Problems for final test

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0) Homework problems and those similar to it.

1) Find the general solution of $(1+x^2)u_x + yxu_y = 0$;
Sketch some of the characteristic curves.
If possible find the the solution of for the auxiliary conditions
   i) $u(0,y) = y^2$,
   ii) $u(x,0) = x^2$.
If there is no solution, explain why!

2) Solve $u_x - 2\,u_y + u = 1 + e^y$,
with $u(x,0) = 0$.

3) Find all eigenvalues and eigenfunctions of the Boundary value Problem
$X''(x) + \lambda X(x) = 0, \quad 0 < x < 2;$
$X'(0) = 0; \quad X(2) = 0.$

4) Find a series representation of the solution of
$u_t - u_{xx} = 0, \quad 0 < x < 2, \;\; t > 0;$
$u(x,0) = x;$
$u_x(0,t) = 0, \;\; u(2,t) = 0.$
Hints: Work problem # 4 first.

5) Solve the linear PDE

$$yu_x + 2xu_y = 0;$$

for $u = u(x,y)$ with the auxiliary condition $u(x,0) = x$.

6) Solve $u_x + u_y + 2u = e^x$, with $u(0,y) = y^2$.

7) Find all eigenvalues and eigenfunctions of the Sturm-Liouville Problem
$X''(x) + \lambda X(x) = 0, \quad 0 < x < 2;$
$X(0) - 4X'(0) = 0; \quad X(2) = 0.$

8) Find a series representation of the solution of
$u_t - u_{xx} = 0, \quad 0 < x < 2, \;\; t > 0;$
$u(x,0) = 1 - (x-1)^2;$
$u(0,t) + 4u_x(0,t) = 0, \;\; u(0,t) = 0 = 0.$
Hint: Work problem # 7 first.

9) For $u = u(x,t)$ find the solution of the Initial Value Problem
$u_{xt} + u_{tt} = 0,$
$u(x,0) = 0, \;\; u_t(x,0) = \cos x.$
Hint: Use the change of variables $\xi = x$, and $\tau = t - x$.

10) Find the Fourier Cosine Series of $f(x) = e^x$ in the interval $(0, \pi)$.

11) Find the Fourier Sine Series of $f(x) = e^{-x}$ in the interval $(0, \pi)$.

12) Find the Fourier Series of
   i) $f(x) = x^2$ in the interval $(-\pi, \pi)$.
   ii) $f(x) = x|x|$ in the interval $(-\pi, \pi)$.
   iii) $f(x) = 2\sin(27x) + 3\cos(4x)$ in the interval $(-\pi, \pi)$.

13) Determine the regions in the xy -plane for which the following PDE is a) elliptic, b) parabolic, c) hyperbolic.
$u_{xx} + (1-x)u_{xy} + u_{yy} + (\sin x)u_x = 0.$

14) Find the Fourier Cosine Series of $f(x) = 1 + \sin x$ in the interval $(0, \pi)$.

15) Find the Fourier Sine Series of $f(x) = 1 + x$ in the interval $(0, \pi)$.

16) Find the Fourier Series of $f(x) = 1 + x$ in the interval $(-\pi, \pi)$.

17) Consider the equation $u_x + u_{xy} = 0.$
   i) What is its type?
   ii) Substituting $v = u_x$, find the general solution.

18) Determine the type of the equation
$$3u_{xx} + 4u_{xy} + u_{yy} = 0,$$
and transform it to its canonical form.

19) Solve
$u_{tt} - u_{xx} = \cos 3x, \quad 0 < x < \pi, \quad t > 0;$
$u(x,0) = 0; \quad u_t(x,0) = 0;$
$u_x(0,t) = 0, \quad u_x(\pi,t) = 0.$

20) Solve
$u_{xx} + u_{yy} = 0, \quad 0 < x < l, \quad 0 < y < m;$
$u(x,0) = \frac{1}{l}(l-x); \quad u(x,m) = 0;$
$u(0,y) = \frac{1}{m}(m-y); \quad u(l,y) = 0.$

21) Find a general solution of
$2y^2 u_{xy} - y u_{yy} + u_y = 0$
for $y > 0$, $-\infty < x < \infty$ using the change of variables $\xi = x, \quad \eta = x + y^2$.

22) Solve
$u_{tt} + u_{xx} = 0. \;\; 0 < x < 2, \quad 0 < y < 2;$
$u(0,y) = y\,, \; u(2,y) = 0;$
$u(x,0) = 2 - x, \;\; u(x,2) = 0.$

23) Solve the linear equation $(1+x^2)yu_x + xu_y = 0;$
Sketch some of the characteristic curves.

24) Solve $u_x + u_y + 2u = x$, with $u(x,0) = 0$.

