $θ_r$-eingliedrig forms, however, some complication shall arise. To clarify the points, we shall classify irreducible representations of a certain Hecke algebra in Appendix, and formulate a plausible Conjecture also for $θ_r$-eingliedrig forms (Conjecture 5.2). Roughly speaking, these Conjectures predict that the global nature of automorphic representations is strongly controlled if their local properties at a place, say $p$, are of special type.

Notation

Let $R$ be a commutative ring. By $M(n, R)$, we denote the associative algebra of all $n×n$ matrices with entries in $R$. For $A∈M(n, R)$, $σ(a)$ denotes the trace of $A$. Put $u=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}∈M(2m, R)$ and $GSp(m, R) = \{ g∈GL(2m, R) | gw = m(g)w \text{ with } m(g)∈R^* \}$.

If $G$ is an algebraic group defined over a global field $k$, $G_A$ denotes the adelicization of $G$, and $G_K$ denotes the group of all $K$-rational points of $G$ for an extension $K$ of $k$. For $z∈C$, we set $e(x)=\text{exp}(2πi z)$ and $\bar{z}$ denotes the complex conjugate of $z$.

Let $G$ be a group (or an associative algebra) and $π$ be a representation of $G$ on a vector space $V$. Let $v∈V$ and $V_1$ be the smallest invariant subspace of $V$ which contains $v$. The representation of $G$ on $V_1$ is called the representation of $G$ generated by $v$. If $G$ is a locally compact group, $G$ denotes the character group of $G$, and $S(G)$ denotes the space of all Schwartz-Bruhat functions on $G$.

§0. Preliminaries

For a positive integer $m$, let $S_m$ denote the Siegel upper half space of genus $m$. Set

$$GS{p^*}(m, R) = \{ g∈GSp(m, R) | m(g) > 0 \},$$

$$GS{p^*}(m, Q) = GSp(m, Q) ∩ GSp^*(m, R);$$

$GS{p^*}(m, R)$ acts on $S_m$ in the usual manner. Let $k$ be an integer. For a function $f$ on $S_m$ and $γ=\begin{pmatrix} a & b \\ c & d \end{pmatrix}∈GSp^*(m, R)$, we set
Let $\Gamma$ be a congruence subgroup of $Sp(m, Z)$. By $G_k(\Gamma)$ (resp. $S_k(\Gamma)$), we denote the space of all holomorphic modular (resp. cusp) forms of weight $k$ with respect to $\Gamma$. For $f \in G_k(\Gamma)$ and $g \in S_k(\Gamma)$, we set

$$
(0.1) \quad \langle f, g \rangle = \frac{1}{\text{vol}(\mathfrak{H}\setminus \mathfrak{H}_m)} \int_{\mathfrak{H}\setminus \mathfrak{H}_m} f(z) \overline{g(z)} (\det y)^k \, dv(z),
$$

where $z = x + \sqrt{-1} y$ with $x, y \in M(m, R)$ and $dv(z)$ denotes the invariant volume element on $\mathfrak{H}_m$ given by $dv(z) = (\det y)^{-m-1} \, dx dy$. Put $\mathfrak{H}_* = \bigcup S_k(\Gamma)$ where $\Gamma$ extends over all congruence subgroups. Then, for $f, g \in \mathfrak{H}_*$, we can define $(f, g)$ by $(0.1)$ since it does not depend on the choice of $\Gamma$; $(\cdot, \cdot)$ is a positive hermitian inner product on $\mathfrak{H}_*$.

**Lemma 0.1.** If $f, g \in \mathfrak{H}_*$ and $\gamma \in GSp^+(m, Q)$, then $(f \mid \gamma \cdot \gamma, g) = (f, g \mid \gamma^{-1} \gamma)$.

This Lemma claims that the operator $f \mapsto f \mid \gamma \cdot \gamma$ is unitary, which is trivial in adelicized definition of cusp forms. The direct proof is also easy, so it is omitted.

Let $N$ be a positive integer. We set

$$
\Gamma^{(m)}(N) = \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp(m, Z) \mid a \equiv d \equiv 1 \mod N, \, b \equiv c \equiv 0 \mod N \right\},
$$

$$
\Gamma^{(m)}_*(N) = \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp(m, Z) \mid c \equiv 0 \mod N \right\}.
$$

As $\begin{pmatrix} 0 & 1 \\ -N & 0 \end{pmatrix} \in GSp^+(m, Q)$ normalizes $\Gamma^{(m)}_*(N)$, we can decompose $S_k(\Gamma^{(m)}_*(N))$ (and $G_k(\Gamma^{(m)}_*(N))$):

$$
(0.2) \quad S_k(\Gamma^{(m)}_*(N)) = S_k(\Gamma^{(m)}(N)) \oplus S_k(\Gamma^{(m)}_*(N)),
$$

where $S_k(\Gamma^{(m)}_*(N)) = \{ f \in S_k(\Gamma^{(m)}_*(N)) \mid f \mid \begin{pmatrix} 0 & 1 \\ -N & 0 \end{pmatrix}_k = \pm f \}$. If $\chi$ is a Dirichlet character modulo $N$, we set

$$
S_k(\Gamma^{(m)}(N), \chi) = \left\{ f \in S_k(\Gamma^{(m)}(N)) \mid f \mid \gamma \chi = \chi(\det a) f \quad \text{for any} \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma^{(m)}(N) \right\}.
$$

Since $\Gamma^{(m)}(N)$ is normal in $Sp(m, Z)$, we get a representation $\pi^{(m)}_\chi(N)$ of $Sp(m, Z/NZ) \cong Sp(m, Z)/\Gamma^{(m)}(N)$ on $S_k(\Gamma^{(m)}(N))$ defined by

$$
([\pi^{(m)}_\chi(N)(\gamma \mod N)]f = f \mid \gamma \cdot \gamma, \ f \in S_k(\Gamma^{(m)}(N)), \ \gamma \in Sp(m, Z)).
$$

Let $G$ be a finite group and $B$ be a subgroup of $G$, and let $\mathcal{H}(G, B)$ denote the Hecke algebra of $G$ with respect to $B$ over $C$. Let $C_i$ (resp. $C_{G_i}$) denote the category of the equivalence classes of all finite dimensional representations of $\mathcal{H}(G, B)$ over $C$ (resp. $G$ over $C$ with non-trivial vectors fixed under $B$).
Lemma 0.2. The functor, \((\pi, V) \mapsto \text{the representation of } \mathcal{A}(G, B) \text{ on } V^B\), is the equivalence of categories \(\mathcal{C}_2\) and \(\mathcal{C}_1\), where \((\pi, V) \in \mathcal{C}_2\) and \(V^B\) denotes the subspace of all \(B\)-fixed vectors of \(V\).

We shall also use the following Lemma.

Lemma 0.3. Let \(G\) and \(B\) be as above. Let \(\pi\) be a representation of \(G\) on a finite dimensional vector space \(V\) over a field \(k\). Let \(W\) be a subspace of \(V\) which is invariant under \(B\), and let \(\sigma\) be the representation of \(B\) realized on \(W\). If \(V\) is generated by \(W\) as a \(G\)-space, then \(\pi\) is a quotient representation of \(\text{Ind}_B^G \sigma\).

Proof. The map \(\varphi : k[G] \otimes_{k[H]} W \rightarrow V\) defined by
\[
\varphi(\sum g_i \otimes w_i) = \sum \pi(g_i)w_i, \quad g_i \in G, \quad w_i \in W,
\]
is a homomorphism of \(k[G]\)-modules. By the assumption, \(\varphi\) is surjective; hence the assertion follows.

§ 1. A review of a theory of Hecke

Let \(p\) be an odd prime. We set
\[
G = \text{SL}(2, F_p), \quad B = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \right\} | a \in F_p^\times, \quad b \in F_p \}, \quad U = \left\{ \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \right\} | u \in F_p \}.
\]
Let \(\chi\) be the quadratic residue character of \(F_p^\times\). We define \(\chi \in B\) by \(\chi \left( \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \right) = \chi(a)\). Let \(\phi\) be a non-trivial additive character of \(F_p\). For \(a \in F_p\), define \(\phi_a \in \hat{U}\) by \(\phi_a \left( \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \right) = \phi(au)\). We see easily that
\[
(1.1) \quad \text{Ind}_B^G \chi \cong \mathcal{G}_{p+1/2} \oplus \mathcal{G}_{p+1/2},
\]
where \(\mathcal{G}_{p+1/2}\) and \(\mathcal{G}_{p+1/2}\) denote irreducible representations of \(G\) of degree \((p+1)/2\), which are not equivalent to each other and satisfy

\[
(1.2) \quad \mathcal{G}_{p+1/2} \mid U \equiv \phi \oplus \bigoplus_{a \in (F_p^*)^2} \phi_a
\]
\[
(1.3) \quad \mathcal{G}_{p+1/2}' \mid U \equiv \phi \oplus \bigoplus_{a \in (F_p^*)^2} \phi_a
\]
We normalize \(\phi\) by
\[
(1.4) \quad \phi(x \mod p) = e(x/p), \quad x \in \mathbb{Z}.
\]
Then the representation \(\mathcal{G}_{p+1/2}\) (resp. \(\mathcal{G}_{p+1/2}'\)) given by (1.2) (resp. (1.3)) is called Rest (resp. Nicht Rest) in Hecke's terminology.
The representation $\pi^{(1)}(p)$ of $G$ is realized on $S_k(\Gamma^{(1)}(p))$. For an irreducible representation $\rho$ of $G$, let $m_\rho(\rho)$ denote the multiplicity of $\rho$ in $\pi^{(1)}(p)$. Hecke obtained the formula

\[
m_\rho(\mathcal{G}_{p+1/2}) - m_\rho(\mathcal{G}_{p+1/2}') = \begin{cases} h(Q(\sqrt{-p})) & \text{if } k > 2 \text{ is odd, and } p \equiv 3 \pmod{4}, \\ 0 & \text{if } k \text{ is even, or } p \equiv 1 \pmod{4}, \end{cases}
\]

by the Riemann–Roch theorem, where $h(Q(\sqrt{-p}))$ denotes the class number of $Q(\sqrt{-p})$ and $p > 3$ is assumed. Hereafter in this section, we shall abbreviate $\Gamma^{(1)}(p)$ to $\Gamma_0(p)$. Let $f \neq 0 \in S_k(\Gamma_0(p), \left(\frac{-1}{p}\right))$ and $\rho_f$ be the representation of $G$ generated by $f$. By Lemma 0.3, $\rho_f$ is a subrepresentation of $\text{Ind}_{G}^{\mathbb{Q}}$. The key points of Hecke’s theory are the following Proposition and Theorem.

**Proposition 1.1.** Assume $f \in S_k(\Gamma_0(p), \left(\frac{-1}{p}\right))$ is a non-zero common eigenfunction of all Hecke operators $T(n)$ for $p \nmid n$. Then $\rho_f \cong \mathcal{G}_{p+1/2}$ or $\mathcal{G}_{p+1/2}' \oplus \mathcal{G}_{p+1/2}'$.

This Proposition states that $\rho_f \cong \mathcal{G}_{p+1/2}'$ cannot occur. For the proof see Satz 26, [6], p. 842.

Hecke called a normalized eigen cusp form $f$ eingliedrig (resp. zweiseigliedrig) if $\rho_f \cong \mathcal{G}_{p+1/2}$ (resp. $\mathcal{G}_{p+1/2} \oplus \mathcal{G}_{p+1/2}'$) ([6], p. 841).

**Theorem 1.2.** $m_k(\mathcal{G}_{p+1/2}) - m_k(\mathcal{G}_{p+1/2}')$ is equal to the number of eingliedrig forms in $S_k(\Gamma_0(p), \left(\frac{-1}{p}\right))$.

For the proof, see [6], p. 841~843.

