A hyperplane in $\mathbb{E}^n$ was already defined to be an $(n-1)$-dimensional flat; that is, a translate of an $(n-1)$-dimensional subspace of $\mathbb{E}^n$. A function $f : \mathbb{E}^n \to \mathbb{E}^m$ is said to be \textit{linear} if

\[ f(x + y) = f(x) + f(y) \]  
\[ f(\lambda x) = \lambda f(x) \]

(additive)  

(homogeneous)

for all $x, y \in \mathbb{E}^n$ and all $\lambda \in \mathbb{R}$.

A linear function $f : \mathbb{E}^n \to \mathbb{R}$ is called a \textit{linear functional}, in which case we denote by $[f : \alpha]$ the set

\[ \{ x \in \mathbb{E}^n : f(x) = \alpha \} \quad (\alpha \in \mathbb{R}). \]

**Theorem 3.2.** Suppose $H$ is a subset of $\mathbb{E}^n$. Then $H$ is a hyperplane if and only if there is a non-trivial linear functional $f$ and a number $\delta$ such that $H = [f : \delta]$.

**Proof.** First assume $H$ is a hyperplane and let $x_0 \in H$. Then $V = H - x_0$ is the $(n-1)$-dimensional subspace of $\mathbb{E}^n$ parallel to $H$. We denote by $v_0$ a unit vector orthogonal to $V$. For each $x \in \mathbb{E}^n$, denote by $x_V$ the orthogonal projection of $x$ onto $V$ (i.e. the nearest point of $V$ to $x$). We know, from linear algebra, that there is a number $\alpha$ such that

\[ x = x_V + \alpha v_0. \]

It is clear that $\alpha$ is uniquely determined by $x$. We define $f : \mathbb{E}^n \to \mathbb{R}$ by $f(x) = \alpha$; so $[f]$ measures the distance of $x$ to $V$.

Next we check that $f$ is a linear functional. If $x, y \in \mathbb{E}^n$, we have

\[ x = x_V + \alpha v_0 \quad \text{and} \quad y = y_V + \beta v_0 \]

and so

\[ x + y = x_V + y_V + (\alpha + \beta)v_0. \]

Consequently,

\[ f(x + y) = \alpha + \beta = f(x) + f(y). \]

Also, if $\lambda \in \mathbb{R}$, $\lambda x = \lambda x_V + \lambda \alpha v_0$ and so $f(\lambda x) = \lambda \alpha = \lambda f(x)$. So $f$ is a linear functional.

Finally we show that $H = [f : \delta]$ where $\delta = f(x_0)$. If $h \in H$ then $h = h_0 + v$ where $v \in V$. Consequently $f(h) = f(x_0) + f(v) = f(x_0)$. Now $x_0 = w_0 + \delta v_0$ where $w_0 \in V$ and so if $f(x) = \delta$ then $x = x_V + \delta v_0$ and therefore $x - x_0 = x_V - w_0 \in V$ and so $x \in H ([f : \delta] \subset H)$. So we have proved a non-trivial linear functional $f$ and a number $\delta$ such that $H = [f : \delta]$.

For the converse, note that, since $f$ is non-trivial, $f : \mathbb{E}^n \to \mathbb{R}$ is surjective. Consequently, dim ker $f = n - 1$. Put $V = \ker f$, an $(n-1)$-dimensional subspace of $\mathbb{E}^n$. Now assume $f(x_0) = \delta$ and complete the proof by showing that $[f : \delta] = V + x_0$. If $v \in V$ then $f(v + x_0) = f(v) + f(x_0) = \delta$ and so $V + x_0 \subset [f : \delta]$. But $[f : \delta]$ is an affine set which is not the whole space. Consequently dim$[f : \delta] \leq n - 1$ and so $[f : \delta] = V + x_0$. □
**Theorem 3.3.** If \( f \) and \( g \) are linear functionals on \( \mathbb{F}^n \) such that \( [f : \alpha] = [g : \beta] \) for some \( \alpha, \beta \in \mathbb{R} \) then there is a number \( \lambda \neq 0 \) such that \( f = \lambda g \) and \( \alpha = \lambda \beta \).

