5. Boundary Value Problems (using separation of variables).

Seven steps of the approach of separation of Variables:

1) Separate the variables:
   (by writing e.g. \( u(x,t) = X(x)T(t) \) etc..

2) Find the ODE for each “variable”.

3) Determine homogenous boundary values to set up a Sturm-Liouville problem.

4) Find the eigenvalues and eigenfunctions.

5) Solve the ODE for the other variables for all different eigenvalues.

6) Superpose the obtained solutions

7) Determine the constants to satisfy the boundary condition.

Second example: Initial boundary value problem for the wave equation with periodic boundary conditions on \( D = (−\pi, \pi) \times (0, \infty) \)

\[
\begin{align*}
\begin{cases}
   u_{tt} - c^2 u_{xx} = 0 & \text{(Hyperb. PDE)} \\
   u(x,0) = f(x) & \text{IC for } u, \\
   u_t(x,0) = g(x) & \text{IC for } u', \\
   u(-\pi,t) = u(\pi,t) & \text{BC for } u \\
   u_x(-\pi,t) = u_x(\pi,t) & \text{BC for } u_x
\end{cases}
\end{align*}
\]

(IVBP)

Step 1.

\[
\begin{align*}
   u(x,t) &= X(x)T(t) \\
   u_{xx}(x,t) &= X''(x)T(t) \\
\end{align*}
\]

and

\[
\begin{align*}
   u_{tt}(x,t) &= X(x)T''(t)
\end{align*}
\]
Step 2.
Substituting that into the equation and and dividing the result by \(c^2 X(x) T(t)\)
yields the ODEs
\[X'' + \lambda X = 0 \text{ and } T'' + \lambda c^2 T = 0.\]

Step 3.
We obtain the B.C:
\[X(-\pi) = X(\pi), \text{ and } X'(-\pi) = X'(\pi)\]

Step 4.
The eigenvalues and eigenfunctions of
\[
\begin{aligned}
X'' + \lambda X &= 0, \\
X(-\pi) &= X(\pi), \\
X'(-\pi) &= X'(\pi)
\end{aligned}
\]
are
\[\lambda_0 = 0 \text{ with eigenfunction } \phi(x) \equiv 1\]
and
\[\lambda_n = n^2 \text{ with eigenfunctions } \phi_n(x) = \cos nx \text{ and } \psi_n(x) = \sin nx.\]

Step 5.
1) For \(\lambda_0 = 0\), the ODE for \(T_0\) reads:
\[T''_0 = 0;\]
with general solution \(T_0 = A_0 + B_0 t.\)

2) For \(\lambda_n = n^2\), the ODE for \(T_n\) reads:
\[ T''_n + n^2 c^2 T_n = 0; \]

with general solution \( \tau_n(t) = A_n \cos nct + B_n \sin nct. \)

Step 6.

We get the following solution of the PDE satisfying the boundary conditions:

\[
u_0(x, t) = (\tilde{A}_0 + \tilde{B}_0 t)C \quad \text{and} \quad u_n(x, t) = (\tilde{A}_n \cos nct + \tilde{B}_n \sin nct)(\tilde{C}_n \cos nx + \tilde{D}_n \sin nx) \]

“Streamlining” the constants, we write these as

\[
u_0(x, t) = (A_0 + B_0 t) \quad \text{and} \quad u_n(x, t) = (A_n \cos nct + B_n \sin nct) \cos nx + (C_n \cos nct + D_n \sin nct) \sin nx.\]

Superposing these functions we get

\[
u(x, t) = (A_0 + B_0 t) + \sum_{n=1}^{\infty} [(A_n \cos nct + C_n \sin nct) \cos nx + (B_n \cos nct + D_n \sin nct) \sin nx] \]

which is a solution of the PDE satisfying the homogeneous boundary condition, provided the coefficients allow for an appropriate convergence of the sequence.

Step 7.

Determine the coefficients so that \( u(x, t) \) satisfies the initial conditions.