We can construct eingliedrig forms from a Grössencharacter $\chi$ of $Q(\sqrt{-p})$. In fact, if $\chi$ is a Grössencharacter of $K = Q(\sqrt{-p})$ of conductor 1 such that

\[
\chi((\alpha)) = \alpha^{k-1}, \quad \alpha \in K^*, \quad k > 1,
\]

and if $p \equiv 3 \pmod{4}$, then

\[
f(z) = \sum_a \chi(a)e(N(a)z) \in S_k(\Gamma_0(p), \left(\frac{-1}{p}\right))
\]

is an eingliedrig form, where $a$ extends over all integral ideals of $K$ and $N(a)$ denotes the norm of $a$ (cf. [6], p. 893, G. Shimura [11], [12]); we obtain $h(Q(\sqrt{-p}))$ eingliedrig forms in this way if $k > 1$, $3 < p \equiv 3 \pmod{4}$. This “explains” the formula (1.5) (and characterizes eingliedrig forms). Hecke further showed that all the eigenvalues of $T(p)$ on $S_k(\Gamma_0(p), \left(\frac{-1}{p}\right))$ are of absolute value $p^{(k-1)/2}$. We shall obtain a generalization of this theorem to Siegel modular forms of genus 2.
§ 2. Representations of finite groups in the space of Siegel modular forms of genus 2

We set \( G = \text{Sp}(2, F_p) \) and define subgroups \( B, P, P' \) of \( G \) as follows.

\[
B = \begin{pmatrix} * & 0 & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix}, \quad P = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix}, \quad P' = \begin{pmatrix} * & 0 & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix}.
\]

\( B \) is a Borel subgroup of \( G \); \( P \) and \( P' \) are all proper parabolic subgroups which contain \( B \). We have

**Lemma 2.1.** \( \text{Ind}_{\mathcal{P}}^G(1_p) \cong 1_p \otimes \theta_3 \otimes \theta_3 \otimes \theta_3 \), \( \text{Ind}_{\mathcal{P}}^G(1_p) \cong 1_p \otimes \theta_3 \otimes \theta_3 \). Here \( H \) denotes the trivial representation of \( H \) for a subgroup \( H \) of \( G \); \( \theta_3, \theta_1, \theta_13, \) and \( \theta_3 \) denote mutually non-equivalent irreducible representations of \( G \) which are labelled according to \( B \). Srinivasan [13] when \( p \neq 2 \).

The structure of the Hecke ring \( \mathcal{S}(G, B) \) is given as follows (cf. [7]). Put

\[
\begin{align*}
(2.1) \quad w_1 &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \\
& w_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \\
& w = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \\
& S_1 = Bw_1B, S_2 = Bw_2B. \end{align*}
\]

Then \( S_1 \) and \( S_2 \) satisfy the relations

\[
(2.2) \quad \begin{cases} S_1^i = (p-1)S_i + p, & i=1, 2, \\
(S_1S_2)^p = (S_2S_1)^p. \end{cases}
\]

We have \( \mathcal{S}(G, B) \cong \mathbb{C}[S_1, S_2] \), the associative algebra generated by \( S_1 \) and \( S_2 \) over \( \mathbb{C} \) with relations (2.2). The (one dimensional) representations of \( \mathcal{S}(G, B) \) which correspond to \( 1_G, \theta_1, \theta_12, \) and \( \theta_13 \) by Lemma 0.2 are given as follows.

\[
(2.3) \quad \begin{array}{cccc} 1_G & \theta_1 & \theta_12 & \theta_13 \\ S_1 & p & p & -1 \\ S_2 & p & -1 & p \end{array}
\]

Corresponding to \( \theta_3 \), we obtain the two dimensional representation \( S_1 \rightarrow \begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix} \), \( S_2 \rightarrow \begin{pmatrix} \gamma & 0 \\ p^{-1} & 1 \end{pmatrix} \), where we may set

\[
(2.4) \quad \alpha = \frac{p-1}{p+1}, \quad \beta = \frac{1}{p+1}, \quad \gamma = \frac{2p(p+1)}{p+1}, \quad \delta = \frac{p(p-1)}{p+1}.
\]

Similarly the representations which occur in \( \text{Ind}_{\mathcal{P}}^G(1_p) \) are classified as follows: \( \mathcal{S}(G, P) \) is generated by three Hecke operators \( P, PwP, Pw_sP \). The eigenvalues of these Hecke operators which correspond to \( 1_G, \theta_3, \) and \( \theta_11 \) are given in the following table.
Let \( \phi : \text{Sp}(2, \mathbb{Z}) \rightarrow G \) be the canonical homomorphism. For a subgroup \( H \) of \( G \), set \( \Gamma_H = \phi^{-1}(H) \). Thus we have \( \Gamma_F = \Gamma_{\phi^3}(p) \). For an irreducible representation \( \rho \) of \( G \), let \( m_k(\rho) \) denote the multiplicity with which \( \rho \) occurs in \( \pi_{\phi^3}(p) \).

T. Yamazaki, R. Tsushima and K. Hashimoto [5] have obtained the following formula:

\[
\begin{align*}
\text{(2.6)} & \quad m_k(\theta_\psi) + m_k(\theta_{10}) - m_k(\theta_{11}) - m_k(\theta_{12}) \\
& = (-1)^{k+1} h(Q(\sqrt{-p}))^2 \times \begin{cases} 
1/4 & \text{if } p \equiv 1, 5 \\
4 & \text{if } p \equiv 3 \pmod{8}, \\
1 & \text{if } p \equiv 7
\end{cases}
\end{align*}
\]

for \( k \geq 4, \ p \geq 5 \). In (2.6), a unipotent cuspidal representation \( \theta_{10} \) of \( G \) appears which is of completely different nature from \( \theta_\psi \), \( \theta_{11} \) and \( \theta_{12} \) representation theoretically; thus (2.6) may not be “explained” as in Hecke’s theory. However, if we consider \( m_k(\theta_\psi) - m_k(\theta_{11}) \), we can develop analogous theory to Hecke’s.

For \( F \in \text{Sp}(2, \mathbb{Z}) \), let \( \rho_F \) denote the representation of \( G \) generated by \( F \). Three Hecke operators \( \Gamma_F, W = \Gamma_p w \Gamma_F, W_z = \Gamma_p w_z \Gamma_F \) act on \( S_k(\Gamma_F) \), where \( w, w_z \in \text{Sp}(2, \mathbb{Z}) \) are given by (2.1).

**Lemma 2.2.** Let \( F \in \text{Sp}(2, \mathbb{Z}), \ F \neq 0 \). Then

1. \( \rho_F \) is a subrepresentation of \( \text{Ind}(\phi(1_p)) \cong 1_\sigma \oplus \theta_\psi \oplus \theta_{11} \).
2. \( \rho_F \cong 1_\sigma \Leftarrow F|W = p^k F, F|W_z = (p^2 + p)F \equiv F|W = p^k F \).
   \( \rho_F \cong \theta_\psi \Leftarrow F|W = -p F, F|W_z = (p - 1)F \equiv F|W = -p F \).
   \( \rho_F \cong \theta_{11} \Leftarrow F|W = p F, F|W_z = -p F \equiv F|W = p F \).

**Proof.** (1) follows from Lemma 0.3, and (2) follows from the table (2.5).

Put \( H = \left( \begin{array}{cc} 0 & 1 \\ -p & 0 \end{array} \right) \).

**Lemma 2.3.** With respect to the Petersson inner product \( ( \cdot , \cdot ) \), \( H, W \) and \( W_z \) are self-adjoint operators acting on \( S_k(\Gamma_F) \).

**Proof.** Let \( F, G \in S_k(\Gamma_F) \). By Lemma 0.1, we have

\[
(F| H, G) = (F \left| \begin{array}{c} 0 \\ -p \end{array} \right\rangle \right),
(G|H) = (F \left| \begin{array}{c} 0 \\ -p^{-1} \end{array} \right\rangle \right)
\]

\[
= (F, G) \left| \begin{array}{c} 0 \\ -p \end{array} \right\rangle \right) = (F, G|H)
\]
\((F|W, G) = (\sum_{u} F \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}, G)\)
\[
= (F \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}, \sum_{u} G \begin{pmatrix} 1 & -u \\ 0 & 1 \end{pmatrix}) = p^G(F \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}, G),
\]
where \(\sum_{u}\) denotes the summation over the equivalence classes modulo \(p\) of \(u \in M(2, \mathbb{Z})\), \(t^u = u\). Similarly we get
\[
(F, G|W) = p^G(F, G \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}) = p^G(F \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}, G).
\]
Hence \(H\) and \(W\) are self-adjoint. We omit the proof for \(W\), which is similar to the above.

**Lemma 2.4.** Let \(V\) be a finite dimensional vector space over \(\mathbb{C}\) and \((,\)\) be a positive hermitian inner product on \(V\). Let \(A\) and \(B\) be endomorphisms of \(V\) which satisfy
\[
A^2 = \varepsilon_A 1_V, \quad B^2 = \varepsilon_B 1_V, \quad A^* = \varepsilon_A A, \quad B^* = \varepsilon_B B.
\]
Here \(\varepsilon_A = \pm 1, \varepsilon_B = \pm 1, 1_V\) is the identical automorphism of \(V\), \(t\) is a positive real number and \(A^*\) (resp. \(B^*\)) denotes the adjoint of \(A\) (resp. \(B\)) with respect to \((,\))
Then
1. \(AB\) is semi-simple and all eigenvalues of \(AB\) have absolute value \(t^{1/2}\).
2. \(V\) is a direct sum of irreducible invariant subspaces under the actions of \(A\) and \(B\).
3. Assume that \(V\) is irreducible. Then \(\dim V \leq 2\). Let \(\lambda\) be an eigenvalue of \(AB\). Then, if \(\varepsilon_A = \varepsilon_B\),
\[
\dim V = 1 \iff \lambda \in \mathbb{R}, \quad \dim V = 2 \iff \lambda \notin \mathbb{R}.
\]

**Proof.** By the assumptions, we see immediately that \(A\) and \(B/\sqrt{t}\) are unitary; hence \(AB/\sqrt{t}\) is unitary and (1) follows. If \(V_1\) is an invariant subspace of \(V\), then the orthogonal complement \(V_2\) of \(V_1\) is an invariant subspace and \(V = V_1 + V_2\) holds. Hence we get (2).

Now we assume that \(V\) is irreducible. Let \(v_1\) (\(\neq 0\)) be an eigenvector of \(AB\). Put \(ABv_1 = \lambda v_1, \lambda \in \mathbb{C}, v_2 = Av_1, W = Cv_1 + Cv_2\). Then we find \(\lambda \neq 0\). Obviously \(W\) is invariant under \(A\). Since
\[
Bv_1 = \varepsilon_A \lambda v_1 \quad \text{and} \quad Bv_2 = (\varepsilon_A \varepsilon_B t/\lambda)v_1,
\]
\(W\) is invariant under \(B\). Hence \(V = W\) and \(\dim V \leq 2\). To prove the latter half of (3), it suffices to show that \(\dim V = 1\) if \(\lambda \in \mathbb{R}\). Put \(\varepsilon = \varepsilon_A = \varepsilon_B\) and take \(\mu \in \mathbb{C}\) so that \(\mu^2 = \varepsilon\). We find
\[
A(v_1 + \mu v_2) = \varepsilon \mu (v_1 + \mu v_2), \quad B(v_1 + \mu v_2) = \mu \lambda (v_1 + \mu v_2),
\]
if \(\lambda = \overline{\lambda}\). This proves (3).
Let $\mathcal{L}$ denote the commutative algebra generated over $\mathbb{C}$ by the Hecke operators $T(1,1,l,l)$ and $T(1,l,l,l^3)$ for all primes $l \neq p$ (cf. Andrianov [1]). The action of $\mathcal{L}$ on $S_\kappa(\Gamma_p)$ is semi-simple; we can take common eigenfunctions of $\mathcal{L}$ as a basis of $S_\kappa(\Gamma_p)$.

**Lemma 2.5.** The operators $W$, $W_1$ and $H$ commute with operators in $\mathcal{L}$.

Since the proof is easy (trivial in adelized definition of automorphic forms), we omit it.

Let $S_\kappa(\Gamma_p)$ denote the smallest invariant subspace under the actions of $H$, $W$ and $W_2$ which contains $S_\kappa(\text{Sp}(2,\mathbb{Z}))$ (the space of “old forms”). Let $S_\kappa(\Gamma_p)'$ denote the orthogonal complement of $S_\kappa(\Gamma_p)$ in $S_\kappa(\Gamma_p)$ (the space of “new forms”). Of course, $S_\kappa(\Gamma_p)'= S_\kappa(\Gamma_p)$ if $S_\kappa(\text{Sp}(2,\mathbb{Z}))=\{0\}$.

**Theorem 2.6.** $S_\kappa(\Gamma_p)'$ is an invariant subspace under the action of $\mathcal{L}$, $H$ and $W$. Put $T(p)= HW$. Then we have

1. As a basis of $S_\kappa(\Gamma_p)'$, we can take common eigenfunctions of operators in $\mathcal{L}$ and $T(p)$.
2. All eigenvalues of $T(p)$ have absolute value $p$.

