CALCULUS III FALL 1999

REVIEW 2

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0. Great fleas have little fleas upon their backs to bite 'em, and little fleas have lesser fleas, and so ad infinitum.

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1. Find the following limits if they exist:

$$\lim_{n\to\infty} \frac{2n^2-1}{3n^2+15n-2}, \qquad \lim_{n\to\infty} (\sqrt{n+1}-\sqrt{n-1}), \qquad \lim_{n\to\infty} \frac{n!}{n^n},$$

$$\lim_{n\to\infty} (\sqrt{n^2+n}-\sqrt{n^2+1}).$$

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2. Do the same thing for the following:

$$\lim_{n\to\infty} \frac{\ln(2n)}{\ln(3n)} \qquad \lim_{n\to\infty} \sqrt[n]{n^2+n} \qquad \lim_{n\to\infty} \sqrt[n]{a^n+b^n} \qquad \lim_{n\to\infty} \frac{\alpha(n)}{n},$$

where $\alpha(n)$ is the number of primes which divide n.

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3. Give an example of a sequence that is:

(a) convergent but not monotonic
(b) bounded but not monotonic
(c) monotone but not convergent
(d) monotone decreasing and unbounded
(e) monotone decreasing and convergent
(f) unbounded but not monotone

4. Show that the following sequences converge and find their limits:

$$a_1 = \sqrt{3}, \quad a_{n+1} = \sqrt{a_n + 2}; \qquad a_1 = \sqrt{2}, \quad a_{n+1} = \sqrt{2a_n}$$

5. Identify the function $f(x) = \lim_{n\to\infty}(\lim_{k\to\infty}(\cos n!\pi x)^{2k})$ - this is a little tricky, but lots of fun.

6. Determine which of the following series are convergent:

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^2-1}} \qquad \sum_{n=1}^{\infty} \frac{n^2}{n!} \qquad \sum_{n=1}^{\infty} \frac{\ln n}{n} \qquad \sum_{n=1}^{\infty} \frac{n^2-1}{n^4-1} \qquad \sum_{n=1}^{\infty} \sin\frac{1}{n} \qquad \sum_{n=1}^{\infty} \frac{5^n n!}{(2n)^n}$$

In the case of one of the convergent series above, estimate the sum accurately to 3 decimal places.

7. For which values of x do the following series converge?

$$\sum_{n=1}^{\infty} (\sin x)^n \qquad \sum_{n=1}^{\infty} (1+x)^n \qquad \sum_{n=1}^{\infty} \frac{1}{(1-x)^n}.$$

8. Determine which of the following series converge:

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n\sqrt{n}} \qquad \sum_{n=1}^{\infty} (-1)^{n+1} \frac{n-1}{n} \qquad \sum_{n=1}^{\infty} n \sin\frac{1}{n} \qquad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n+1}+\sqrt{n}}$$

9. Find the interval of convergence for each of the following power series:

$$\sum_{n=0}^{\infty} \frac{x^n}{n^3+1} \qquad \sum_{n=0}^{\infty} \frac{(-2)^{n+1} x^n}{2n+3} \qquad \sum_{n=0}^{\infty} \frac{3^n x^{3n}}{(3n)!}.$$

REVIEW 2 – ANSWERS

1.

$$\frac{2n^2-1}{3n^2+15n-2} \to \frac{2}{3}$$

$$\sqrt{n+1}-\sqrt{n-1} = \frac{1}{\sqrt{n+1}+\sqrt{n-1}} \to 0$$

$$\frac{n!}{n^n} = \frac{1.2\ldots n}{n\ldots n} \leqslant \frac{1}{n} \to 0$$

$$\sqrt{n^2+n}-\sqrt{n^2+1} = \frac{n-1}{\sqrt{n^2+n}+\sqrt{n^2+1}} \to \frac{1}{2}.$$

2.

$$\frac{\ln(2n)}{\ln(3n)} = \frac{\ln 2+\ln n}{\ln 3+\ln n} \to 1$$
$$\sqrt[n]{n^2+n} = n^{1/n}(n+1)^{1/n} = n^{1/n}\left(\frac{n+1}{n}\right)^{1/n} n^{1/n} \to e.$$

If $a \leqslant b$ then

$$\sqrt[n]{a^n+b^n} = b\left(1+\left(\frac{a}{b}\right)^n\right)^{1/n} \to b, \qquad \text{in general } \sqrt[n]{a^n+b^n} \to \max\{a,b\}.$$

Note that if $\alpha(n)=m$ then there are m primes $p_1,\ldots,p_m$ which all divide n. Hence $p_1\ldots p_m \leqslant n$, on the other hand $p_1\ldots p_m \geqslant 2^m$. Hence $m\ln 2 \leqslant \ln n$. Thus

$$\frac{\alpha(n)}{n} \leqslant \frac{\ln n}{n\ln 2} \to 0,$$

by L'Hôpital's rule.

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3. a) $(-1)^n/n$ b) $(-1)^n$ c) n d) $-n$ e) $1/n$ f) $(-1)^n n$.

