Section 6. Elliptic equations in general domains

Firstly we recall Gauss theorem:

$$\int_D \text{div} F = \int_{\partial D} F \cdot \mathbf{n} \, d\omega.$$ 

Since for two twice differentiable functions $u$ and $v$ we have

$$\text{div}(v \nabla u) = \nabla v \nabla u + u \Delta u,$$

Gauss’ theorem applied to $F = v \nabla u$ yields

**First Greens identity:**

$$\int_D \nabla v \nabla u \, dx + \int_D v \Delta u \, dx = \int_{\partial D} (v \nabla u) \cdot \mathbf{n} \, d\omega$$

$$= \int_{\partial D} v \frac{\partial}{\partial n} u \, d\omega.$$ 

There are some immediate consequences for solutions of the Laplace equation.

1) Uniqueness of the Dirichlet problem:

There is at most one continuous function defined on $\overline{D}$, with integrable derivatives up to order two solving

$$\begin{cases}
  \Delta u = f, \\
  u|_{\partial D} = g.
\end{cases}$$

To see this note that with $v = u$ Green’s identity implies

$$\int_D \nabla u \nabla u \, dx + \int_D u \Delta u \, dx = \int_{\partial D} u \frac{\partial}{\partial n} u \, d\omega.$$ 

Setting $u$ to be the difference of two solutions we get that $\| \nabla u \|^2 \equiv 0$, and hence is a constant which by definition is zero at the boundary, so it is identical zero.

2) Setting $v = 1$ in Greens first identity we get for harmonic functions $u$

$$0 = \int_{\partial D} \frac{\partial}{\partial n} u \, d\omega.$$
Second Green’s identity:

Interchanging $u$ and $v$ in First Green’s identity

\[ \int_{D} \nabla v \nabla u \, dx + \int_{D} v \Delta u \, dx = \int_{\partial D} v \frac{\partial}{\partial n} u \, d\omega \]

we get

\[ \int_{D} \nabla v \nabla u \, dx + \int_{D} u \Delta v \, dx = \int_{\partial D} u \frac{\partial}{\partial n} v \, d\omega ; \]

then subtracting from this equation the original one we get

\[ \int_{D} (u \Delta v - u \Delta v) \, dx = \int_{\partial D} (u \frac{\partial}{\partial n} v - v \frac{\partial}{\partial n} u) \, d\omega . \]

Applications:

As the uniqueness theorem indicates the solution of the Laplace equation is determined solely by its values on the boundary. Indeed we have the following representation formula in $\mathbb{R}^3$.

If $u$ is harmonic function continuous up to the boundary in a domain $D$, then we have for all $x_0 \in D$:

\[ u(x_0) = \frac{1}{4\pi} \int_{\partial D} \left( -u \frac{\partial}{\partial n} \frac{1}{\|x - x_0\|} + \frac{1}{\|x - x_0\|} \frac{\partial}{\partial n} u \right) d\omega \]

Proof:

First we note, where defined, both $u$ and $\frac{1}{\|x - x_0\|}$ are harmonic. Then again, since the differentiation and the integration are invariant under translation we can assume that $x_0$ is the origin.

We cut out from the domain $D$ a small ball $B_\varepsilon \subset D$, centered at the origin of radius $\varepsilon$, and denote the remaining set $D_\varepsilon$.

Since both $u$ and $\frac{1}{\|x - x_0\|}$ are harmonic in $D_\varepsilon$ Green”s second identity gives

\[ 0 = \int_{\partial D_\varepsilon} \left( -u \frac{\partial}{\partial n} \frac{1}{\|x\|} + \frac{1}{\|x\|} \frac{\partial}{\partial n} u \right) d\omega \]

that is
\[
\int_{\partial B_\epsilon} \left( -u \frac{\partial}{\partial n} \frac{1}{\| x \|} + \frac{1}{\| x \|} \frac{\partial}{\partial n} u \right) \, d\omega \\
= \int_{\partial D} \left( -u \frac{\partial}{\partial n} \frac{1}{\| x \|} + \frac{1}{\| x \|} \frac{\partial}{\partial n} u \right) \, d\omega.
\]

For the above calculation, note that we have a sign change if we integrate over \( \partial b_\epsilon \) as a part of the boundary of \( D_\epsilon \) to the usual integration over \( \partial B_\epsilon \) as the boundary of the ball \( \partial B_\epsilon \).