1) First initial condition gives:

\[ f(x) = A_0 + \sum_{n=1}^{\infty} [A_n \cos nx + B_n \sin nx] \]
If we choose these constants to be the coefficients of the Fourier Series of \( f \), then the first IC is satisfied:

\[
A_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f \, dx, \quad A_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f \cos nx \, dx \quad \text{and} \quad B_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f \sin nx \, dx
\]

2. Second initial condition gives:

\[
g(x) = u_t(x, 0) = B_0 + \sum_{n=1}^{\infty} [(A_n - nc \sin nct + C_n nc \cos nct) \cos nx + (B_n - nc \sin nct + D_n nc \cos nct) \sin nx]_{t=0}
\]

\[
= B_0 + \sum_{n=1}^{\infty} [ncC_n \cos nx + ncD_n \sin nx]
\]

If we choose the constants \( B_0 \), \( ncC_n \) and \( ncD_n \), to be the coefficients of the Fourier Series of \( g \) then the second IC is satisfied: so

\[
B_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} g \, dx, \quad C_n = \frac{1}{nc\pi} \int_{-\pi}^{\pi} g \cos nx \, dx \quad \text{and} \quad D_n = \frac{1}{nc\pi} \int_{-\pi}^{\pi} g \sin nx \, dx
\]

With the above choice of constants \( u(x, t) \) is the solution of the problem.

Example 3.

Now we deal with a mixed boundary value problem for the Laplace equation.
\( \begin{align*}
\{ & u_{xx} + u_{yy} = 0 \quad \text{Laplace EQ. in } (0, l) \times (0, m) \\
& u_y(x, 0) = 0 \quad \text{NC at } y = 0, \\
& u(x, m) = 0 \quad \text{DC at } y = m, \\
& u(0, y) = g(y) \quad \text{nonhomogeneous DC at } x = 0 \\
& u(l, y) = 0 \quad \text{DC at } x = l 
\end{align*} \)

Step 1.

\begin{align*}
& u(x, y) = X(x)Y(y) \\
& u_{xx} = X''Y
\end{align*}

and

\begin{align*}
& u_{yy} = XY''
\end{align*}

Step 2.

Substituting that into the equation and and dividing the result by \( XY \) yields the ODEs

\begin{align*}
X'' + \lambda X = 0 \text{ and } Y'' \mp \lambda Y = 0.
\end{align*}

Step 3.

We obtain the B.C:

\begin{align*}
& Y'(0) = Y(m) = 0, \text{ and } X(l) = 0, \quad X(0)Y(y) = g(y).
\end{align*}

So we have \( Y \) as the unkown in the Sturm- Liouville problem.

Step 4.

We have to find the eigenvalues and eigenfunctions of

\begin{align*}
\begin{cases} 
Y'' + \lambda Y = 0, \\
Y'(0) = Y(m) = 0 
\end{cases}
\end{align*}

i ) \( \lambda < 0 \)

No negative eigen values since \( B = 0 \).
ii ) $\lambda = 0$

The general solution is $\phi(y) = A + By$, and we get from the first BC

$$0 = \phi'(0) = B$$

and then from the second

$$0 = \phi(m) = A$$

hence zero is not an eigenvalue.

iii ) $\lambda > 0$

The general solution is $\phi(y) = A \cos \nu y + B \sin \nu y$. with $\nu = \sqrt{\lambda}$ and we get from the first B.C

$$0 = \nu (A(- \sin \nu y) + B \cos \nu y)|_{y=0} = \nu B , \Rightarrow B = 0 ,$$

and then from the second BC

$$0 = A \cos \nu m ,$$

which allows for eigenvalues if $\cos \nu m = 0$ or $\nu m = (n - \frac{1}{2})\pi$, that is

$$\lambda_n = \left(\frac{(2n - 1)\pi}{2m}\right)^2, \quad n = 1, 2, \ldots ;$$

with eigenfunction

$$\phi_n(y) = \cos\left(\frac{(2n - 1)\pi y}{2m}\right).$$

Step 5.

With $\lambda_n = \left(\frac{(2n - 1)\pi}{2m}\right)^2 = : \nu_n^2$, the ODE

$$X_n'' - \lambda X_n = 0;$$

has the general solution $\psi_n(x) = A_n \cosh \nu_n x + B_n \sinh \nu_n x$.

Step 6.

So the solutions

$$u_n(x, t) = (A_n \cosh \nu_n x + B_n \sinh \nu_n x)(\cos \nu_n y)$$
of the PDE also satisfy the boundary conditions $u_y(x,0) = u(x,m) = 0$.

Superposing these functions we get

$$u(x,y) = \sum_{n=1}^{\infty} (A_n \cosh \nu_n x + B_n \sinh \nu_n x)(\cos \nu_n y)$$

which is a solution of the PDE satisfying the homogeneous boundary condition for $y$, provided the coefficients allow for an appropriate convergence of the sequence.

Step 7.

