Problems for final test

0) Homework problems and those similar to it.

1) Find the general solution of \((1 + x^2)u_x + yxu_y = 0\);
   Sketch some of the characteristic curves.
   If possible find the solution of for the auxiliary conditions
   i) \(u(0, y) = y^2\),
   ii) \(u(x, 0) = x^2\).
   If there is no solution, explain why!

2) Solve \(u_x - 2u_y + u = 1 + e^y\),
   with \(u(x, 0) = 0\).

3) Find all eigenvalues and eigenfunctions of the Boundary value Problem
   \(X''(x) + \lambda X(x) = 0, \quad 0 < x < 2;\)
   \(X'(0) = 0; \quad X(2) = 0\).

4) Find a series representation of the solution of
   \(u_t - u_{xx} = 0, \quad 0 < x < 2, \quad t > 0;\)
   \(u(x, 0) = x;\)
   \(u_x(0, t) = 0, \quad u(2, t) = 0.\)
   Hints: Work problem # 4 first.

5) Solve the linear PDE
   \(yu_x + 2xu_y = 0;\)
   for \(u = u(x, y)\) with the auxiliary condition \(u(x, 0) = x\).

6) Solve \(u_x + u_y + 2u = e^x\), with \(u(0, y) = y^2\).

7) Find all eigenvalues and eigenfunctions of the Sturm-Liouville Problem
   \(X''(x) + \lambda X(x) = 0, \quad 0 < x < 2;\)
   \(X(0) - 4X'(0) = 0; \quad X(2) = 0\).

8) Find a series representation of the solution of
   \(u_t - u_{xx} = 0, \quad 0 < x < 2, \quad t > 0;\)
   \(u(x, 0) = 1 - (x - 1)^2;\)
   \(u(0, t) + 4u_x(0, t) = 0, \quad u(0, t) = 0 = 0.\)
   Hint: Work problem # 7 first.

9) For \(u = u(x, t)\) find the solution of the Initial Value Problem
   \(u_{xt} + u_{tt} = 0,\)
   \(u(x, 0) = 0, \quad u_t(x, 0) = \cos x.\)
   Hint: Use the change of variables \(\xi = x\), and \(\tau = t - x.\)
10) Find the Fourier Cosine Series of \( f(x) = e^x \) in the interval \((0, \pi)\).

11) Find the Fourier Sine Series of \( f(x) = e^{-x} \) in the interval \((0, \pi)\).

12) Find the Fourier Series of
   i) \( f(x) = x^2 \) in the interval \((-\pi, \pi)\).
   ii) \( f(x) = x|x| \) in the interval \((-\pi, \pi)\).
   iii) \( f(x) = 2 \sin(27x) + 3 \cos(4x) \) in the interval \((-\pi, \pi)\).

13) Determine the regions in the \(xy\)-plane for which the following PDE is
   a) elliptic, b) parabolic, c) hyperbolic.
   \[ u_{xx} + (1 - x)u_{xy} + u_{yy} + (\sin x)u_x = 0. \]

14) Find the Fourier Cosine Series of \( f(x) = 1 + \sin x \) in the interval \((0, \pi)\).

15) Find the Fourier Sine Series of \( f(x) = 1 + x \) in the interval \((0, \pi)\).

16) Find the Fourier Series of \( f(x) = 1 + x \) in the interval \((-\pi, \pi)\).

17) Consider the equation \( u_x + u_{xy} = 0. \)
   i) What is its type?
   ii) Substituting \( v = u_x \), find the general solution.

18) Determine the type of the equation
   \[ 3u_{xx} + 4u_{xy} + u_{yy} = 0, \]
   and transform it to its canonical form.

19) Solve
   \[ u_{tt} - u_{xx} = \cos 3x, \quad 0 < x < \pi, \quad t > 0; \]
   \[ u(x, 0) = 0; \quad u_t(x, 0) = 0; \]
   \[ u_x(0, t) = 0, \quad u_x(\pi, t) = 0. \]

20) Solve
   \[ u_{xx} + u_{yy} = 0, \quad 0 < x < l, \quad 0 < y < m; \]
   \[ u(x, 0) = \frac{1}{2}(l - x); \quad u(x, m) = 0; \]
   \[ u(0, y) = \frac{1}{m}(m - y); \quad u(l, y) = 0. \]

21) Find a general solution of
   \[ 2y^2u_{xy} - yu_{yy} + u_y = 0 \]
   for \( y > 0, \quad -\infty < x < \infty \) using the change of variables \( \xi = x, \quad \eta = x + y^2. \)

22) Solve
   \[ u_{tt} + u_{xx} = 0. \quad 0 < x < 2, \quad 0 < y < 2; \]
   \[ u(0, y) = y, \quad u(2, y) = 0; \]
   \[ u(x, 0) = 2 - x, \quad u(x, 2) = 0. \]

23) Solve the linear equation \((1 + x^2)y u_x + x u_y = 0;\)
    Sketch some of the characteristic curves.