**Proof.** The invariance under $\mathcal{L}$ follows from Lemma 2.5 and the fact that the operators in $\mathcal{L}$ are hermitian. The invariance for $H$ and $W$ follows from Lemma 2.3. Take $f \in S_\kappa(\Gamma_p)$. By Lemma 2.2, we have $f \mid \mid W^2- p^2 f \in S_\kappa(\text{Sp}(2,\mathbb{Z}))$. Hence $W^2= p^2$ on $S_\kappa(\Gamma_p)'$. Obviously we have $H^2=1$. Now, by Lemma 2.1, (1) and Lemma 2.5, the operators in $\mathcal{L}$ and $T(p)$ are mutually commutative and semi-simple. Hence we obtain (1); (2) follows from Lemma 2.3 and Lemma 2.4, (1).

**Remark.** (1) The eigenvalues of $T(p)$ can be both real and non-real. The examples shall be given in § 4.
2. We can prove Hecke’s original theorem by the same method.

Let $F \in S_\kappa(\Gamma_p)$ and let $F(z)=\sum N A(N)e(\sigma(Nz))$ be its Fourier expansion. Then, by a direct computation, we have

$$\langle F|T(p)\rangle(z)= p^{s-k} \sum_{N \equiv \sigma \mod p} A(p^{-1}N)e(\sigma(Nz)).$$

Now we are going to look the space $S_\kappa(\Gamma_p)'$ more closely. The representation of $\mathcal{L}$ on $S_\kappa(\Gamma_p)'$ decomposes into a direct sum of one dimensional representations. For a one dimensional representation $\lambda$ of $\mathcal{L}$, let $S_\kappa(\Gamma_p)_\lambda$ denote the $\lambda$-isotypic component of $S_\kappa(\Gamma_p)'$. By Lemma 2.5, $S_\kappa(\Gamma_p)_\lambda$ is a $C[H,W]$-module. Since $H$ and $W$ are hermitian with respect to the Petersson inner product and $H^2=1, W^2= p^2$ on $S_\kappa(\Gamma_p)'$, we have, by Lemma 2.4,

$$S_\kappa(\Gamma_p)'= \bigoplus V_\lambda,$$
where \( V_i \) is an irreducible \( C[H, W] \)-module such that \( \dim_C V_i \leq 2 \). If \( \dim_C V_i = 2 \), the eigenvalues of \( T(p) \) on \( V_i \) are not real and mutually complex conjugate by Lemma 2.4, (3). Thus the action of \( T(p) \) on \( V_i \) is semi-simple. This consideration proves (1) of Theorem 2.6 again and also justifies the following definition.

**Definition 2.7.** Let \( F(\pm 0) \in SL(\Gamma \rho) \) be a common eigenfunction of operators in \( \mathcal{L} \) and of \( T(p) \). We call \( F \) **ziegliedrig** (resp. **eingliedrig**) if \( F \) generates a two (resp. one) dimensional irreducible \( C[H, W] \)-module. Assume that \( F \) is eingliedrig. If \( \rho_F \cong \theta_s \) (resp. \( \theta_{11} \)), \( F \) is called \( \theta_s \) (resp. \( \theta_{11} \))-eingliedrig.

**Remark.** Let \( F \) be as above. Then \( \rho_F \cong \theta_s \oplus \theta_{11} \) or \( \theta_s \) or \( \theta_{11} \); \( F \) is eingliedrig if and only \( \rho_F \cong \theta_s \) or \( \theta_{11} \); \( F \) is zweigliedrig if and only if \( \rho_F \cong \theta_s \oplus \theta_{11} \). Put \( F|T(p) = \mu F, \mu \in C, |\mu| = p \). Then \( F \) is eingliedrig (resp. zweigliedrig) if and only if \( \mu = \pm p \) (resp. \( \mu \notin R \)).

**Theorem 2.8.** Assume \( S_4(Sp(2, \mathbb{Z})) = \{0\} \). Then \( m_4(\theta_s) - m_4(\theta_{11}) = \dim_C \langle \theta_s \text{-eiblingedrig forms} \rangle - \dim_C \langle \theta_{11} \text{-eiblingedrig forms} \rangle \).

**Proof.** By the assumption and Lemma 2.2, we have \( W^2 = p^2 \) on \( S_4(\Gamma \rho) \) and \( m_4(\theta_s) \) (resp. \( m_4(\theta_{11}) \)) is the multiplicity of the eigenvalue \(-p\) (resp. \( p \)) of \( W \). We consider the decomposition (2.8). If \( \dim V_i = 2 \), the set of eigenvalues of \( W \) on \( V_i \) is \( \{ p, -p \} \), since otherwise \( W \) acts as a scalar on \( V_i \) and \( V_i \) cannot be irreducible. Thus a two dimensional component \( V_i \) gives no contribution to \( m_4(\theta_s) - m_4(\theta_{11}) \). Assume \( \dim V_i = 1 \) and \( V_i \) is spanned by \( F_i \in S_4(\Gamma \rho) \). Then we have \( F_i|W = -p F_i \) (resp. \( p F_i \)) if and only if \( F_i \) is a \( \theta_s \) (resp. \( \theta_{11} \))-eiblingedrig form. We see, by the Jordan-Hölder theorem, that the number of such \( V_i \)'s does not depend on the particular choice of the decomposition (2.8). This completes the proof.

**Remark.** We can obtain analogous results for \( S_4(\Gamma_{\rho'}) \) and \( m_4(\theta_s) - m_4(\theta_{12}) \).

We are going to study arithmetic properties of Fourier coefficients of eiblingedrig forms. Let \( U \) be the subgroup of \( G \) defined by

\[
U = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \biggm| b \in M(2, F_\rho), b^2 = b \right\}.
\]

Hereafter in this section, we assume \( p \neq 2 \). For a symmetric matrix \( S \in M(2, F_\rho) \), define \( \eta_S \in \hat{U} \) by \( \eta_S \left( \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \right) = \phi(\sigma(\tau S b)) \), where \( \phi \) is given by (1.4); we see easily that all characters of \( U \) are of this form. For symmetric matrices \( S_1, S_2 \in M(2, F_\rho) \), let us write \( S_1 \sim S_2 \) if there exists \( T \in GL(2, F_\rho) \) such that \( ^t T S_1 T = S_2 \). We have

\[
(2.9) \quad \theta_s | U \cong \bigoplus_{S_1 \sim S_2} \eta_S \bigoplus \bigoplus_{S \cong S_2} \eta_S \bigoplus \bigoplus_{S \cong S_1} \eta_S \bigoplus (p+1) \eta_0,
\]
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\[ \theta_{11} | U \equiv \left( \bigoplus_{\mathbf{s} \in \mathbb{Z}^2} \eta_{\mathbf{s}} \bigoplus \left( \bigoplus_{\mathbf{s} \in \mathbb{Z}^2} \eta_{\mathbf{s}} \right) \bigoplus \left( \bigoplus_{\mathbf{s} \in \mathbb{Z}^4} \eta_{\mathbf{s}} \right) \bigoplus 1_{\alpha} \right), \]

where \( S_1 = \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right), \ S_2 = \left( \begin{array}{cc} 1 & 0 \\ -\alpha & 1 \end{array} \right), \ S_3 = \left( \begin{array}{cc} \alpha & 0 \\ 0 & 1 \end{array} \right), \ S_4 = \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) \) with \( \alpha \in \mathbb{F}_p - \left( \mathbb{F}_p^\times \right)^2 \).

These formulas are similar to (1.2) and (1.3). We need the following.

**Lemma 2.9.** Let \( \nu_1 \) (resp. \( \nu_2 \)) be a non-zero \( P \)-fixed vector in a representation space of \( \theta_{11} \) (resp. \( \theta_3 \)). Then

\[ \sum_{u \in \mathbb{M}(2, \mathbb{Z}_p), \ t_u = \eta} \theta_{11} \left( \begin{array}{cc} 1 & u \\ 0 & 1 \end{array} \right) \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right) \eta(u) \nu_1 = 0 \]

if \( \eta = \eta_3 \) with \( -\det S \in \mathbb{F}_p - \left( \mathbb{F}_p^\times \right)^3 \),

\[ \sum_{u \in \mathbb{M}(2, \mathbb{Z}_p), \ t_u = \sigma} \theta_3 \left( \begin{array}{cc} 1 & u \\ 0 & 1 \end{array} \right) \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right) \eta(u) \nu_2 = 0 \]

if \( \eta = \eta_3 \) with \( -\det S \in \left( \mathbb{F}_p^\times \right)^3 \).

Since the proof is easy, it is omitted.

**Proposition 2.10.** Let \( F \in \mathbb{S}_k(\Gamma_p) \), \( F \neq 0 \) and \( F(z) = \sum \mathcal{A}(N) e(\sigma(Nz)) \) be the Fourier expansion of \( F \), where \( N \) extends over positive definite half integral symmetric matrices. We assume that \( F \) is an eigenfunction of \( H \) and of \( \sigma \) operators in \( \mathbb{C} \) and \( \mathbb{R} \).

Then

(1) \( F \) is \( \theta_x \)-einh"eldigdreg if and only if \( \mathcal{A}(N) = 0 \) whenever \( \left( -\frac{\det 2N}{p} \right) = -1 \).

(2) \( F \) is \( \theta_{11} \)-einh"eldigdreg if and only if \( \mathcal{A}(N) = 0 \) whenever \( \left( -\frac{\det 2N}{p} \right) = 1 \).

**Proof.** Since \( F|H = cF \) with \( c^2 = 1 \), we get

\[ (F| \left[ w \right]_k)(x) = (F| \left( \begin{array}{cc} 0 & 1 \\ -p & 0 \end{array} \right)_k \left( \begin{array}{cc} 0 & 1 \end{array} \right)_k)(x) = c p^{-k} \sum \mathcal{A}(N) e(\sigma(Nz/p)) \]

For \( \eta = \eta_3 \), put

\[ F_\eta = \sum_{u \in \mathbb{M}(2, \mathbb{Z}_p), \ t_u = \eta} F \left[ \left[ w \right]_k \right] \left( \begin{array}{cc} 1 & u \\ 0 & 1 \end{array} \right)_k \eta(u)^{-1} \]

where \( \bar{u} \) denotes a symmetric matrix in \( \mathbb{M}(2, \mathbb{Z}) \) such that \( \bar{u} \equiv \eta \mod p = u \). Then we have

\[ F_\eta \left( \begin{array}{cc} 1 & u \\ 0 & 1 \end{array} \right)_k = \eta(\eta \mod p) F_\eta, \ u \in \mathbb{M}(2, \mathbb{Z}), \ t\eta = u \].

By a direct computation using (2.11), we obtain

\[ F_\eta(x) = c p^{-k} \sum_{N \equiv \eta \mod p = \sigma} \mathcal{A}(N) e(\sigma(Nz/p)) \]

Now assume \( F \) is \( \theta_{11} \)-einh"eldigdreg, i.e. \( \rho_F \equiv \theta_{11} \). By (2.10) and (2.12), we see that \( F_\eta = 0 \) if \( -\det S \in \left( \mathbb{F}_p^\times \right)^3 \). Therefore, by (2.13), we obtain \( \mathcal{A}(N) = 0 \) if
\((-\det \frac{2N}{p}) = 1\). Conversely assume \(A(N) = 0\) if \((-\det \frac{2N}{p}) = 1\). Assume \(\rho_{\pi} \cong \theta_{\eta} \oplus \theta_{\eta'}\) or \(\theta_{\eta}\). By Lemma 2.9, we get \(F_{\pi} \neq 0\) for some (actually any) \(\eta \neq \eta_{\pi}\) with \(-\det S \in (F_{\pi})^N\). Hence we get \(A(N) \neq 0\) for some \(N\) such that \((-\det \frac{2N}{p}) = 1\) by (2.13). This is a contradiction. Therefore we must have \(\rho_{\pi} \cong \theta_{\eta}\). This proves (2). The assertion (1) can be proved in a similar way.