Determine the coefficients so that the PDE satisfies the other boundary conditions:

In order to deal firstly with the homogeneous boundary condition we write

$$u(x,y) = \sum_{n=1}^{\infty} (A_n \cosh \nu_n (x-l) + B_n \sinh \nu_n (x-l))(\cos \nu y).$$

(Recall: Solutions of ODEs with constant coefficients are translation invariant.) We get from the second B.C:

$$0 = u(l,y) = \sum_{n=1}^{\infty} A_n$$

Hence $A_n = 0$, $n = 1, 2, \ldots$, and the first BC requires

$$g(y) = \sum_{n=1}^{\infty} B_n \sinh \nu_n (-l)](\cos \nu_n y)$$

So if we choose the constants $B_n \sinh \nu_n (-l)$ to be the coefficients of the generalized Fourier Series of $g$, then the BC for $x = 0$ is satisfied.

Consequently:

$$B_n = \frac{2}{m \sinh \nu_n (-l)} \int_0^m g(y) \cos \left(\frac{2n-1}{2m} \pi y \right) dy.$$
value problem for the Laplace equation is:

\[ u(x, y) = \sum_{n=1}^{\infty} \sinh\left( \frac{(2n-1)\pi}{2m} (x - l) \right) \cos\left( \frac{(2n-1)\pi}{2m} y \right). \]

Remark:

If the boundary conditions are inhomogeneous at more than one side of the rectangle \((0, l) \times (0, m)\) then we separate the problem into problems with inhomogeneous BC given at one side only, and we obtain the solution by superposing the solution of the separated problems.

Inomogeneous problems and higher dimensional problems

As a first example let us consider

\[
\begin{cases}
  u_t - k u_{xx} = f(x, t) & \text{(inhom. PDE)} \\
  u(x, 0) = 0 & \text{Initial condition for } u , \\
  u(0, t) = 0 & \text{BC at } x = 0 \\
  u(\pi, t) = 0 & \text{BC at } x = \pi
\end{cases}
\]

Our first approach is to find solution of the kind

\[ u(x, t) = \sum_{n=1}^{\infty} A_n(t) \varphi_n(x) , \]

Where \((\varphi_n)\) is a sequence of eigenfunctions of Sturm- Liouville problem of the associated homogeneous problem.

In our case that is the sequence of eigenfunctions of the Fourier Sine Series.

So here we have:

\[ u(x, t) = \sum_{n=1}^{\infty} A_n(t) \sin nx . \]

Assuming that we can interchange the differentiation with the infinite sum, we calculate
\[ u_t(x, t) = \sum_{n=1}^{\infty} A'_n(t) \sin nx. \]
\[ u_{xx}(x, t) = \sum_{n=1}^{\infty} A_n(t)(-n^2 \sin nx). \]

With those derivatives the PDE reads
\[ \sum_{n=1}^{\infty} A'_n(t) \sin nx + k \sum_{n=1}^{\infty} A_n(t)n^2 \sin nx = f(x, t) \]

Now we multiply the inhomogeneous PDE with an eigenfunction \( \varphi_k = \sin(kx) \), say, and integrate over the domain \((0, \pi)\). Because of the orthogonality of the eigenfunction we get
\[ A'_n(t) \int_0^\pi \sin^2 nxdx + kn^2 A_n(t) \int_0^\pi \sin^2 nxdx = \int_a^b f(x, t) \sin nx. \]

Dividing by \( \int_0^\pi \sin^2 nxdx \) we get on the right hand side a function in \( t \) say,
\[ \psi_n(t) = \left( \int_0^\pi \sin^2 nxdx \right)^{-1} \int_0^\pi f(x, t) \sin nx dx. \]

and we get the first order ODE for \( A_n(t) \) as
\[ A'_n(t) + kn^2 A_n(t) = \psi_n(t). \quad (*) \]

The initial condition
\[ u(x, 0) = 0, \]

is satisfies if we add the initial condition
\[ A_n(0) = 0 \]

to the ODE. This is a first order ODE for which we can write down the
solution in the form
\[ A_n(t) = e^{-n^2kt} \int_0^t \psi_n(\tau)e^{n^2k\tau}d\tau, \]
which in turn we can write as
\[ A_n(t) = \int_0^t \psi_n(\tau)e^{-n^2k(t-\tau)}d\tau. \]

Example: \( f(x,t) = t, \) so
\[ \psi_n(t) = \frac{2}{\pi} \int_0^\pi t \sin nx dx = \begin{cases} 0 & \text{if } n, \text{ is even} \\ \frac{4t}{n\pi} & \text{if } n, \text{ is odd} \end{cases}. \]

In that case a more direct approach to solve the ODE’s is possible. We have that \( A_n(t) = C_n e^{-kn^2t} \) and is a solution of the homogeneous equation for \((*)\). That is the case if \( n \) is even.