**Proof.** First assume \( g \) is trivial. Then \( [g : \beta] = \mathbb{F}^n \) if \( \beta = 0 \) and \( [g : \beta] = \emptyset \) if \( \beta \neq 0 \). So, if \( \beta = 0 \) we have \([f : \alpha] = \mathbb{F}^n \). Consequently \( \alpha = f(0) = 0 \) and \( f \) is trivial. In this case any \( \lambda \neq 0 \) works. If \( \beta \neq 0 \) we have \([f : \alpha] = \emptyset \) and so \( \alpha \neq f(0) = 0 \) and \( f \) is trivial since, if there were an \( x \in \mathbb{F}^n \) with \( f(x) \neq 0 \) then \( f(\frac{\alpha x}{f(x)}) = \alpha \), which is impossible. In this case put \( \lambda = \alpha/\beta \).

Now assume \( g \) is not trivial and choose \( x_0 \in \mathbb{F}^n \) with \( g(x_0) \neq 0 \). Put \( \lambda = f(x_0)/g(x_0) \) and \( V = [g : 0] \), the kernel of \( g \). Note that \( V \) is an \((n-1)\)-dimensional subspace \( \mathbb{F}^n \). We have

\[
v + \frac{\beta}{g(x_0)} x_0 \in [g : \beta] \quad \text{for all } v \in V.
\]

Hence

\[
v + \frac{\beta}{g(x_0)} x_0 \in [f : \alpha] \quad \text{for all } v \in V,
\]

equivalently

\[
f(v) + \frac{\beta}{g(x_0)} f(x_0) = \alpha \quad \text{for all } v \in V.
\]

Thus

\[
f(v) + \lambda \beta = \alpha \quad \text{for all } v \in V.
\]

The fact that \( v \) is a subspace means \( v \in V \) and therefore \( \alpha = \lambda \beta \). Furthermore, if \( x \in \mathbb{F}^n \), there is a \( \mu \in \mathbb{R} \) such that \( x = v + \mu x_0 \) for some \( v \in V \); this follows from the fact that \( \dim V = n - 1 \) and \( x_0 \in V \). Hence

\[
f(x) = f(v) + \mu f(x_0) = \mu f(x_0) = \mu \lambda g(x_0) = \lambda (g(v) + \mu g(x_0)) = \lambda g(x)
\]

as required.

\( \square \)

**Theorem 3.4 and 3.5.** Let \( f \) be a linear functional defined on \( \mathbb{F}^n \).

a) There is a \( z \in \mathbb{F}^n \) such that \( f(x) = \langle x, z \rangle \) for all \( x \in \mathbb{F}^n \);

b) \( f \) is continuous;

c) Each set of \([f : \alpha]\) is closed and therefore every hyperplane is closed.

**Proof.**

a) If \( f \) is trivial put \( z = o \). Otherwise put \( V = [f : 0] \). Then \( V \) is an \((n-1)\)-dimensional subspace of \( \mathbb{F}^n \) and we may choose a vector \( u \) orthogonal to \( V \). We put \( g(x) = \langle x, u \rangle \) for each \( x \in \mathbb{F}^n \). Then \( g \) is a linear functional on \( \mathbb{F}^n \) and \([f : 0] = [g : 0]\). It follows from Theorem 3.3 that there is a \( \lambda \in \mathbb{R} \) with \( f = \lambda g \). If we put \( z = \lambda u \) then \( f(x) = \langle x, z \rangle \) for each \( x \in \mathbb{F}^n \).

b) It follows from a) that

\[
|f(x) - f(y)| = |\langle x - y, z \rangle| \leq \|x - y\| \|z\| \quad \text{for all } x, y \in \mathbb{F}^n.
\]
So if \( \varepsilon > 0 \) is given, we choose \( \delta > 0 \) so that

\[
\delta \|z\| < \varepsilon.
\]

Then, if \( y \in B(x, \delta) \) we have \( f(y) \in B(f(x), \varepsilon) \). Consequently \( f \) is continuous.

c) It follows from b) that

\[
\{ x : f(x) > a \} \cup \{ x : f(x) < a \} = f^{-1}(a, \infty) \cup f^{-1}(-\infty, a,)
\]

is open. Now \([f : a] \) is the complement of this set and must therefore be closed. We learned in Theorem 3.2 that each hyperplane is of the form \([f : a] \) for some linear functional \( f \) and some member \( a \). So each hyperplane is closed.

\[\square\]

We note that we have shown that if \( H \) is a hyperplane then there is a \( \gamma \in \mathbb{R} \) and a vector \( z \in \mathbb{E}^n \) such that

\[
H = \{ x \in \mathbb{E}^n : \langle x, z \rangle = \gamma \}.
\]

We also know that \( z \) is orthogonal to all vectors parallel to \( H \) since it is orthogonal to all vectors in the subspace parallel to \( H \). We could choose \( z \) to be a unit vector, in which case \( \gamma \) measures that distance from \( o \) to \( H \).