§ 3. A construction of \(\theta_{\eta}\)-eingliedrig forms

In this section, we shall show that some of Siegel's modular forms constructed in our previous paper [15] are \(\theta_{\eta}\)-eingliedrig. First we shall recall this construction briefly. Let \(p\) be a prime and \(D\) be the quaternion algebra over \(Q\) which ramify only at \(p\) and at the archimedean prime \(\infty\) (called \((p, \infty)\)-quaternion algebra); \(*\) denotes the main involution of \(D\). Let \(D_{\chi}\) denote the adelization of \(D^*\). We take a maximal order \(R\) of \(D\) and set \(D_{\chi} = D \otimes_{Q} Q_{1}\), \(R_{l} = R \otimes_{Q} \mathbb{Z}_{l}\) for a prime \(l\), \(K = \prod_{l} R_{l} \times H^*\). Here \(H = D \otimes_{Q} R\) is (isomorphic to) the Hamilton quaternion algebra. For \(0 \leq n \in \mathbb{Z}\), let \(\sigma_{\chi}^{n}\) denote the symmetric tensor representation of degree \(n\) of \(GL(2, C)\) on \(V \cong C^{2n+1}\). Fixing a splitting \(H \otimes_{Q} C \cong M(2, C)\), we consider \(\sigma_{\chi}^{n}\) as a representation of \(H^*\) and put \(\sigma_{\chi}(g) = \sigma_{\chi}^{n}(g) N(g)^{-n}\). We set \(S(R, 2\pi) = \{ \varphi | \varphi\text{ is a }V\text{-valued function of }D_{\chi}^*\text{ which satisfies }\varphi(\gamma k h) = \varphi(k \sigma_{\chi}(k_{m})\text{ for any }\gamma \in D_{\chi}^*, h \in D_{\chi}^*, k \in K\}\).

For any prime \(l \neq p\), we can define Hecke operator \(T'(/l)\) which acts on \(S(R, 2\pi)\) (cf. [15], p. 210). If \(\varphi\) is a common eigenfunction of \(T'(/l)\) and \(T'(/l)\varphi = \lambda_{l} \varphi\), we put

\[L(s, \varphi) = \prod_{p \neq l} (1 - \lambda_{l} | l^{t} + l^{-t} |)^{-1} \]

Let \(\varphi_{p}\) be a prime element of \(D_{\pi}\). As \(\varphi_{p}\) normalizes \(R_{p}\), we have

\[S(R, 2\pi) = S^{*}(R, 2\pi) \oplus S^{*}(R, 2\pi),\]

where \(S^{*}(R, 2\pi) = \{ \varphi \in S(R, 2\pi) | \varphi(h \varphi_{p}) = \pm \varphi(h) \text{ for any } h \in D_{\chi}^* \}\). We consider \(D\) as a quadratic space over \(Q\) by the reduced norm \(N\). Set

\[X = D \oplus D, \ G = Sp(2), \ H = \{(a, b) \in D^* \times D^* | N(a) = N(b) = 1\};\]

\(G\) and \(H\) are considered as algebraic groups over \(Q\). We let \(D^* \times D^*\) act on \(X\) on the right by

\[\rho(a, b)(x_{1}, x_{2}) = (a^{-1}x_{1}, b, a^{-1}x_{2}b), \ x_{1}, x_{2} \in D, \ a, b \in D^* .\]

Take an additive character \(\phi\) of \(Q_{A}\) so that \(\phi = \prod_{\chi} \phi_{\chi}\)

\[\phi_{\chi}(x) = e(x), \ x \in R \cdot Q_{\chi}, \ \phi_{\chi}(x) = e(-Fr(x)), \ x \in Q_{l} ,\]

where \(Fr(x)\) denotes the fractional part of \(x\). Then \(\phi\) is trivial on \(Q\). Let \(\pi\) denote the Weil representation of \(G_{A}\) realized on \(S(X_{A})\) associated with \(D\) and \(\phi\). Take \(\varphi_{1} \in S(R, 0), \ \varphi_{\pi} \in S(R, 2\pi)\) and let \(V_{1}\) be the representation space of
Let $\langle , \rangle$ be a hermitian inner product on $V$, such that $\sigma_0 \otimes \sigma_{2n}$ is unitary with respect to $\langle , \rangle$. For $f \in S(X_\Lambda) \otimes V_1$, set

$$\Phi_f(g) = \sum_{x \in X_\Lambda} \langle \pi(g)f(\rho(h)x), \varphi(h) \rangle dh, \quad g \in G_A,$$

where $\varphi = \varphi_1 \otimes \varphi_2$ is the $V_1$-valued function on $D \times D$. We choose $f$ more explicitly in the following form:

$$f = r_1 f_1 x f_2,$$

then a function $F_0$ on $\Phi_0$ can be defined by

$$F_0(g(\sqrt{-1} 1_2)) = \Phi_0(\tilde{g}) \det(c \sqrt{-1} + d)^k$$

for $g = (a \ b \ c \ d)^T \in G_{V'}$. We set $F_0 = \Psi(\Phi_0)$.

Put $F = \Psi(\Phi_0)$. By the choice of $f$ as above, we have $F \in G_{n+1}(\Gamma^{(p)}(p))$;

$$F \in S_{n+1}(\Gamma^{(p)}(p))$$

if $n > 0$. We can easily prove the following

**Lemma 3.1.** Let $\gamma \in Sp(2, \mathbb{Z})$. Then we have $F|_{\gamma} = \Psi(\Phi_f)$, where $f' \in S(V_\Lambda) \otimes V_1$ is given by $f' = \prod_p f_p \pi_p(\gamma^{-1}) f_p$. Here $\pi_p$ denotes the local Weil representation of $G_{q_p}$ on $S(X_{q_p})$.

The explicit form of $F$ is given as follows. Let $D' = \bigcup_{i=1}^b D_q y_i K$ be a double coset decomposition with

$$N(y_i) = 1 \in \mathbb{Q}, \quad (y_i)_{2n} = 1.$$ 

Define a lattice on $D$ by $L_i = D_q \cap y_i \prod R_i y_j^{-1}$. Then $L_i$ is a maximal order of $D$; put $e_i = |L_i|$. Up to a constant multiple, we have

$$F(z) = \sum_{i=1}^b \sum_{j=1}^b \sum_{x \in D_q \cap y_i \prod_{j \neq i} R_j y_j^{-1}} P(x \cdot x) e(\sigma(Q(x)z)),

\varphi_1(y_i) \otimes \varphi_2(y_j)(e_1 e_2), \quad z \in \mathbb{F}_2,$$

where $Q(x) = \left( Tr(x \cdot x) / 2, N(x) \right)$ and $Tr$ denotes the reduced trace. We
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denote this \( F \) by \( F(\varphi_1, \varphi_2) \). If \( \varphi_1 \in S^+(R, 0), \varphi_2 \in S^+(R, 2n) \), then we have \( F(\varphi_1, \varphi_2) = 0 \). Now we shall prove that \( F(\varphi_1, \varphi_2) \) are \( \theta_{11} \)-eingenliebig forms.

**Theorem 3.2.*** Let \( \varphi_1 \in S^+(R, 0), \varphi_2 \in S^+(R, 2n) \) and put \( F=F(\varphi_1, \varphi_2), k=n+2 \). We set \( \varepsilon=1 \) (resp. \(-1\)) if \( \varphi_1 \in S^+(R, 0) \) (resp. \( S^-(R, 0) \). Then we have

1. \( F|W=pF \).
2. \( F|H=\varepsilon F \).
3. If \( F \neq 0 \), then \( \rho_F \cong \theta_{11} \).
4. If \( F \) is a cusp form.

Now we shall prove that \( F(\varphi_1, \varphi_2) \) are \( \theta_{11} \)-eingenliebig forms.

**Proof.** First we shall prove (1). Let \( w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \) as in (2.1). Since

\[
F|W = \sum_{u \in M(t, z), \; u \equiv t_u \mod p} F[w] \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix},
\]

it suffices to show

\[ \pi_p(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix})w_f = pf_f, \]

by Lemma 3.1. We get, by the definition of the Weil representation \( \pi_p \) (cf. [14], p. 403 and Remark 1), \( \pi_p(w)f_p = f_p^\ast \) where \( f_p^\ast \) denotes the Fourier transformation of \( f_p \) with respect to the self dual measure. Set \( \tilde{R}_p = \varpi_p^{-1} R_p = R_p \varpi_p^{-1} \) (the dual lattice of \( R_p \)). We have \( f_p^\ast = \varpi^{-1} \times \text{the characteristic function of } \tilde{R}_p \). Since \( \text{vol}(R_p \oplus R_p) = [\tilde{R}_p \oplus \tilde{R}_p : R_p \oplus R_p]^{-1/2} = \varpi^{-1} \). As

\[ \pi_p(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix})f_p^\ast(x) = \varphi_p(\sigma(u^t x S x))f_p^\ast(x) \]

and \( u \rightarrow \varphi_p(\sigma(u^t x S x)) \) defines the trivial character of \( \{u \in M(2, \mathbb{Z}_p) \mid \{u, u\} \} \) if and only if \( x \in R_p \oplus R_p \) for \( x \in \tilde{R}_p \oplus \tilde{R}_p \), we obtain (3.7). This prove (1).

Now we shall prove (2). We have \( F[w]_* = \varphi(\Phi_f) \) with \( f' = \prod f \times f_p^\ast \). From this, we get by a little computation that

\[ (F | [w]_*)(z) = \varpi^{-2} \sum_{i=1}^6 \sum_{j=1}^6 \sum_{x \in (x_1, x_2) \in L_i \times L_j} P(x_1 x_2) e(\sigma(Q(x)z)), \]

where \( L_{ij} = D_{ij} \cap y_i(\prod R_i \times \tilde{R}_p) y_j^{-1} \). For each \( i, 1 \leq i \leq h \), let \( y_i \varpi_p^{-1} = y_i y_k \) with
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1 \leq s \leq h, \gamma_i \in D_q, k_s \in K. Then we find \( N(\gamma_i) = p^{-s} \) and \( L_i = \gamma_i L_i \); so the map \( L_i \ni x \rightarrow \gamma_i x, x \in L_i \) is a bijection. As \( (F \mid H)(z) = p^{-s}(F \mid [w])_s(pz) \), we get

\[
(F \mid H)(z) = p^{s+2} \sum_{i=1}^{h} \sum_{j=1}^{h} \sum_{x=(x_1, x_2) \in L_i \otimes L_j} P(x_i^* \gamma_i^* x_j) e(\sigma(Q(x)z)),
\]

\[
\varphi_i(y_0 w^0) \otimes \varphi_0(y_0) / e_i(e_f).
\]

We have \( P(x_i^* \gamma_i^* x_j) = P(p^{-s} x_i x_j) = p^{-s} P(x_i x_j) \). Hence we get \( F \mid H = \epsilon F \), which is (2).

By Lemma 2.2 and (1), we have \( \rho_F = \theta_1 \) if \( F \neq 0 \); hence we get (3). To prove (4), it suffices to show \( (f, F) = 0 \) for any \( f \in S_4(\text{Sp}(2, \mathbb{Z})) \). But this is clear since \( f | W = pf \), \( F | W = pF \) and \( W \) is hermitian. The assertion (5), except for Euler 2-factors, is proved in [15]; the results of [16] show that this holds also for Euler 2-factors. This completes the proof of Theorem 3.2.

The restrictions (3.2) and (3.5) made on the choices of \( y \) and \( P \) are sometimes inconvenient for numerical computations. Drop the assumption (3.5) and define the lattice \( L_i \) by the same formula; also assume simply that \( P \) is a (scalar valued) homogeneous harmonic polynomial of degree \( n \) on \( H \) of three variables depending on the pure quaternion part. Put

\[
\theta_{ij}^P = \sum_{x=(x_1, x_2) \in L_i \otimes L_j} P(x_i^* x_j) e(\sigma(Q(x)z)),
\]

where \( Q(x) = \frac{1}{N(L_{ij})} \left( \frac{N(x_i)}{Tr(x_1 x_2^*)/2} \frac{Tr(x_1 x_2^*) \sqrt{N(x_2)}}{2} \right) \) with the norm \( N(L_{ij}) \) of the lattice \( L_{ij} \). Then we have

**Proposition 3.3.** Let \( \Theta \) be the space spanned by \( \theta_{ij}^P \) for \( 1 \leq i, j \leq h \) and all \( P \) as above. Then \( \Theta \subset G_{n+1}(\Gamma_0) \) if \( n > 0 \), and is invariant under \( \mathcal{L} \). As a basis of \( \Theta \), we can take functions of the form \( F(\varphi_1, \varphi_2) \), where \( \varphi_i \) may be assumed to be a common eigenfunction of all \( T(l \mid l) \), \( l \neq p \).

This is an easy consequence of Theorem 3.2, (5); we omit the proof.