If \( n \) is odd, we use as trial solutions for a particular solution of \((*)\)
\[ (A_n)_p(t) = \alpha t + \beta, \text{ with } (A_n)'_p(t) = \alpha \]
From the Ode we get
\[ \alpha + kn^2(\alpha t + \beta) = \frac{4t}{n\pi} \]
Comparing coefficients we get
\[ kn^2\alpha t = \frac{4t}{n\pi}, \text{ and } \alpha + kn^2\beta = 0 \]
or
\[ \alpha = \frac{4}{\pi kn^3}, \text{ and } \beta = -\frac{4}{\pi k^2 n^5}. \]
Finally the initial condition \( A_n(0) = 0, \) gives
\[ C_n = 0 \text{ if } n \text{ even and } C_n = -\beta_n = \frac{4}{\pi k^2 n^5}, \text{ if } n \text{ is odd.} \]
Altogether, with \( 2m - 1 = n \) we have
\[ u(x, t) = \sum_{m=1}^{\infty} A_n(t) \sin(2m - 1)x. \]

with
\[ A(t) = \left( \frac{4}{\pi k^2(2m - 1)^5} e^{-k(2m-1)^2t} + \frac{4}{\pi k(2m - 1)^3} t - \frac{4}{\pi k^2(2m - 1)^5} \right). \]

**Inhomogeneous boundary conditions**

As an example let us deal here with the one-dimensionnal wave equation (vibrating string equation) on \((0, l) \times (0, \infty)\):

\[
\begin{cases}
  u_{tt} - c^2 u_{xx} = 0 & \text{(inhom. PDE)} \\
  u(x, 0) = 0 & \text{Initial cond. for } u, \\
  u_t(x, 0) = 0 & \text{Initial cond. for } u_t, \\
  (u(0, t) = g(t)) & \text{BC at } x = 0 \\
  u(l, t) = h(t) & \text{BC at } x = l
\end{cases}
\]

The most straight forward approach here is to write \(u(x, t) = w(x, t) + v(x, t)\) where \(v(x, t)\) is a “simple function” satisfying the boundary conditions.

Then \(w(x, t) = u(x, t) - v(x, t)\) satisfies the inhomogeneous PDE with homogeneous B.C.:

Usually we set
\[ v(x, t) = g(t)(1 - x \frac{1}{l}) + h(t)(x \frac{1}{l}). \]

We calculate
\[ v_{tt} = g''(t)(1 - x \frac{1}{l}) + h''(t)(x \frac{1}{l}), \]
and
\[ v_{xx} = 0. \]

So we need \(w\) to be a solution of...
\[
\begin{aligned}
    \begin{cases}
        w_{tt} - c^2 w_{xx} &= -\left(g''(t)(1 - \frac{1}{l} x) + h''(t)(\frac{1}{l} x)\right); \\
        w(x, 0) &= -(g(0)(1 - \frac{1}{l} x) + h(0)(\frac{1}{l} x)), \\
        w_t(x, 0) &= -(g'(0)(1 - \frac{1}{l} x) + h'(0)(\frac{1}{l} x)), \\
        (w(0, t) = w(l, t) = 0).
    \end{cases}
\end{aligned}
\]

which we can solve using the discussed methods.

**Elliptic equations on spherical domains.**

(1) \[
\begin{aligned}
    &u_{xx} + u_{yy} = 0 \quad \text{on} \quad \sqrt{x^2 + y^2} < R \\
    &f(x, y) = (x, y) \quad \text{for} \quad \sqrt{x^2 + y^2} = R
\end{aligned}
\]

We introduce a change of variables setting \[U(r, \varphi) = u(x, y)|_{x = r \cos \varphi, y = r \sin \varphi} \]

then \[u_{xx} + u_{yy} = 0 \quad \text{for} \quad \sqrt{x^2 + y^2} < R, \]

imply that

(2) \[
\begin{aligned}
    &U_{rr} + \frac{1}{r} U_r + \frac{1}{r^2} U_{\varphi\varphi} = 0 \quad \text{for} \quad 0 < r < R, -\pi < \varphi < \pi, \\
    &U(R, \varphi) = F(\varphi), \quad \text{with} \quad F(\varphi) = f(R \cos \varphi, R \sin \varphi)
\end{aligned}
\]

Somehow the set of auxiliary conditions does not seem to be complete but if we separate the variables first, this will give us a better picture of what will be needed.