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4. $a_{n+1}^2 - a_n^2 = -(a_n - 2)(a_n + 1)$ and $a_{n+1}^2 - 4 = a_n - 2$. So $0 < a_n < 2$ for all n since $a_1 < 2$. Furthermore $a_{n+1} > a_n$ for all n and so we have a bounded monotonic sequence. It follows that the sequence is convergent and the limit ℓ satisfies $\ell^2 = \ell + 2$. This implies $\ell = -1$ or 2. Clearly the limit is 2.
$a_{n+1}^2 - a_n^2 = a_n(2 - a_n)$ and $a_{n+1}^2 - 4 = 2(a_n - 2)$. So $0 < a_n < 2$ for all n since $a_1 < 2$. Furthermore $a_{n+1} > a_n$ for all n and so we have a bounded monotonic sequence. It follows that the sequence is convergent and the limit ℓ satisfies $\ell^2 = 2\ell$. This implies $\ell = 0$ or 2. Clearly the limit is 2.

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5. If x is rational, then $n!\pi x$ is an integer multiple of π for all sufficiently large values of n. Hence $(\cos n!\pi x)^{2k} = 1$ for all k and all sufficiently large n. Hence $f(x) = 1$ if x is rational. If x is irrational, then $|\cos n!\pi x|^{2k} < 1$ for each n. Hence $f(x) = 0$ if x is irrational.

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6. Comparison with $\sum 1/n^{2/3}$ shows that the series diverges.

$$\frac{(n+)^2 n!}{(n+1)!n^2} = \frac{n+1}{n^2} \to 0,$$

so the ratio test shows we have convergence.

$$\frac{\ln n}{n} > \frac{1}{n}, \qquad \text{for sufficiently large } n$$

so the comparison test shows that the series diverges.

$$\frac{n^2 - 1}{n^4 - 1} = \frac{1}{n^2 + 1},$$

so comparison with $\sum 1/n^2$ shows that this converges.

$$\frac{1/n}{\sin 1/n} \to 1,$$

so the comparison test shows that this series diverges.

$$\frac{5^{n+1}(n+1)!(2n)^n}{(2n+2)^{n+1}5^n n!} = \frac{5n^n}{2(n+1)^n} \to \frac{5}{2e} < 1,$$

So the ratio test shows that the series converges.

We'll estimate $\sum_{n=2}^{\infty}(n^2 - 1)/(n^4 - 1) = \sum_{n=2}^{\infty} 1/(n^2 + 1)$. We can apply the integral test to this to see that

$$R_n \leqslant \int_n^{\infty} \frac{dx}{1 + x^2} = \lim_{A \to \infty} \Big[\tan^{-1} x\Big]_n^A = \frac{\pi}{2} - \tan^{-1} n = \tan^{-1} \frac{1}{n}.$$

So we choose n with $\tan^{-1}(1/n) < 0.0005$. We know that $\tan^{-1} x < x$ so it suffices to choose $n = 2,000$. The required estimate is $s_{2,000} = 0.576$.

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7. For convergence, we require $|\sin x| < 1$, which means x is not an odd multiple of π. For the second series, we require $|1 + x| < 1$ which means x is in the interval $(-2, 0)$. Finally, we require $|1 - x| > 1$ for convergence. This means either $x < 0$ or $x > 2$.

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8. The corresponding absolute series is $\sum_{n=1}^{\infty} 1/n^{3/2}$ which converges. So we have absolute convergence.
This series diverges since its terms do not approach zero.
Note that $n \sin(1/n) \to 1$ and so this series diverges too.
This series does not converge absolutely, by comparison with $\sum_{n=1}^{\infty} 1/\sqrt{n}$. On the other hand, the alternating series test can be applied to show the series converges conditionally.

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9. We use the ratio test in all three cases. For the first series, we have

$$\frac{|x|^{n+1}(n^3+1)}{((n+1)^3+1)|x|^n} = \frac{|x|(n^3+1)}{(n+1)^3+1} \to |x|.$$

When $x = \pm 1$, the corresponding series of absolute values is $\sum_{n=1}^{\infty} 1/(n^3+1)$ which converges. So the interval of convergence is $[-1, 1]$.

For the second series,

$$\frac{2^{n+2}|x|^{n+1}(2n+3)}{(2n+5)2^{n+1}|x|^n} = \frac{2|x|(2n+3)}{2n+5} \to 2|x|.$$

When $x = 1/2$, the series is $-2\sum_{n=1}^{\infty}(-1)^n/(2n+3)$ which converges (alternating series test). When $x = -1/2$ the series is $-2\sum_{n=1}^{\infty} 1/(2n+3)$ which diverges. So the interval of convergence is $(-1/2, 1/2]$.

For the third series

$$\frac{3^{n+1}|x|^{3n+3}(3n)!}{(3n+3)!3^n|x|^{3n}} = \frac{3|x|^3}{(3n+3)(3n+2)(3n+1)} \to 0.$$

So the series converges for all x.