**Remark.** Let \( f_1 \in G_6(\Gamma_0(p)) \), \( f_2 \in G_7(p)(p)(\Gamma_0(p)) \) be the modular forms which correspond to \( \varphi_1 \) and \( \varphi_2 \) respectively ([15], Prop. 7.1). We have \( L(s, f_1) = L(s, \varphi_1) \), \( L(s, f_2) = L(s-n, \varphi_2) \),

\[
(3.9) \quad L(s, F) = L(s-n, f_1) L(s, f_2)
\]

except for Euler \( p \)-factors. We also get

\[
f_1 \begin{pmatrix} 0 & 1 \\ -p & 0 \end{pmatrix} = -\varepsilon f_1, \quad f_2 \begin{pmatrix} 0 & 1 \\ -p & 0 \end{pmatrix} = -\varepsilon f_2.
\]
§ 4. Numerical examples

Let $H$ denote the Hamilton quaternion algebra and $1, i, j, k$ be the standard quaternion basis. We shall use the following harmonic polynomials $P_n$ of degree $n$:

$$
P_0(x) = 1, \quad P_1(x) = c^4 - d^4, \quad P_2(x) = c^4 - 6c^2d^2 + d^4, \quad P_3(x) = c^4 - 15c^2d^2 + 15cd^4 - d^8, \quad P_4(x) = b^4 - 15b^2c^2 + 15bc^4 - c^8,
$$

where $x = a + bi + cj + dk \in H$. The data concerning the dimension of $S_k(\Gamma_0(3))$ are taken from K. Hashimoto [4]. Some of the formulas in this section are conjectural; we shall mark them by the subscript $c$. The equality of Euler products means the identity up to the Euler $p$-factor, where $p$ is the level of modular forms.

(1) The case of level 3

The $(3, \infty)$-quaternion algebra $D$ is given explicitly by $D = Q + Qi' + Qj' + Qk'$ with $i'' = -1, j'' = -3, i'j' = -j'i' = k'$. A maximal order $R$ of $D$ is given by $R = Z\omega_1 + Z\omega_2 + Z\omega_3 + Z\omega_4$, where $\omega_1 = (1 + j')/2, \omega_2 = (i' + k')/2, \omega_3 = j', \omega_4 = k'$. We have $h = 1$; so $S(R, 0)$ consists of constant functions. Put

$$
x = (x_1, x_2, x_3, x_4), \quad y = (y_1, y_2, y_3, y_4) \in \mathbb{Z}^4,
\quad \tilde{x} = x_1\omega_1 + x_2\omega_2 + x_3\omega_3 + x_4\omega_4, \quad \tilde{y} = y_1\omega_1 + y_2\omega_2 + y_3\omega_3 + y_4\omega_4.
$$

We define a symmetric matrix $S$ by $N(\tilde{x}) = \tilde{x}^t S \tilde{x}$; we have

$$
S = \begin{pmatrix}
1 & 0 & 3/2 & 0 \\
0 & 1 & 0 & 3/2 \\
3/2 & 0 & 3 & 0 \\
0 & 3/2 & 0 & 3
\end{pmatrix}.
$$

Set $Q(x, y) = (\tilde{x}^t S \tilde{x}, \tilde{x}^t S \tilde{y})$.

Put $X = \tilde{x}^* \tilde{y}$ and let

$$
b = 2 \times \text{the coefficient of } i' \text{ in } X,
\quad c = 2 \times \text{the coefficient of } j' \text{ in } X,
\quad d = 2 \times \text{the coefficient of } k' \text{ in } X.
$$

The explicit forms are

$$
b = (x_2y_1 - y_2x_1) + 3(x_3y_3 - y_3x_3 + x_4y_1 - y_4x_1) + 6(x_4y_3 - y_4x_3),
\quad c = (x_1y_1 - y_1x_1) + (x_2y_4 - y_2x_4),
\quad d = (x_1y_4 - y_1x_4) + (y_2x_3 - x_2y_3) + (x_3y_2 - y_3x_2).
$$

We define five theta series by

$$
\theta_1(z) = \sum_{(x, y)} e(\sigma(Q(x, y)z)),
\quad \theta_2(z) = \sum_{(x, y)} (c^4 - 6c^2d^2 + d^4) e(\sigma(Q(x, y)z)),
\quad \theta_3(z) = \sum_{(x, y)} (c^4 - 6c^2d^2 + d^4) e(\sigma(Q(x, y)z)),
$$
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\[ \theta_{i}^{(1)}(z) = \sum_{\gamma \in \Gamma_0(3)} e^{\gamma z}, \]
\[ \theta_{i}^{(2)}(z) = \sum_{\gamma \in \Gamma_0(3)} e^{\gamma z}, \]

where \( z \in \mathbb{H} \) and \((x, y)\) extends over \( \mathbb{Z} \oplus \mathbb{Z}. \) By the results stated in § 3, we see \( \theta_{i} \in G_{\alpha}(\Gamma_0^+(3)), \) \( \theta_{i} \in S_{\alpha}(\Gamma_0^+(3)), \) \( \theta_{i} \in S_{\alpha}(\Gamma_0^+(3)), \) \( \theta_{i}^{(1)}, \theta_{i}^{(2)} \in S_{\alpha}(\Gamma_0^+(3)). \) We have

\[ (4.1) \quad L(s, \theta_{i}) = \zeta(s) \zeta(s-1). \]
\[ (4.2) \quad L(s, \theta_{i}) = \zeta(s-2) \zeta(s-3) L(s, f_{10}). \]
\[ (4.3) \quad L(s, \theta_{i}) = \zeta(s-4) \zeta(s-5) L(s, f_{10}). \]

where \( f_{10} \) is a one-dimensional \( \theta_{i} \)-eigenspace. \( \theta_{i} \) is a \( \theta_{i} \)-eigenspace which satisfies \( \theta_{i} H = \theta_{i}, \)

First we are going to decompose \( S_{\alpha}(\Gamma_0^+(3)) \) into eigenspaces. We have \( \dim \, S_{\alpha}(\Gamma_0^+(3)) = 2 \); \( \theta_{i} \) is a \( \theta_{i} \)-eigenspace which satisfies \( \theta_{i} \mid H = \theta_{i}, \)

Let \( \phi_{\alpha}(z) = \sum_{\alpha} a(n) \sigma(\alpha(n)z), z \in \mathbb{H}, f_{10}(z) = \sum_{n=1}^{\infty} a(n) \sigma(nz), a(1) = 1, z \in \mathbb{H}, \) be Fourier expansions, where \( f_{10} \) is a one-dimensional. In tables (I) and (II), we can observe the relation

\[ \zeta(s-4) \zeta(s-5) L(s, \phi_{\alpha}) = \zeta(s-4) \sum_{n=1}^{\infty} A(n) n^{-s}, \]

which suggests

\[ (4.4) \quad L(s, \phi_{\alpha}) = \zeta(s-4) \zeta(s-5) L(s, f_{10}). \]

It is almost certain that (4.4) can be proven by the method of H. Maass [10] and D. Zagier. The eigenvalues of \( T(3) \) on \( S_{\alpha}(\Gamma_0^+(3)) \) are 3 and -3. Incidentally we get \( S_{\alpha}(\Gamma_0^+(3)) = S_{\alpha}(\Gamma_0^+(3)). \)

Now we are going to decompose \( S_{\alpha}(\Gamma_0^+(3)) \) into eigenspaces. We have \( \dim S_{\alpha}(\Gamma_0^+(3)) = 5. \) Put

\[ \theta_{i}^{(1)}(z) = \sum_{\alpha} B(\alpha) \sigma(\alpha(Nz)), \quad \theta_{i}^{(2)}(z) = \sum_{\alpha} C(\alpha) \sigma(\alpha(Nz)), \]

be the Fourier expansions. Put \( V_{\alpha} = \langle \theta_{i}^{(1)}, \theta_{i}^{(2)} \rangle_{C}. \) By the table (II), we see that \( \dim V_{\alpha} = 2. \) Therefore, by Proposition 3.3, \( V_{\alpha} \) is stable under \( \mathcal{C}. \) Set
By tables (II) and (III), we find

\[
\begin{align*}
\theta_4^{(1)} | T(2)(x) &= -1224 \theta_4^{(1)} - 50 \theta_4^{(2)}, \\
\theta_4^{(2)} | T(2)(x) &= 38232 \theta_4^{(1)} + 1554 \theta_4^{(2)}.
\end{align*}
\]

Put

\[
\begin{align*}
\phi_4^{(1)} &= (2535 - 33\sqrt{d}) \theta_4^{(1)} + (87 - \sqrt{d}) \theta_4^{(2)}, \\
\phi_4^{(2)} &= (2535 + 33\sqrt{d}) \theta_4^{(1)} + (87 + \sqrt{d}) \theta_4^{(2)},
\end{align*}
\]

where \( d = 11 \times 179 \). Then we have

\[
\begin{align*}
\phi_4^{(1)} | T(2)(x) &= (165 + 3\sqrt{d}) \phi_4^{(1)}, \\
\phi_4^{(2)} | T(2)(x) &= (165 - 3\sqrt{d}) \phi_4^{(2)}.
\end{align*}
\]

Let \( f_4(z) = \sum_{n=1}^{\infty} b(n) e(nz) \in S_6(\Gamma_0^{(1)}(3)) \) be normalized so that \( b(1) = 1, b(2) = -27 + 3\sqrt{d} \). By Proposition 3.3, we have

\[
L(s, \phi_4^{(1)}) = \zeta(s-6) \zeta(s-7) L(s, f_4(z)).
\]

Comparing tables (I) and (II), we can observe the relation

\[
\zeta(s-6) \zeta(s-7) L(s, f_4(z)) = \zeta_{q(\sqrt{-1})(s-6) \sum_{n=1}^{\infty} D(n(1 \ 0 \ 0)) n^{-s}},
\]

which is a consequence of (4.7), where we put

\[
\phi_4^{(1)}(x)/(48 \times 336) = \sum_{n} D(N) e(\sigma(Nz)).
\]

By Theorem 3.2, \( \phi_4^{(1)} \) and \( \phi_4^{(2)} \) are \( \theta_1 \)-eignogliedrige forms and we have \( \phi_4^{(i)} | H = \phi_4^{(i)}, \phi_4^{(1)} | W = 3 \phi_4^{(1)}, \# = 1, 2 \).

Next we consider \( \theta_4^{(1)} \times \theta_6^{(1)} \in S_6(\Gamma_0^{(3)}(3)) \). Put

\[
\begin{align*}
\theta_4^{(1)}(z) &= \sum_{N} E_4(N) e(\sigma(Nz)), \\
(\theta_4^{(1)} \times \theta_6^{(1)})(z) &= \sum_{N} F(N) e(\sigma(Nz)), \\
(\theta_4^{(1)} | T(2))(x) &= \sum_{N} E_4(N) e(\sigma(Nz)), \\
((\theta_4^{(1)} \times \theta_6^{(1)} | T(2))(x) &= \sum_{N} F(N) e(\sigma(Nz)),
\end{align*}
\]

Some of these values are given in tables (IV) and (V). Put \( V = \langle \theta_4^{(1)}, \theta_4^{(2)}, \theta_4^{(1)} \times \theta_6^{(1)} \rangle_c \). As the Fourier coefficients of \( \theta_4^{(1)} \) and of \( \theta_4^{(2)} \) vanish for \( n \left( \begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix} \right) \) (cf. Prop. 2.10), we have \( \dim V = 4 \). We see \( V \subset S_6(\Gamma_0^{(3)}(3)) \) by (2) of Theorem 3.2. Let \( E_6 \in G_6(S \times (2, Z)) \) be the Eisenstein series and put \( \eta_4^{(1)} = \theta_4 \times (E_6 - E_4 | H). \) Obviously \( \eta_4^{(1)} \neq 0 \) and \( \eta_4^{(1)} \in S_6(\Gamma_0^{(3)}(3)) \). Therefore we have \( S_6(\Gamma_0^{(3)}(3)) = V, S_6(\Gamma_0^{(3)}(3)) = \langle \eta_4^{(1)} \rangle_c \). In particular, \( V \) is invariant under \( \mathcal{L} \) and \( \eta_4^{(1)} \) is a common eigenfunction of all operators in \( \mathcal{L} \) (cf. Lemma 2.5). Using the tables (IV) and (V) for \( n \cdot N_h \), we find
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\[ \theta_1 \mid T(2) \equiv -72 \theta_1^4 + 24 \theta_4 \theta_6 \mod V_0, \]
\[ (\theta_2 \theta_0) \mid T(2) \equiv 756 \theta_1^4 + 108 \theta_4 \theta_6 \mod V_0. \]