Step 1: We set

\[U(r, \varphi) = \mathcal{R}(r) \Phi(\varphi), \quad \text{and get} \]

\[U_r = \mathcal{R}' \Phi, \quad U_{rr} = \mathcal{R}'' \Phi \quad \text{and} \quad U_{\varphi\varphi} = \mathcal{R} \Phi''.\]

The PDE of (2) implies

\[
\mathcal{R}'' \Phi + \frac{1}{r} \mathcal{R} \Phi + \frac{1}{r^2} \mathcal{R} \Phi'' = 0,
\]

Step 2: Multiplying with \[\frac{r^2}{\mathcal{R} \Phi} .\] yields
\begin{align*}
\frac{r^2 \mathcal{R}'' + r \mathcal{R}'}{\mathcal{R}} + \frac{\Phi''}{\Phi} &= 0,
\end{align*}
which gives us the ODE’s
\begin{align}
(3) \quad r^2 \mathcal{R}'' + r \mathcal{R}' &\mp \lambda \mathcal{R} = 0, \quad (2) \quad \Phi'' \pm \lambda \Phi = 0.
\end{align}

Step 3. We would like to work with $\Phi$ as the ODE for a Sturm-Liouville problem. In order to do this we observe that the “reasonable” requirement that the solution is continuously differentiable implies
\begin{align*}
U(r, -\pi) &= U(r, \pi) \quad \text{and} \quad U_\phi(r, -\pi) = U_\phi(r, \pi).
\end{align*}

Step 4 For $\Phi$, we get the Sturm-Liouville problem
\begin{align*}
\text{ST.L.} \quad \begin{cases}
\Phi'' + \lambda \Phi = 0 \\
\Phi(-\pi) = \Phi(\pi) \\
\Phi'(-\pi) = \Phi'(\pi)
\end{cases}
\end{align*}
with the eigen values $\lambda_0 = 0$, $\lambda_n = n^2$, $n = 1, 2, \ldots$ and the eigen function $1, \cos x, \sin x, \cos 2x \sin 2x \ldots$.

Step 5 We have to consider the ODE’s
\begin{align*}
r^2 \mathcal{R}'' + r \mathcal{R}' &= 0 \quad \text{(for } \lambda = 0)\quad \\
\quad \text{and} \quad r^2 \mathcal{R} + r \mathcal{R}' - n^2 \mathcal{R} &= 0 \quad \text{(for } \lambda_n = n^2)\quad .
\end{align*}
For $\lambda = 0$ we get with $w = \mathcal{R}'$
\begin{align*}
w'(\pi) + \frac{1}{\pi} w(r) &= 0.
\end{align*}
The solution can be found by a “separation of variables approach”. We get
\begin{align*}
\frac{w'}{w} &= -\frac{1}{r} \\
\ln w &= \ln(-r) + C \\
w &= C_1 \frac{1}{r}.
\end{align*}
or
\[ \phi_0(r) = A_0 + B_0 \ln(r) \]

The second ODE an Euler equation has the solutions \[ \phi_n(r) = A_n r^n + B_n r^{-n} \]

Step 6

Now superposing the solution of (1) and (2) we get
\[ U(r, \phi) = A_0 + \sum_{n=1}^{\infty} r^n \left( A_n \cos n\varphi + B_n \sin n\varphi \right) \]
\[ + C_0 (\ln(r)) + \sum_{n=1}^{\infty} \left( \frac{1}{r} \right)^n \left( C_n \cos n\varphi + D_n \sin n\varphi \right) \]