Now we verify that the left hand side of the equality above is \( 4\pi u(0) \), as stated by the representation formula:

For points \( x \in S \) we have that the outernormal

\[ n = \frac{1}{\| x \|} x, \]

and therefore

\[ \frac{\partial}{\partial n} \frac{1}{\| x \|} = -\frac{1}{\| x \|^2} = -\frac{1}{r^2} \]

for every point \( x \) on a sphere centered at the origin with radius \( r \). Hence

\[ \int_{\partial B_\epsilon} \left( -u \frac{\partial}{\partial n} \frac{1}{\| x \|} + \frac{1}{\| x \|} \frac{\partial}{\partial n} u \right) \, d\omega \\
= \int_{\partial B_\epsilon} \frac{u}{\epsilon^2} + \frac{1}{\epsilon} \frac{\partial}{\partial r} u \, d\omega_S = 4\pi \int_{\partial B_\epsilon} u \, d\omega_S + 4\pi \epsilon \int_{\partial B_\epsilon} \frac{\partial}{\partial r} u \, d\omega_S
\]

where

\[ \int_{\partial B_\epsilon} u \, d\omega_S \quad \text{and} \quad \int_{\partial B_\epsilon} \frac{\partial}{\partial r} u \, d\omega_S
\]

are the averages of \( u \) and of \( \frac{\partial}{\partial r} u \), over \( \partial B_\epsilon \), respectively.

Since both functions are assumed to be continuous, they approach \( u(0) \) and \( \frac{\partial}{\partial r} u(0) \), respectively, as \( B_\epsilon \), becomes small, and so

\[ 4\pi \int_{\partial B_\epsilon} u \, d\omega_S \to 4\pi u(0), \quad \text{and} \quad 4\pi \epsilon \int_{\partial B_\epsilon} \frac{\partial}{\partial r} u \, d\omega_S \to 0,
\]

as \( \epsilon \to 0 \). Hence

\[ \int_{\partial B_\epsilon} \left( -u \frac{\partial}{\partial n} \frac{1}{\| x \|} + \frac{1}{\| x \|} \frac{\partial}{\partial n} u \right) \, d\omega = 4\pi u(0),
\]
as claimed.
As an immediate consequence we get the Mean value property

Let $S \subset D$ be a sphere with center $x_0$ then we have

$$u(0) = \frac{1}{A(S)} \int_S \frac{\partial}{\partial n} u d\omega,$$

where $A(S)$ is the area of the sphere $S$.

Proof:
Because of the translation invariance of the differentiation and integration we can assume that $S$ is a sphere with center $x_0 = 0$, and radius $R$.
So we have

$$u(0) = \frac{1}{4\pi} \int_S \left( -u \frac{\partial}{\partial n} \frac{1}{\|x\|} + \frac{1}{\|x\|} \frac{\partial}{\partial n} u \right), d\omega$$

Since $\|x\| = R$, constant on the sphere we have

$$\int_S \frac{1}{\|x\|} \frac{\partial}{\partial n} u d\omega = 0,$$

by the above remark 2) and $\frac{\partial}{\partial n} \frac{1}{\|x\|} = \frac{1}{R^2}$. With $A(S) = 4\pi R^2$, the statement follows.

3) Maximum principle;

Either a (smooth) solution of the Laplace equation is constant or for all $x$ in $D$ we have $u(x) < \max\{u(y) \mid y \in \overline{D}\}$.

Proof:
Let $m = \max\{u(y) \mid y \in \overline{D}\}$. Suppose the statement is not true, then the set $\{x \in D \mid u(x) = m\}$ has a boundary point $x_0$ in $D$, with $u(x_0) = m$.
Hence there is a sphere $S \subset D$ on which $u$ has values less than the maximum. Since $u$ is continuous, that implies that the average over the sphere can not be the maximum.