Then by tables (V) for \( n \cdot N \) and (II), we obtain

\[ (4.8) \theta_1 \mid T(2) = -72 \theta_1^4 + 24 \theta_4 \theta_6 + 216 \theta_1^{(1)} + 8 \theta_4^{(3)}, \]
\[ (4.9) (\theta_2 \theta_0) \mid T(2) = 756 \theta_1^4 + 108 \theta_4 \theta_6 + 756 \theta_1^{(1)} + 26 \theta_4^{(3)}. \]

Put

\[ \phi_s^{(3)} = 3 \theta_1^2 + \theta_4 \theta_6 + \theta_8^{(3)}, \]
\[ \eta_8^{(3)} = -21 \theta_1^2 + 2 \theta_4 \theta_6 + 9 \theta_8^{(3)} + (\theta_4^{(3)}/3). \]

By (4.5), (4.6), (4.8) and (4.9), we have

\[ \phi_s^{(3)} \mid T(2) = 180 \phi_s^{(3)}, \]
\[ \eta_8^{(3)} \mid T(2) = -144 \eta_8^{(3)}. \]

On the other hand, by table (IV), we have \( \eta_8^{(3)} \mid T(2) = -144 \eta_8^{(3)} \). Therefore \( \phi_s^{(3)} \) must be an einengliedrig form by Theorem 2.6, (1). By table (IV), the Fourier coefficient of \( \phi_s^{(3)} \) for \( \left( \begin{array}{c} 1 \\ 0 \\ 2 \end{array} \right) \) is 576. Hence \( \phi_s^{(3)} \) must be \( \theta_s \)-einfeldrig. We have

\[ \phi_s^{(3)} \mid H = \phi_s^{(3)}, \quad \phi_s^{(3)} \mid W = -3 \phi_s^{(3)}. \]

For \( f_+(x) = \sum_{n=1}^{\infty} c(n) e(nz) \in S_4(\Gamma_0^{(3)}(3)) \) (one dimensional), \( c(1) = 1 \), we observe the relation (cf. table (1) and (IV))

\[ \text{(4.10)} \quad L(s, f_+) = \zeta(s-6) \zeta(s-7) L(s, f_+). \]

Now we turn to the most interesting part of \( S_8(\Gamma_0^{(3)}(3)) \). Let \( V_1 \) denote the subspace \( \langle \eta_8^{(3)}, \phi_s^{(3)} \rangle_c \). As an orthogonal complement of \( \langle \phi_4^{(3)}, \phi_4^{(3)}, \phi_8^{(3)} \rangle_c \), \( V_1 \) is a two dimensional \( C[\chi, W] \)-module. We shall show that \( V_1 \) is irreducible. For this purpose, it is sufficient to prove that the eigenvalues of \( T(3) \) on \( V_1 \) are not real, by Lemma 2.4, (3). Let

\[ \eta_8^{(3)}(x) = \sum N G^{(3)}(N) e(\sigma(Nz)), \quad \eta_8^{(3)}(x) = \sum N G^{(3)}(N) e(\sigma(Nz)), \]

be the Fourier expansions. By table (IV) and (2.7), we find

\[ (4.11) \quad \eta_8^{(3)} \mid T(3) = -\eta_8^{(3)} + (\eta_8^{(3)})/5, \]
\[ (4.12) \quad \eta_8^{(3)} \mid T(3) = -40 \eta_8^{(3)} - \eta_8^{(3)}. \]

Hence the characteristic roots of \( T(3) \) are \(-1 \pm 2\sqrt{-2}\). Therefore \( V_1 \) is irreducible. Put \( \phi_s^{(4)} = 20 \eta_8^{(3)} - \sqrt{-2} \eta_8^{(3)}, \phi_4^{(3)} = 20 \eta_4^{(3)} + \sqrt{-2} \eta_8^{(3)}. \) By (4.11) and (4.12), we see

\[ (4.13) \quad \phi_4^{(3)} \mid T(3) = -1 + 2 \sqrt{-2} \phi_4^{(3)}. \]

Thus \( \phi_4^{(3)} \) and \( \phi_4^{(3)} \) are zweigliedrig forms. This completes the decomposition of \( S_9(\Gamma_0^{(3)}(3)) \). The eigenvalues of \( T(3) \) on this space are 3, 3, -3 and \(-1 \pm 2\sqrt{-2}\). Incidentally we can prove...
Proposition 4.1. \( \dim S_4(\Gamma_0^{(3)}(3))=0 \) and \( \dim S_4(\Gamma_0^{(3)}(3))=1 \).

Proof. Considering the injection
\[
S_4(\Gamma_0^{(3)}(3)) \ni f \mapsto f \times \theta_4 \in S_4(\Gamma_0^{(3)}(3)),
\]
we see \( \dim S_4(\Gamma_0^{(3)}(3)) \leq 2 \). Assume \( \dim S_4(\Gamma_0^{(3)}(3))=2 \). As we have shown \( \dim S_4(\Gamma_0^{(3)}(3))=S_4(\Gamma_0^{(3)}(3)) \) and \( \theta_4| H=\theta_4 \), we must have \( \dim S_4(\Gamma_0^{(3)}(3))=S_4(\Gamma_0^{(3)}(3)) \).

Now consider another injection
\[
S_4(\Gamma_0^{(3)}(3)) \ni f \mapsto f \times (E_4-1)| H \in S_4(\Gamma_0^{(3)}(3)).
\]
Since \( \dim S_4(\Gamma_0^{(3)}(3))=1 \) as we have shown, we get a contradiction. We obtain \( \dim S_4(\Gamma_0^{(3)}(3))=1 \) since it contains \( \theta_4 \). If \( f(\neq 0) \in S_4(\Gamma_0^{(3)}(3)) \), then \( f^2 \) and \( f \times \theta_4 \) are linearly independent cusp forms in \( S_4(\Gamma_0^{(3)}(3)) \). This is a contradiction. Hence we obtain \( \dim S_4(\Gamma_0^{(3)}(3))=0 \).

(11) The case of level 11

The \((11, \infty)-\)quaternion algebra \( D \) is given by \( D=Q+Q'Q+Q'Q'Qk' \) with \( i'^2=-1, j'^2=-11, l'^2=-j'i'=k' \). We have \( h=2 \) and \( S(R, 0)=S(R, 0) \). We are going to construct theta series from lattices \( L_{ij}(1 \leq i, j \leq 2) \).

We use the following notation. The lattice \( L_{ij} \) is written as \( L_{ij}=Zw_1+Zw_2+Zw_3+Zw_4 \), and \( N(L_{ij}) \) denotes the norm of \( L_{ij} \); an explicit form is due to A. Pizer. We put
\[
x=(x_1, x_2, x_3, x_4), \quad y=(y_1, y_2, y_3, y_4) \in \mathbb{Z}^4,
\]
and set \( \bar{x}=\sum_{i=1}^4 x_iw_i, \quad \bar{y}=\sum_{i=1}^4 y_iw_i \). Define \( S_{ij} \in M(4, \mathbb{Q}), \quad S_{ij}=S_{ij} \) by \( N(\bar{x})/N(L_{ij}) =^4xS_{ij}x \) and set \( Q_{ij}(x,y)=(i^4xS_{ij}x^4xS_{ij}y^4xS_{ij}y) \). Put \( X_{ij}=\bar{x}\bar{y} \) and let
\[
\begin{align*}
b &= \text{the coefficient of } i' \text{ in } X_{ij}, \\
c &= \text{the coefficient of } j' \text{ in } X_{ij}, \\
d &= \text{the coefficient of } k' \text{ in } X_{ij}.
\end{align*}
\]
Thus \( b, c \) and \( d \) are polynomials of \( x \) and \( y \) which may depend on \( L_{ij} \).

First we consider \( L_{11} \). We have \( \omega_1=(1+j)/2, \quad \omega_2=(i'+j')/2, \quad \omega_3=j', \quad \omega_4=k' \) and \( N(L_{11})=1 \). We set
\[
\begin{align*}
\theta_2^{(1)}(z) &= \sum_{(x,y)} \sigma(Q_{11}(x,y)z), \\
\theta_4^{(1)}(z) &= \frac{1}{8} \sum_{(x,y)} (b^2-11c^2) \sigma(Q_{11}(x,y)z), \\
\theta_4^{(2)}(z) &= \frac{1}{4} \sum_{(x,y)} (c^2-d^2) \sigma(Q_{11}(x,y)z),
\end{align*}
\]
where \( (x, y) \) extends over \( \mathbb{Z}^2 \) and \( z \in \mathbb{Z} \).

We consider \( L_{23} \). We have \( \omega_1=(1+3j)/2, \quad \omega_2=(i'+16j'+3k')/6, \quad \omega_3=3j', \quad \omega_4=k' \) and \( N(L_{23})=1 \). We set...
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$$\theta_1^{(i)}(z) = \sum_{(x, y)} e(\sigma(Q_{12}(x, y)z)),$$

$$\theta_2^{(i)}(z) = \frac{3}{4} \sum_{(x, y)} (c^3 - d^3)e(\sigma(Q_{12}(x, y)z)).$$

We consider $L_{12}$. We have $\omega_1 = (1 + 3j' + 4k')/12$, $\omega_2 = (i' + 4j' + 3k')/12$, $\omega_3 = j'/2$, $\omega_4 = k'/2$, $N(L_{12}) = 1/12$. We set

$$\theta_1^{(i)}(z) = \sum_{(x, y)} e(\sigma(Q_{12}(x, y)z)),$$

$$\theta_2^{(i)}(z) = 18 \sum_{(x, y)} (b^2 - 11c^2)e(\sigma(Q_{12}(x, y)z)),$$

$$\theta_3^{(i)}(z) = 9 \sum_{(x, y)} (c^3 - d^3)e(\sigma(Q_{12}(x, y)z)).$$

We consider $L_{21}$. We have $\omega_1 = 1 + 3j' + 2k'$, $\omega_2 = i' + 4j' + 3k'$, $\omega_3 = 6j'$, $\omega_4 = 6k'$, $N(L_{21}) = 12$. We set

$$\theta_0^{(i)}(z) = \sum_{(x, y)} (c^3 - d^3)e(\sigma(Q_{12}(x, y)z)).$$

We have $\theta_1^{(i)} \in G_2(\Gamma_0(11))(1 \leq i \leq 3)$, $\theta_2^{(i)} \in S_4(\Gamma_0(11))(1 \leq i \leq 6)$; $\dim G_2(\Gamma_0(11)) = \dim S(R, 0) = \dim G_2(\Gamma_0(11)) = \dim S^*(R, 0) = 2$, $\dim S_4(\Gamma_0(11)) = \dim S^*(R, 4) = 3$, $\dim S_2(\Gamma_0(11)) = \dim S^*(R, 4) = 1$. Let $E_2$ and $f_2$ denote the Eisenstein series and the normalized cusp forms in $G_2(\Gamma_0(11))$ respectively.

As is shown in [15], §8, we have

$$L(s, \theta_1^{(i)} + \theta_2^{(i)} - 2\theta_3^{(i)}) = L(s, f_2)^2,$$

$$L(s, 3\theta_1^{(i)} - 2\theta_2^{(i)} - \theta_3^{(i)}) = \zeta(s)\zeta(s-1)L(s, f_2) = L(s, E_2)L(s, f_2),$$

$$L(s, 9\theta_1^{(i)} + 4\theta_2^{(i)} + 12\theta_3^{(i)}) = \zeta(s)^2\zeta(s-1)^2 = L(s, E_2)^2,$$

and $3\theta_1^{(i)} - 2\theta_2^{(i)} - \theta_3^{(i)} \in S_2(\Gamma_0(11))$; these are modular forms which correspond to the pairing $G_2(\Gamma_0(11)) \times G_2(\Gamma_0(11))$.

Now let us consider the pairing $G_2(\Gamma_0(11)) \times S_4(\Gamma_0(11))$. Corresponding to the six pairs, we can obtain six linearly independent cusp forms in $S_4(\Gamma_0(11))$ by linear combinations of $\theta_1^{(i)}(1 \leq i \leq 6)$. These are given explicitly as follows.