Step 7

Since \( \ln(r) \) and \( \frac{1}{r^n} \) are unbounded the functions \( u_n(r, \theta) \) which need to superpose are not solutions of the PDE if those terms a part of \( u_n(r, \theta) \). So we set the coefficients \( C_n \), \( n = 0, 1, 2, \ldots \) and \( D_n \), \( n = 1, 2, \ldots \) zero. Then
\[ u(r, \varphi) = A_0 + \sum_{n=1}^{\infty} r^n (A_n \cos n\varphi + B_n \sin n\varphi) \]
Is a solution of the PDE in the disk and we get we get from the initial condition
\[ U(R, \varphi) = F(\varphi) = A_0 + \sum_{n=1}^{\infty} R^n A_n \cos n\varphi + \sum_{n=1}^{\infty} R^n B_n \sin n\varphi \]
which are satisfied if \( A_0 \), \( R^n A_n \), \( R^n B_n \) are the Fourier coefficients of \( F \), i.e.:
\[ A_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(\varphi) d\varphi \]
\[ A_n = \frac{1}{\pi R^n} \int_{-\pi}^{\pi} F(\varphi) \cos n\varphi d\varphi \]

\[ B_n = \frac{1}{\pi R^n} \int_{-\pi}^{\pi} F(\varphi) \sin n\varphi d\varphi \]

Note, that we can not set the coefficients \( C_n, n = 0, 1, 2, \ldots \) and \( D_n, n = 1, 2, \ldots \) to be zero in case the domain is a “washer”, say

\[ D = \{(x, y) \mid 0 < a^2 < x^2 + y^2 < R^2\} \]

In that case there will be an additional boundary condition for \( x^2 + y^2 = a^2 \).

and we get from the BC’s two equation for each pair of coefficients \( A_n, C_n \) and two equation for the pair of coefficients \( B_n, D_n \), respectively, c.f. the related Homework problem.

We want to consider the solution for the Disk somewhat closer:

We have

\[ u(r, \phi) = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(\theta) d\theta \]

\[ + \sum_{n=1}^{\infty} \frac{r^n}{\pi R^n} \int_{-\pi}^{\pi} F(\theta)(\cos n\phi \cos n\theta + \sin \phi \sin n\theta) d\theta \]

\[ = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(\theta)(1 + 2 \sum_{n=1}^{\infty} \left( \frac{r}{R} \right)^n \cos n(\phi - \theta)) d\theta \]

\[ = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(\theta)(1 + \sum_{n=1}^{\infty} \left( \frac{r}{R} \right)^n (e^{in(\phi-\theta)} + e^{-in(\phi-\theta)}) d\theta \]
\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} F(\theta)(1 + \left( \frac{re^{i(\phi-\theta)}}{R - re^{i(\phi-\theta)}} + \frac{re^{-i(\phi-\theta)}}{R - re^{-i(\phi-\theta)}} \right))d\theta
\]

and so we get Poisson's integral formula

\[
u(r, \phi) = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(\theta) \frac{R^2 - r^2}{R^2 + r^2 - 2rR \cos(\phi - \theta)} d\theta
\]

For \( r = 0 \) we get

\[
u(0, \phi) = \frac{1}{2\pi R} \int_{-\pi}^{\pi} U(R, \theta) R d\theta,
\]

That is the value of \( u \) at the center is the average of the values of \( u \) at the boundary of the disk.

Remark:

Consequences: Mean Value Theorem and Maximum Principle for solutions of the Laplace equations.

**Higher dimensional problems**

**Cooling of the sphere:**

Under the assumption of a rotational symmetric initial temperature distribution and a constant outer temperature distribution (normalized to zero the initial boundary value problem for heat distribution in a sphere

\[
\begin{aligned}
& u_t - k \Delta u = 0, \quad t > 0, \quad \| x \| < \pi, \\
& u(x, 0) = f(\| x \|), \\
& u(x, t) = 0, \quad \text{if } \| x \| = \pi,
\end{aligned}
\]

becomes
\[
\begin{align*}
\begin{cases}
  u_t - k(u_{\rho\rho} + \frac{2}{\rho} u_\rho) = 0, & t > 0, \rho < \pi, \\
  u(\rho,0) = f(\rho), \\
  u(\pi,t) = 0.
\end{cases}
\end{align*}
\]

(Assuming that the heat distribution is only a matter of the distance to the center the derivatives with respect to \( \phi \) are vanishing.)

Separating the variables \( u(\rho,t) = \mathcal{R}T \) we get
\[
T' \mathcal{R} + k(\mathcal{R}'' T + \frac{2}{\rho} (\mathcal{R}' T)) = 0, \quad \text{and so}
\]
\[
\frac{T'}{kT} = -\frac{\mathcal{R}'' + \frac{2}{\rho}(\mathcal{R}')}{\mathcal{R}} = \lambda
\]

Since we have now choice for which variable we might use for a Sturm - Liouville problem, we need to consider
\[
\mathcal{R}'' + \frac{2}{\rho} \mathcal{R}' + \lambda \mathcal{R} = 0, \quad 0 < \rho < \pi
\]

But this is not in the form
\[
(\rho X')' - qX + \lambda m X = 0.
\]

Further more the boundary conditions \( 0 = u(\pi,t) = \mathcal{R}(\pi) T(t) \), provides one homogeneous BC
\[
\mathcal{R}(\pi) = 0
\]

but not a second one.

so we are not dealing with a regular Sturm - Liouville problem, here.