25) Consider the initial-boundary value problem for the diffusion equation
$u_t \ = \ u_{xx}. \ \ -2 < x < 2;$
$u_x(t, -2) = u_x(t, 2) = 0;$
$u(0, x) \ = \ x^2.$

26) Solve
$u_{tt} - c^2 u_{xx} = 0. \ \ 0 < x < \infty;$
$u(t, 0) = te^t;$
$u(0, x) \ = \ 0, \ \ u_t(0, x) = x.$

27) Solve $u_t \ - \ ku_{xx} \ + \ bu \ = \ 0$, for $-\infty < x < \infty, \quad t \geq 0, \quad k > 0, \quad b > 0.$
$u(x, 0) \ = \ \phi(x),$
Hint: Make the change of variables $u(x, t) = e^{-bt}v(x, t).$

28) Solve
$u_{tt} - u_{xx} = \sin x; \quad 0 < x, \quad 0 < t;$
$u(x, 0) = \dfrac{1}{x + 1}, \quad u_t(x, 0) = 0;$
$u_x(0, t) = 0.$

29) Solve
$u_t - u_{xx} = 0, \quad 0 < x, \quad 0 < t;$
$u(x, 0) = e^{-x};$
$u(0, t) = 0.$

30) Find all eigenvalues and eigenfunctions of the Sturm-Liouville Problem

$(xX'(x))' + \lambda \frac{1}{x} X(x) = 0, \quad 1 < x < b;$
$X'(1) = 0; \quad X'(b) = 0.$
(Answer: $\lambda_0 = 0$, $X(x) \equiv 1$; $\lambda_k = \nu_k^2 = (\frac{k\pi}{\ln b})^2$, $X_k(x) = \cos(\nu_k \ln x)$; $k = 1, 2, 3, \ldots$.) Hint: Use the change of variables $x = e^s$, $s = \ln x$ to transform the above Sturm- Liouville Problem to an equivalent but easier one.

31) Let $\phi(x) \ = \ \begin{cases} 0 & \text{if } \ x \leq 0 \\ e^{-x} - 1 & \text{if } x \geq 0 \end{cases}.$ Solve
$u_t \ - ku_{xx} \ = \ 0, \ -\infty \ < \ x \ < \ \infty, \ 0 \ < \ t;$
$u(0, x) \ = \ \phi(x)$

32) Solve
$u_{tt} - u_{xx} = xt^2; \quad 0 < x < a, \quad t > 0;$
$u(x, 0) = 0, \quad u_t(x, 0) = x;$
$u(0, t) = 0, \quad u(a, t) = 0.$

33) Find the series expansion of the solution of
$u_{tt} - 4u_{xx} + u_t = 0. \ \ 0 < x < 1;$
$u_x(t, 0) = u_x(t, 1) = 0;$
$u(0, x) \ = \ \cos(\pi x), \ \ u_t(0, x) = 0.$

34) Solve

$u_{tt} - \Delta u = ty, \quad 0 < x < l,\ 0 < y < m, \quad t > 0;$
$u(x, y, 0) = yx;$
$u_t(x, y, 0) = 0;$
$u_x(0, y, t) = 0, \quad u(l, y.t) = 0.$
$u(x, 0, t) = 0, \quad u(x, m, t) = 0.$

35) Solve

$r^2 u_{rr} + r u_r + u_{\varphi\varphi} = 0, \quad 1 < r < b,\ 0 < \varphi < \pi;$
$u_r(1, \varphi) = u_r(b, \varphi) = 0;$
$u(r, 0) = 0,\ u(r, \pi) = f(r).$

36) Solve the following Neumann problem for the diffusion equation on the half- line

$u_t - k u_{xx} = 0,\ 0 < x < \infty,\ 0 < t < \infty;$
$u_x(t, 0) = 0;$
$u(0, x) = \phi(x).$

37) Solve

$\Delta u = 0,$ for $a^2 < x^2 + y^2 < b^2,$
$u = \sin 2\theta + \cos\theta$, for $r = a$,
$u = ,1$ for $r = b.$

38) Let $\| \mathbf{x} \|^2 = \sum_{i=1}^{n} x_1^2$, for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. Show that

i) $\Delta \ln(\| \mathbf{x} - \mathbf{y} \|) = 0$, for $\mathbf{x} \neq \mathbf{y}$, and $n = 2$.

ii) $\Delta(\frac{1}{\| \mathbf{x} - \mathbf{y} \|})^{n-2} = 0$, for $\mathbf{x} \neq \mathbf{y}$, and $n = 3, 4, \ldots$.

39) Let u be a twice differentiable function on a smooth bounded domain D in $\mathbb{R}^n$, solving the Neumann boundary value problem for the Poisson equation:

$$\begin{cases} \Delta u = f\,, & \text{in } D\,, \\ \dfrac{\partial}{\partial n} u = h\,, & \text{on } \partial D\,. \end{cases}$$

Show that then $\int_{\partial D} h d\omega = \int_D f dx$.

40) Let u and v be two solution of the Neumann boundary value problem for the Poisson equation above. Show that then $u - v$ is constant in D.