Let $f_2(z) = \sum_{n=1}^{\infty} a_ne(nz)$ be any normalized eigen cusp form of all Hecke operators in $S_4(\Gamma_0(11))$. First determine $\alpha_4$, $\alpha_2$, $\alpha_3$, $\alpha_5$ and $\alpha_6$ successively by

$$\alpha_4 = \frac{1}{264} \{50 + 8(a_4 + 4) + 9a_5 - a_6\},$$

$$\alpha_2 = 5\alpha_4 - \frac{11}{2} - a_3 - \frac{7}{3}(a_4 + 4),$$

$$\alpha_3 = 4\alpha_4 - \frac{1}{3}(a_3 + 4),$$

$$12\alpha_5 = 45 + 6a_5 + 48\alpha_4 - 12a_6 - (a_5 + 8)^2 + 8a_5 + 32,$$

$$27\alpha_6 = 2758 - 116a_5 + 252a_4 + 4266\alpha_4 + 908\alpha_6 - 4(a_5 + 27)(a_4 + 36) + 108a_4 + 972.$$


and set \( F = \theta_i^{(1)} + \sum_{i=1}^{k} \alpha_i \theta_i^{(1)} \). Then we have \( F \neq 0 \) and

\[
L(s, F) = \zeta(s-2)\zeta(s-3)L(s, f_\ell) = L(s-2, E_2)L(s, f_\ell).
\]

Next determine \( \beta_1, \beta_2, \beta_3, \beta_4, \beta_5 \) and \( \beta_6 \) successively by

\[
\beta_1 = \frac{1}{264} \left[ 50 + 8(a_3 - 16) + 9(a_3 - 45) - (a_3 - 125) \right],
\]
\[
\beta_2 = \frac{11}{2} (a_3 - 45) - \frac{7}{3} (a_3 - 16),
\]
\[
\beta_3 = 4\beta_4 - \frac{1}{3} (a_3 - 16),
\]
\[
12\beta_3 = 45 + 6\beta_2 + 48\beta_4 - 12\beta_5 - (a_3 - 12)(a_3 - 8) + 8a_3 + 64,
\]
\[
27\beta_5 = 2758 - 116\beta_3 + 252\beta_4 + 4266\beta_4 + 908\beta_6 - 4(a_3 - 9)^3
\]
\[-36a_3 + 2268.
\]

and set \( G = \theta_i^{(1)} + \sum_{i=1}^{k} \beta_i \theta_i^{(1)} \). Then we have \( G \neq 0 \) and

\[
L(s, G) = L(s-2, f_\ell)L(s, f_\ell).
\]

According to Hashimoto [4], it is very plausible that \( \dim S_4(\Gamma_6^{(1)}(11)) = 7 \). Hereafter in this example, we shall assume this value of the dimension. As we have constructed six \( \theta_1 \)-eingliedrig forms, a form \( \phi_i(\neq 0) \) in the orthogonal complement of \( \langle \theta_i^{(1)} | 1 \leq i \leq 6 \rangle_c \) must be an eingliedrig form. Put

\[
\eta_4 = (3\theta_i^{(1)} - 2\theta_i^{(2)} - \theta_i^{(3)}) (9\theta_i^{(1)} + 4\theta_i^{(2)} + 12\theta_i^{(3)}) \in S_4(\Gamma_6^{(1)}(11)).
\]

The Fourier coefficient of \( \eta_4 \) for \( \left( \frac{1}{2}, \frac{1}{2} \right) \) is \(-1440\). Hence \( \phi_4 \) must be \( \theta_1 \)-eingliedrig. As the Fourier coefficients of \( \phi_4 \) for \( n \cdot \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \) must vanish, we obtain

\[
\phi_4 = \eta_4 + \frac{25}{2} \left\{ -2\theta_i^{(1)} + \frac{585}{2} \theta_i^{(2)} + 100\theta_i^{(3)} + \frac{632}{33} \theta_i^{(1)} + 888\theta_i^{(2)} + \frac{388}{11} \theta_i^{(3)} \right\}
\]

up to a constant multiple. Thus \( S_4(\Gamma_6^{(1)}(11)) = S_4(\Gamma_6^{(1)}(11)) \) and this space is spanned by six \( \theta_1 \)-eingliedrig forms and by one \( \theta_1 \)-eingliedrig form. By computing more Fourier coefficients of \( \phi_4 \), we can observe the relation

\[
L(s, \phi_4) = \zeta(s-2)\zeta(s-3)L(s, f_\ell),
\]

where \( f_\ell \) is the normalized cusp form in \( S_4^0(\Gamma_6^{(1)}(11)) \). Here the following observation seems very interesting: There is no form whose Euler product corresponds to the pair \( (f_2, f_\ell) \); i.e. no form \( H \in S_4(\Gamma_6^{(1)}(11)) \) such that \( L(s, H) = L(s-2, f_\ell)L(s, f_\ell) \). We shall take into account of this fact when we shall formulate Conjecture 5.2.

(III) The case of level 7

The \( (7, \infty) \)-quaternion algebra \( D \) is given explicitly by \( D = Q + Qj + Qj' + Qk' \) with \( i^2 = -1, j^2 = -7, i'j' = -j'i' = k' \). A maximal order \( R \) of \( D \) is given by...
\( R = Z \omega_0 + Z \omega_2 + Z \omega_3 + Z \omega_4 \), where \( \omega_0 = (1 + j')/2, \omega_2 = (i' + k')/2, \omega_3 = j', \omega_4 = k' \). We have \( h = 1 \). Put

\[
x = ^t(x_1, x_2, x_3, x_4), y = ^t(y_1, y_2, y_3, y_4) \in Z^4,
\]

\[
x = \sum_{i=1}^4 x_i \omega_i, \quad y = \sum_{i=1}^4 y_i \omega_i.
\]

Define a symmetric matrix \( S \) by

\[
(x, y) = \begin{pmatrix} ^tSx & ^tSy \\ ^tSy & ^tSx \end{pmatrix},
\]

\( X = x^*y \) and let

\[
b = 2x \text{ the coefficient of } i' \text{ in } X,
\]

\[
c = 2x \text{ the coefficient of } j' \text{ in } X,
\]

\[
d = 2x \text{ the coefficient of } k' \text{ in } X.
\]

We define six theta series by

\[
\theta_\sigma(z) = \sum_{(x, y)} e(\sigma(Q(x, y)z)),
\]

\[
\theta_1^{(1)}(z) = \sum_{(x, y)} (b^5 - 7c^5)e(\sigma(Q(x, y)z)),
\]

\[
\theta_1^{(2)}(z) = \sum_{(x, y)} (c^3 - d^3)e(\sigma(Q(x, y)z)),
\]

\[
\theta_\sigma^{(3)}(z) = \sum_{(x, y)} (b^4 - 42b^2c^2 + 49c^4)e(\sigma(Q(x, y)z)),
\]

\[
\theta_\sigma^{(4)}(z) = \sum_{(x, y)} (c^6 - 6c^4d^2 + d^4)e(\sigma(Q(x, y)z)).
\]

We have \( \theta_\sigma \in G_4(\Gamma_0^{(2)}(7)), \theta_i^{(1)} = S_i(\Gamma_0^{(2)}(7))(1 \leq i \leq 2), \theta_i^{(2)} = S_i(\Gamma_0^{(3)}(7))(1 \leq i \leq 3). \) Put

\[
\phi_4^{(i)} = \theta_4^{(i)} + (8 - \sqrt{57})\theta_4^{(i)} + (8 + \sqrt{57})\theta_4^{(i)}.
\]

Let \( f_\sigma(z) = \sum_{n=1}^\infty a_n e(nz) \in S(\Gamma_0^{(2)}(7))(\text{two dimensional}) \) be the normalized eigen cusp form such that \( a_1 = 1, a_2 = (9 + \sqrt{57})/2. \) Then we have \( \phi_4^{(i)} \neq 0 \) and

\[
(4.16) \quad L(s, \phi_4^{(i)}) = \zeta(s-2)\zeta(s-3)L(s, f_\sigma).
\]

Now we are going to consider the decomposition of \( S_6(\Gamma_0^{(3)}(7)). \) We have \( \dim S_6(\Gamma_0^{(3)}(7)) = 11. \) First the modular forms which correspond to the pairing

\[
S^*(R, 0) \times S^*(R, 9) \cong \Gamma_0^{(3)}(7) \times \Gamma_0^{(3)}(7)
\]

can be constructed as follows. Put \( V_3 = \langle \theta_3^{(3)}, \theta_5^{(2)}, \theta_5^{(3)} \rangle \); we find \( \dim V_3 = 3, \dim S_6(\Gamma_0^{(3)}(7)) = 3. \) Let \( f_{10}(z) = \sum_{n=1}^\infty b_n e(nz) \in S_6(\Gamma_0^{(3)}(7)) \) be a normalized eigen cusp form; \( b_3 \) satisfies the irreducible equation \( X^2 - 21X^2 - 1326X + 19080 = 0. \) Put

\[
\phi_6^{(i)} = \frac{1}{8} \theta_6^{(i)} + (b_3 + 15)\theta_6^{(i)}
\]

\[
+ \frac{1}{5824}(14b_3 + b_3 + 208)(\theta_6^{(i)} - \theta_6^{(i)} - 329\theta_6^{(i)}).
\]
Then $\phi^{(1)} \neq 0$ and we can prove
\begin{equation}
L(s, \phi^{(1)}) = \zeta(s-4)\zeta(s-5)L(s, f_{i_0}),
\end{equation}
by Proposition 3.3. Put
\begin{align*}
\eta_i^{(1)} &= \theta_2 \theta_i^{(1)}, \quad \eta_{i_0}^{(1)} = \theta_2 \theta_i^{(1)} \bigg| T(2), \quad \eta_i^{(2)} = \eta_i^{(3)} \bigg| T(2), \\
\eta_i^{(3)} &= \eta_i^{(2)} \bigg| T(3), \quad V_1 = \langle \eta_i^{(1)}, \eta_i^{(2)}, \eta_i^{(3)}, \eta_i^{(4)}, \eta_i^{(5)}, \eta_i^{(6)} \rangle c, \\
V_2 &= V_0 + V_1.
\end{align*}
By computing Fourier coefficients of $\eta_i^{(3)}$ for $n \cdot \begin{pmatrix} 1 \\ 1/2 \\ 1/2 \end{pmatrix}$, we find $\dim V_1 = 5$, $\dim V_2 = 8$. Applying $T(7)$ and computing Fourier coefficients using (2.7), we get $S_6(T_7^{((7),(7))}) = V_2 + V_3 | T(7)$. More precisely, by computing the Fourier coefficients for $n \cdot \begin{pmatrix} 1 \\ 1/2 \\ 1/2 \end{pmatrix}$, we can see that $V_1$ and $\sum_{i=1}^5 C \eta_i^{(1)} | T(7)$ generate a 8-dimensional space. Then, by the theory in §2, we can conclude that $S_6(T_7^{((7),(7))})$ contains at least 3-pairs of independent zweigliedrig forms. As $V_2 \subset S_6(T_7^{((7),(7))})$, $\dim S_6(T_7^{((7),(7))}) \leq 3$, it must contain exactly 3-pairs of independent zweigliedrig forms. Thus we obtain
\begin{align*}
V_1 &= S_6(T_7^{((7),(7))}), \\
S_6(T_7^{((7),(7))}) &= 8, \quad \dim S_6(T_7^{((7),(7))}) = 3.
\end{align*}
In particular, $V_2$ is invariant under $L$. It must contain two more eingliedrig forms. The characteristic polynomial of $T(2)$ on $V_2$ is $(X^4 - 165X^3 + 7602X - 76248)(X^2 - 90X + 1832)(X^4 + 4X^3 - 636X + 4656)$. Put $d = 193$ and
\begin{align*}
\phi_i^{(2)} &= 2(8013 - 861\sqrt{d})\eta_i^{(1)} + 2(475195 - 5915\sqrt{d})\eta_i^{(2)} + (9969 + 51\sqrt{d})\eta_i^{(3)} \\
&+ (36785 - 973\sqrt{d})\eta_i^{(4)} + 2(1515 - 51\sqrt{d})\eta_i^{(5)} + (66711 + 1461\sqrt{d})\eta_i^{(6)} \\
&+ 4(411143 + 284645\sqrt{d})\eta_i^{(7)} + \left( -\frac{6224}{7} + 3088\sqrt{d} \right) (\theta_i^{(1)} - \theta_i^{(2)} - 329\theta_i^{(3)}).
\end{align*}
Then we can prove that $\phi_i^{(2)}$ and its conjugate $\phi_i^{(3)}$ by $\text{Gal}(Q(\sqrt{d})/Q)$ are $\theta_i$-eingliedrig forms. We have observed the relation
\begin{equation}
L(s, \phi_i^{(2)}) = \zeta(s-4)\zeta(s-5)L(s, f_{i_0}),
\end{equation}
where $f_{i_0}(x) = \sum_{n=1}^\infty c_n e(nx) \in S_6(T_7^{((7),(7))})$(two dimensional) is the normalized cusp form such that $c_1 = 1$, $c_2 = -3 + \sqrt{d}$.