Introducing the functions \( X(\rho) = \rho \mathcal{R}(\rho) \) we get for \( X \) the ODE
\[
X'' + \lambda X = 0
\]

and so for \( \nu^2 = \lambda > 0 \) we have the solutions
\[ R(\rho) = \frac{1}{\rho} C_1 \cos(\nu \rho) + \frac{1}{\rho} C_1 \sin(\nu \rho). \]

Since we can use only bounded solutions we would like to disregard the unbounded functions in this sum. So replacing the second boundary condition by the condition that the solution need to be bounded might replace the second boundary condition. Then multiplying the ODE with \( \rho^2 \) gives the singular Sturm-Liouville problem

\[
\begin{cases}
\rho^2 R'' + 2\rho R' + \rho^2 \lambda R = 0, & 0 < x < \pi, \\
R(\pi) = 0, \\
R \text{ a bounded function in } (0, \pi)
\end{cases}
\]

whose eigenfunctions have the properties discussed in the last chapter with respect to the weight function \( m(\rho) = \rho^2 \).

Is it an exercise using the above transformation to show that all eigen values are \( \lambda_n = n^2 \), \( n = 1, 2, 3, \ldots \) with eigenfunctions

\[ \phi_n(\rho) = \frac{\sin(n\rho)}{\rho}. \]

The solutions of the ODE for \( T \) are \( e^{-n^2 kt} \) so that we have

\[ u_n(\rho, t) = A_n e^{-n^2 kt} \frac{\sin(n\rho)}{\rho}. \]

are bounded solutions of the PDE with \( u_n(\pi, t) = 0 \).

For \( u(\rho, t) = \sum_{n=1}^{\infty} u_n(\rho, t) \), we determine the coefficients from the remaining auxiliary condition:

\[ f(\rho) = u(\rho, 0) = \sum_{n=1}^{\infty} \frac{A_n}{\rho} \sin(n\rho). \]

With the generalized Fourier coefficient of the eigenfunctions \( \phi_n(\rho) \) that is

\[ A_n = \left( \int_0^\pi \left( \frac{\sin(\rho)}{\rho} \right)^2 \rho^2 d\rho \right)^{-1} \int_0^\pi f(\rho) \frac{\sin \rho}{\rho} \rho^2 d\rho. \]
\[
\int_0^\pi \sin^2(\rho) \, d\rho \int_0^\pi f(\rho) \rho \sin \rho \, d\rho.
\]

Note that that \( A_n \) are the Fourier Sine coefficients of \( \rho f \), which we would get if we work the entire problem with the transformed functions.

**Two dimensional Wave equation**

Example

\[
\begin{cases}
  u_{tt} - c^2 \Delta w = 0, & (x, y, t) \in (0, L) \times (0, M) \times \mathbb{R}, \\
  u(x, y, 0) = f(x, y), \\
  u_t(x, 0) = g(x, y), \\
  u(0, y, t) = u(L, y, t) = 0, \\
  u_y(x, 0, t) = u_y(0, M, t) = 0,
\end{cases}
\]

Separation of variables:

Substituting \( u(x, y, t) = X(x)Y(y)T(t) \) into the PDE we get

\[
X(x)Y(y)T''(t) - c^2(X''(x)Y(y)T(t) + X(x)Y''(y)T(t)) = 0,
\]

or

\[
\frac{T''}{c^2T} = \frac{X''}{X} + \frac{Y''}{Y} = -\lambda
\]

this gives us first the ODE’s for \( T, X \) and \( Y \):

\[
T'' + c^2\lambda T = 0
\]

\[
X'' + \mu X = 0
\]

\[
Y'' + \kappa Y = 0 \quad \text{with} \quad \lambda = \mu + \kappa.
\]

From the boundary condition we get the Sturm-Liouville problems

\[
\begin{cases}
  X'' + \mu X = 0 \\
  X(0) = X(L) = 0
\end{cases}
\]
and
\[
\begin{aligned}
Y'' + \kappa Y &= 0 \\
Y'(0) &= Y'(M) = 0.
\end{aligned}
\]

with are the Sturm-Liouville problems for the Fourier Sine and Fourier Cosine Series, respectively. So we have the eigen values \( \mu_m = \left( \frac{m\pi}{L} \right)^2 \) with eigen functions
\[
\phi_m = \sin(\sqrt{\mu_m} x), \quad m = 1, 2, 3, 4, \ldots
\]
for the first Sturm-Liouville problem.