24) Solve \( u_x + u_y + 2u = x, \) with \( u(x, 0) = 0. \)
25) Consider the initial-boundary value problem for the diffusion equation
\[ u_t = u_{xx}, \quad -2 < x < 2; \]
\[ u_x(t, -2) = u_x(t, 2) = 0; \]
\[ u(0, x) = x^2. \]

26) Solve
\[ u_{tt} - c^2 u_{xx} = 0, \quad 0 < x < \infty; \]
\[ u(t, 0) = te^t; \]
\[ u(0, x) = 0, \quad u_t(0, x) = x. \]

27) Solve \( u_t - ku_{xx} + bu = 0 \), for \( -\infty < x < \infty, \ t \geq 0, \ k > 0, \ b > 0 \).
\[ u(x, 0) = \phi(x), \]
Hint: Make the change of variables \( u(x, t) = e^{-bt} v(x, t) \).

28) Solve
\[ u_{tt} - u_{xx} = \sin x; \quad 0 < x < \infty; \]
\[ u(x, 0) = 1, \quad u_t(x, 0) = x; \]
\[ u(0, t) = 0. \]

29) Solve
\[ u_t - u_{xx} = 0, \quad 0 < x < \infty; \]
\[ u(x, 0) = e^{-x}; \]
\[ u(0, t) = 0. \]

30) Find all eigenvalues and eigenfunctions of the Sturm-Liouville Problem
\[ (x X'(x))' + \lambda \frac{1}{x} X(x) = 0, \quad 1 < x < b; \]
\[ X'(1) = 0; \quad X'(b) = 0. \]
(Answer: \( \lambda_0 = 0 \), \( X(x) \equiv 1 \); \( \lambda_k = \nu_k^2 = (\frac{k\pi}{\ln b})^2 \), \( X_k(x) = \cos(\nu_k \ln x); \ k = 1, 2, 3, \ldots . \) Hint: Use the change of variables \( x = e^s, \ s = \ln x \) to transform the above Sturm- Liouville Problem to an equivalent but easier one.

31) Let \( \phi(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ e^{-x} - 1 & \text{if } x > 0 \end{cases} \). Solve
\[ u_t - ku_{xx} = 0, \quad -\infty < x < \infty, \ 0 < t; \]
\[ u(0, x) = \phi(x) \]

32) Solve
\[ u_{tt} - u_{xx} = xt^2; \quad 0 < x < a, \ t > 0; \]
\[ u(x, 0) = 0, \quad u_t(x, 0) = x; \]
\[ u(0, t) = 0, \quad u(a, t) = 0. \]

33) Find the series expansion of the solution of
\[ u_{tt} - 4u_{xx} + u_t = 0, \quad 0 < x < 1; \]
\[ u_x(t, 0) = u_x(t, 1) = 0; \]
\[ u(0, x) = \cos(\pi x), \ u_t(0, x) = 0. \]
34) Solve
\[ u_{tt} - \Delta u = ty, \quad 0 < x < l, \quad 0 < y < m, \quad t > 0; \]
\[ u(x, y, 0) = yx; \]
\[ u_t(x, y, 0) = 0; \]
\[ u_x(0, y, t) = 0, \quad u(l, y, t) = 0. \]
\[ u(x, 0, t) = 0, \quad u(x, m, t) = 0. \]

35) Solve
\[ r^2 u_{rr} + ru_r + u_{\varphi\varphi} = 0, \quad 1 < r < b, \quad 0 < \varphi < \pi; \]
\[ u_r(1, \varphi) = u_r(b, \varphi) = 0; \]
\[ u(r, 0) = 0, \quad u(r, \pi) = f(r). \]

36) Solve the following Neumann problem for the diffusion equation on the half-line
\[ u_t - ku_{xx} = 0, \quad 0 < x < \infty, \quad 0 < t < \infty; \]
\[ u_x(t, 0) = 0; \]
\[ u(0, x) = \phi(x). \]

37) Solve
\[ \Delta u = 0, \quad \text{for} \quad a^2 < x^2 + y^2 < b^2, \]
\[ u = \sin 2\theta + \cos \theta, \quad \text{for} \quad r = a, \]
\[ u = 0, \quad \text{for} \quad r = b. \]

38) Let \( \| x \|_2^2 = \sum_{i=1}^{n} x_i^2 \), for \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \). Show that
\[ i) \quad \Delta \ln(\| x - y \|) = 0, \quad \text{for} \quad x \neq y, \quad \text{and} \quad n = 2. \]
\[ ii) \quad \Delta \left( \frac{1}{\| x - y \|} \right)^{n-2} = 0, \quad \text{for} \quad x \neq y, \quad \text{and} \quad n = 3, 4, \ldots. \]

39) Let \( u \) be a twice differentiable function on a smooth bounded domain \( D \) in \( \mathbb{R}^n \), solving the Neumann boundary value problem for the Poisson equation:
\[ \begin{cases} 
\Delta u = f, & \text{in } D, \\
\frac{\partial u}{\partial n} = h, & \text{on } \partial D.
\end{cases} \]

Show that then \( \int_{\partial D} h d\omega = \int_D f dx \).

40) Let \( u \) and \( v \) be two solution of the Neumann boundary value problem for the Poisson equation above. Show that then \( u - v \) is constant in \( D \).