Finally we have calculated eigenvalues of $T(7)$ using Fourier coefficients of $\eta_i^{(1)}$, $1 \leq i \leq 6$.

\textbf{Proposition 4.2.} The eigenvalues of $7 \cdot T(7)$ on $S_6(T_7^{((7),(7))})$ are $7^2$, $7^2$, $7^2$, $-7^2$, $-7^2$ and the six roots of the equation $X^8 + 10X^7 + 230X^6 + 57428X^5 + 5529503X^4 + 57648010X + 13841287201 = 0$. 


}\]
Representations in the space of Siegel modular forms

In the Table below, we set $N_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $N_2 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$.

**Table (I)**

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<tr>
<th>$p$</th>
<th>$a(p)$</th>
<th>$b(p)$, $d=11 \times 179$</th>
<th>$c(p)$</th>
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**Table (II)**

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<th>$A(n N_2)/576$</th>
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**Table (III)**

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## Table (VI)

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5. Conjectures

We now come to the place to meditate upon our experiments and to formulate conjectures.

Conjecture 5.1. Let $F \in S_{\kappa}(\Gamma_\rho)$ be a $\theta_{11}$-eingliedrig form. Let $F|H=\varepsilon F$, $\varepsilon=\pm 1$. Then $F=F(\varphi_1, \varphi_2)$ with $\varphi_1 \in S^+(R,0), \varphi_2 \in S^+(R,2n)$, where $n=k-2$ and $\pm$ is the signature of $\varepsilon$. The Euler product of $F$ corresponds to a pair of forms in $G_2(\Gamma_0^1(p)) \times S_{2k-2}(\Gamma_0^1(p))$ or forms in $S_2^+(\Gamma_0^1(p)) \times S_{2k-4}(\Gamma_0^1(p))$ as (3.9).

This conjecture, which accords with all experiments that we have made, characterizes $\theta_{11}$-eingliedrig forms. One may expect similar characterization for $\theta_3$-eingliedrig forms. However, by the data

$$\dim S_8(\Gamma_0^1(7))=26, \quad \dim S_4(\Gamma_0^1(7))=7,$$

$$\dim S_8(\Gamma_0^1(13))=46, \quad \dim S_4(\Gamma_0^1(13))=9,$$

we see that this expectation cannot be satisfied, since $26 \equiv 7 \pmod{2}$, etc. To formulate more accurate conjectures, we need information about representations of a certain Hecke algebra. We use the notation of §2. Consider three Hecke operators

$$S_1=\Gamma_B w_1 \Gamma_B, \quad S_2=\Gamma_B w_2 \Gamma_B, \quad S_\rho=\Gamma_B w_\rho \Gamma_B,$$

where $w_1$ and $w_2$ are given by (2.1) and $w_\rho=w_1\begin{pmatrix} 0 & 1 \\ -p & 0 \end{pmatrix}$. Then $S_1, S_2$ and $S_\rho$ satisfy the relations

$$\begin{cases} S_i^2=(\rho-1)S_i+p, & i=1, 2, \\ (S_iS_2)^2=(S_2S_i)^2, \\ S_\rho^2=1, \quad S_1S_\rho=S_\rho S_1, \quad (S_2S_\rho)^2=(S_\rho S_2)^2. \end{cases}$$

These relations follow from Iwahori-Matsumoto [8]. In fact, set

$$\tilde{\mathcal{G}}=GSp(2, \mathbb{Q}_p)/\text{the center}, \quad \tilde{\mathcal{K}}=GSp(2, \mathbb{Z}_p)/\text{the center},$$

$$\tilde{B} = \{ k \in \tilde{\mathcal{K}} \mid k \text{ mod } p = \begin{pmatrix} * & * & \star & \star \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & \star \end{pmatrix} \}.$$
Hiroyuki Yoshida

The point is that, if $F$ is $\theta$-eingliedrig, there are three possibilities for the representations of $\mathcal{H}$ generated by $F$.

To determine irreducible representations of $\mathcal{H}$ explicitly is a somewhat laborious task, although there is a work of Kazhdan-Lusztig [9] on general case. In the Appendix, we list explicit realizations of representations $\pi$ of $\mathcal{H}$; if $\dim \pi > 2$, we assumed that $\pi |_{\mathcal{H}_1}$ does not contain the "trivial representations $\pi_1$": $S_1 \mapsto p, S_2 \mapsto p$. This condition is satisfied if $\pi$ is generated by a form in $S_{2\pi}(\Gamma_p)$.

Assume $F \in S_{2\pi}(\Gamma_p)$ is a $\theta$-eingliedrig form which generates an irreducible representation $\pi$ of $\mathcal{H}$. Then there are three possibilities: $\pi \cong \Pi_3$, or $\Pi_4$, or the two dimensional representation

$$\Pi_{\pi}: S_1 \rightarrow \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad S_2 \rightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad S_p \rightarrow \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

of $\mathcal{H}$ (cf. Appendix). By the examples in §4, we see that the "lifted" $\theta$-eingliedrig forms generate the representation $\Pi_{\pi}$. We can now formulate the following (cf. §4. (II))

**Conjecture 5.2.** Let $F \in S_{2\pi}(\Gamma_p)$ be a $\theta$-eingliedrig form which generates the irreducible two dimensional representation $\Pi_{\pi}$ of $\mathcal{H}$. Then the Euler product of $F$ corresponds to a pair of the Eisenstein series in $G(\Gamma_{p}^{(1)}(\mathfrak{p}))$ and of a form in $S_{2\pi-\delta}(\Gamma_{p}^{(1)}(\mathfrak{p}))$.

**Appendix**

1. The one dimensional representations of $\mathcal{H}$. These are

$$S_1 \rightarrow p \text{ or } -1, \quad S_2 \rightarrow p \text{ or } -1, \quad S_p \rightarrow 1 \text{ or } -1.$$ 

According as the above choices, we get eight irreducible representations of $\mathcal{H}$.

Let $\alpha, \beta, \gamma, \delta$ be given by (2.4).

2. The two dimensional representations of $\mathcal{H}$. These are

$$S_1 \rightarrow \begin{pmatrix} p & 0 \\ 0 & -1 \end{pmatrix}, \quad S_2 \rightarrow \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad S_p \rightarrow \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

or

$$S_1 \rightarrow p \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$ 

According as the above choices, we get four irreducible representations of $\mathcal{H}$.

3. The irreducible representations of $\mathcal{H}$ whose dimensions are higher than 2 and whose restriction to $\mathcal{H}_1$ does not contain the "trivial representation": $S_1 \mapsto p, S_2 \mapsto p$. 
Representations in the space of Siegel modular forms

We label the representations of \( \mathcal{G}_1 \) by the corresponding representations of \( \text{Sp}(2, F_p) \) (cf. (2.3)). Set

\[
\lambda = -\frac{(p-1)^3}{2p}, \quad \mu = \frac{(p-1)^3}{p^3+1}.
\]

\[
\Pi_1 : S_1 \rightarrow \begin{pmatrix} \alpha & 0 & \beta \\ 0 & -1 & 0 \\ \gamma & 0 & \delta \end{pmatrix}, \quad S_2 \rightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & -1 \end{pmatrix},
\]

\[
S_\rho \rightarrow \pm \begin{pmatrix} -\lambda & 1 & -1/2p \\ -\lambda^2 \mu & \lambda \mu^{-1} & \lambda/2p \\ -(p^3+1) & -2p & 0 \end{pmatrix}, \quad \Pi_1 \mid \mathcal{G}_1 \equiv \theta_6 \oplus \theta_{15}.
\]

\[
\Pi_2 : S_1 \rightarrow \begin{pmatrix} \alpha & \beta & 0 \\ \gamma & \delta & 0 \\ 0 & 0 & p \end{pmatrix}, \quad S_2 \rightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix},
\]

\[
S_\rho \rightarrow \pm \begin{pmatrix} 0 & 1/(p^3+1) & -1/2p \\ -(p^3+1) \lambda & \lambda \mu^{-1} & -\lambda \mu^{-1} \\ (p^3+1) \lambda & -\lambda^2 \mu & -\lambda \mu^{-1} \end{pmatrix}, \quad \Pi_2 \mid \mathcal{G}_1 \equiv \theta_6 \oplus \theta_{11}.
\]

\[
\Pi_3 : S_1 \rightarrow \begin{pmatrix} \alpha & \beta & 0 \\ \gamma & \delta & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad S_2 \rightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix},
\]

\[
S_\rho \rightarrow \pm \begin{pmatrix} 2p & \mu & 1 \\ -2p \mu & -\mu^2 \lambda & -\mu \lambda^{-1} \\ -2p \mu^{-1} t(1+\lambda^{-1}) \mu^{-1} t(1+\lambda^{-1}) & -\lambda^{-1} t(1+\lambda^{-1}) & -\lambda^{-1} t \end{pmatrix}, \quad \Pi_3 \mid \mathcal{G}_1 \equiv \theta_6 \oplus \theta_{13}.
\]

\[
\Pi_4 : S_1 \rightarrow \begin{pmatrix} \alpha & 0 & \beta \\ 0 & 1 & 0 \\ \gamma & 0 & \delta \end{pmatrix}, \quad S_2 \rightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & 0 \end{pmatrix},
\]

\[
S_\rho \rightarrow \pm \begin{pmatrix} 1-\mu^{-1} t & (1-\mu^{-1} t)/2p & t/(p-1)^3 & -1/2p \\ 2p \mu^{-1} t & \mu^{-1} t & \lambda^{-1} t & 1 \\ 2p \mu^{-1} t & -(1-\mu^{-1} t) & 1+\lambda^{-1} t & 0 \\ -2p \mu^{-1} t(1+\lambda^{-1} t) & (1-\mu^{-1} t)(1+\lambda^{-1} t) & -\lambda^{-1} t(1+\lambda^{-1} t) & -\lambda^{-1} t \end{pmatrix},
\]

\[t \in C, \ t \neq 0, \ \mu, -\lambda. \quad \Pi_4 \mid \mathcal{G}_1 \equiv \theta_6 \oplus \theta_{15} \oplus \theta_{13}.
\]

\[
\Pi_5 : S_1 \rightarrow \begin{pmatrix} \alpha & \beta & 0 \\ \gamma & \delta & 0 \\ 0 & 0 & p \end{pmatrix}, \quad S_2 \rightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\]

\[
S_\rho \rightarrow \pm \begin{pmatrix} 0 & t/(p-1)^3 & 1/(p^3+1) & -1/2p \\ 2p \mu^{-1} t & 1 & 1 \\ 2p \mu^{-1} t & \mu^{-1} t & \mu^{-1} t & 0 \\ -2p \mu^{-1} t(1-\mu^{-1} t) & \mu^{-1} t(1-\mu^{-1} t) & -\mu^{-1} t & 0 \end{pmatrix},
\]

\[t \in C, \ t \neq 0, \ \mu, -\lambda. \quad \Pi_5 \mid \mathcal{G}_1 \equiv \theta_6 \oplus \theta_{15} \oplus \theta_{13}.
\]
where $\nu = \frac{p^2+1}{(p+1)^2}$, $\eta = \frac{2p}{(p+1)^2}$, $t \in \mathbb{C}$, $t \neq 0$, $\pm \mu$, $\pm \lambda$, $\Pi_5 | \mathcal{H}_1 \cong \theta_8 \oplus \theta_{11} \oplus \theta_{15}$.

Let $\pi$ be an irreducible finite dimensional representation of $\mathcal{H}$ on a vector space $V$ over $\mathbb{C}$. For $\pi$ to occur in the representation of $\mathcal{H}$ on $S_k(\Gamma_p)$, there must exist a positive hermitian inner product $(\cdot, \cdot)$ on $V$ such that $S_1$, $S_2$ and $S_\rho$ become hermitian with respect to $(\cdot, \cdot)$. When this is the case, let us call $\pi$ unitarizable. We obtain

**Proposition A.1.** All representation $\pi$ such that $\dim \pi \leq 2$ are unitarizable. $\Pi_1$ and $\Pi_2$ are not unitarizable. $\Pi_3$ is unitarizable. $\Pi_4$ is unitarizable if and only if $0 < t < \mu$. $\Pi_5$ is unitarizable if and only if $-\mu < t < \mu$.

The proof, which is not difficult, is omitted.

**Example A.2.** The zweigliedrig form $20 \eta^{(t)} + \sqrt{-2} \eta^{(t)} \in S_6(\Gamma_9^{(t)}(3))$ generates the irreducible representation $\Pi_5$ with $t = \mu/3 = 2/15$.

**References**