For the second Sturm-Liouville problem we have the eigenvalues \( \kappa_k = \left( \frac{k\pi}{M} \right)^2 \) with eigen functions
\[
\psi_k = \cos(\sqrt{\kappa_k} y), \quad k = 1, 2, 3, 4, \ldots
\]
and
\[
\psi_0 \equiv 1.
\]

For each combination \( \lambda_{mk} = \mu_m + \kappa_k \) we get the solution
\[
\tau_{mk} = A_{mk} \cos(c\sqrt{\mu_m + \kappa_k} t) + B_{m,k} \sin(c\sqrt{\mu_m + \kappa_k} t)
\]

of the ODE’s for \( T \)
\[
T'' + c^2 \lambda_{mk} T = 0.
\]

(Step 6) So the functions
\[
u_{m,k}(x, y, t) = A_{m,k} \cos(c\sqrt{\mu_m + \kappa_k} t) \sin(\sqrt{\mu_m} x) \cos(\sqrt{\kappa_k} y)
\]
\[
+ B_{m,k} \sin(c\sqrt{\mu_m + \kappa_k} t) \sin(\sqrt{\mu_m} x) \cos(\sqrt{\kappa_k} y)
\]

for \( m, k = 1, 2, 3, \ldots \) and
\[ u_{m,0} = A_{m,0} \cos(c \mu_m t) \sin(\sqrt{\mu_m}x) + B_{m,0} \sin(c \sqrt{\mu_m} t) \sin(\sqrt{\mu_m}x) \]

for \( m = 1, 2, 3, \ldots \) are solutions of the PDE together with the homogeneous boundary condition. We superpose those function to get

\[ u(x, y, t) = \sum_{m=1}^{\infty} [A_{m,0} \cos(c \sqrt{\mu_m} t) \sin(\sqrt{\mu_m}x) + B_{m,0} \sin(c \sqrt{\mu_m} t) \sin(\sqrt{\mu_m}x)] + \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} [A_{m,k} \cos(c \sqrt{\mu_m + \kappa_k t} \mu_m x) \cos(\sqrt{\kappa_k}y) + B_{m,k} \sin(c \sqrt{\mu_m + \kappa_k t} \mu_m x) \cos(\sqrt{\kappa_k}y)] . \]

We determine the constants using the orthogonality relations of those eigenfunctions. We get from the first initial conditions:

\[ f(x) = u(x, y, 0) = \sum_{m=1}^{\infty} A_{m,0} \sin(\sqrt{\mu_m}x) + \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} A_{m,k} \sin(\sqrt{\mu_m}x) \cos(\sqrt{\kappa_k}y) \]

so

\[ A_{m,k} = \frac{4}{LM} \int_0^L \int_0^M f(x, y) \sin(\sqrt{\mu_m}x) \cos(\sqrt{\kappa_k}y) dy dx , \]

and

\[ A_{m,0} = \frac{2}{LM} \int_0^L \int_0^M f(x, y) \sin(\sqrt{\mu_m}) dy dx . \]

The second initial condition gives

\[ g(x) = u_t(x, y, 0) = \]
\[
= \sum_{m=1}^{\infty} c\sqrt{\mu_m}B_{m,0}\sin(\sqrt{\mu_m}x) \\
+ \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} c\sqrt{\mu_m + \kappa_k}B_{m,k}\sin(\sqrt{\mu_m}x)\cos(\sqrt{\kappa_k}y) ,
\]

hence

\[
B_{m,k} = \frac{4}{c\sqrt{\mu_m + \kappa_k} LM} \int_0^L \int_0^M g(x, y) \sin(\sqrt{\mu_m}x)\cos(\sqrt{\kappa_k}y) dydx ,
\]

for \( m, k = 1, 2, 3, \ldots \) and

\[
B_{m,0} = \frac{2}{LM\sqrt{\mu_m}} \int_0^L \int_0^M g(x, y) \sin(\sqrt{\mu_m}x) dydx ,
\]

for \( m = 1, 2, 3, \ldots . \)