CALCULUS III FALL 1999
HOMEWORK 10 – ANSWERS

§10.11 Question 6,8,12,16,20; §10.12 Questions 10,12

6.

$$\frac{1}{\sqrt{2+x}} = \frac{1}{\sqrt{2}}\left(1+\frac{x}{2}\right)^{-1/2} = \frac{1}{\sqrt{2}}\sum_{n=0}^{\infty}\frac{(-\frac{1}{2})(-\frac{3}{2})\cdots(-\frac{1}{2}-n+1)}{n!}\left(\frac{x}{2}\right)^{n}$$
$$= \frac{1}{\sqrt{2}}\sum_{n=0}^{\infty}(-1)^n\frac{1.3\ldots(2n-1)}{2^{2n}n!}x^n = \frac{1}{\sqrt{2}}\sum_{n=0}^{\infty}\frac{(-1)^n(2n)!}{2^{3n}(n!)^2}x^n.$$

We applied the binomial theorem to the variable $x/2$ so the possible range of values is given by $|x/2| < 1$ which gives $|x| < 2$ and so the radius of convergence is 2.

8.

$$\frac{x^2}{\sqrt{1-x^3}} = x^2(1-x^3)^{-1/2} = x^2\sum_{n=0}^{\infty}\frac{(-1)^n(2n)!}{2^{2n}(n!)^2}(-x^3)^n$$
$$= \sum_{n=0}^{\infty}\frac{(2n)!}{2^{2n}(n!)^2}x^{3n+2}.$$

We applied the binomial theorem to the variable $-x^3$ so the possible range of values is given by $|-x^3| < 1$ which is the same as $|x| < 1$ and so the radius of convergence is 1.

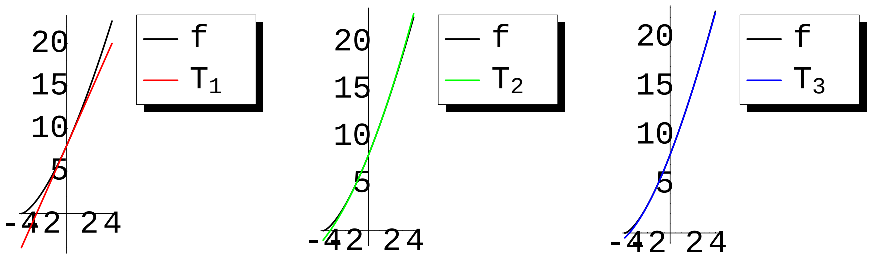
12.

$$(4+x)^{3/2} = 8\Big(1+\frac{x}{4}\Big)^{3/2} = 8\sum_{n=0}^{\infty}\frac{(\frac{3}{2})(\frac{1}{2})(-\frac{1}{2})\cdots(\frac{3}{2}-n+1)}{n!}\Big(\frac{x}{4}\Big)^n$$
$$= 24\sum_{n=0}^{\infty}(-1)^n\frac{1.3\ldots(2n-5)}{2^n n!}\Big(\frac{x}{4}\Big)^n = 24\sum_{n=0}^{\infty}\frac{(2n-4)!}{2^{4n-2}n!(n-2)!}x^n.$$

It follows that the Taylor polynomials are

$$T_1(x) = 8\Big(1+\frac{3x}{8}\Big) = 8+3x,$$
$$T_2(x) = 8\Big(1+\frac{3x}{8}+\frac{(\frac{3}{2})(\frac{1}{2})}{2}\Big)\frac{x^2}{16} = 8+3x+\frac{3x^2}{16},$$
$$T_3(x) = 8\Big(1+\frac{3x}{8}+\frac{(\frac{3}{2})(\frac{1}{2})}{2}+\frac{(\frac{3}{2})(\frac{1}{2})(-\frac{1}{2})}{3!}\frac{x^3}{64}\Big)\frac{x^2}{16} = 8+3x+\frac{3x^2}{16}$$
$$= 8+3x+\frac{3x^2}{16}-\frac{x^3}{128}.$$

We applied the binomial theorem to the variable $x/4$ so the possible range of values is given by $|x/4|<1$ which gives $|x|<4$ and so the radius of convergence is 4. The graphs are given below



16.

$$(8+x)^{1/3} = 2\Big(1+\frac{x}{8}\Big)^{1/3} = 2\sum_{n=0}^{\infty}\frac{(\frac{1}{3})(-\frac{2}{3})\cdots(\frac{1}{3}-n+1)}{n!}\frac{x^n}{8^n}$$
$$= 2\sum_{n=0}^{\infty}(-1)^{n-1}\frac{2.5\ldots(3n-4)}{2^{3n}3^n n!}x^n.$$

We applied the binomial theorem to the variable $x/8$ so the possible range of values is given by $|x/8|<1$ which gives $|x|<8$ and so the radius of convergence is 8.

For part b). we put $x=0.2$ to get

$$2\sum_{n=0}^{\infty}(-1)^{n-1}\frac{2.5\ldots(3n-4)}{2^{3n}3^n n!}\frac{1}{5^n}.$$

The alternating nature of this series suggests that the required answer is given by the partial sum s_n where

$$\frac{2.5\ldots(3n-1)}{2^{3n+3}3^{n+1}n!}\frac{1}{5^{n+1}} < 0.00005.$$

We find that this is satisfied when $n=2$ and so the required estimate is

$$\sqrt[3]{8.2} \simeq 2\Big(1+\frac{1}{120}-\frac{1}{14400}\Big) \simeq 2.0165.$$

20.

$$(1+x^3)^{-1/2} = \sum_{n=0}^{\infty}\frac{(-\frac{1}{2})(-\frac{3}{2})\cdots(-\frac{1}{2}-n+1)}{n!}\big(x^3\big)^n$$
$$= \sum_{n=0}^{\infty}(-1)^n\frac{1.3\ldots(2n-1)}{2^n n!}x^{3n} = \sum_{n=0}^{\infty}(-1)^n\frac{(2n)!}{2^{2n}(n!)^2}x^{3n}.$$

It follows that

$$\frac{f^{(3n)}(0)}{(3n)!} = (-1)^n\frac{(2n)!}{2^{2n}(n!)^2} \qquad \text{and so} \qquad f^{(9)}(0) = -9!\frac{6!}{2^6 r} = -113,400.$$

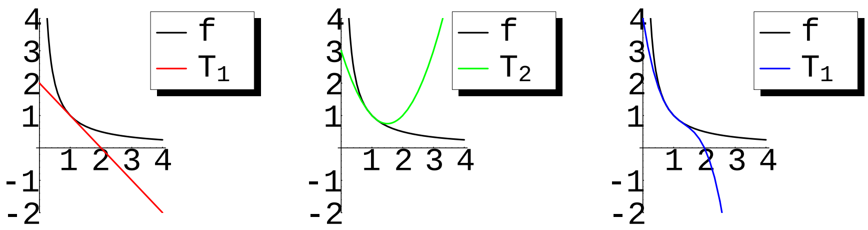
10. We have $f'(x) = -1/x^2$, $f''(x) = 2/x^3$ and, in general, $f^{(n)}(x) = (-1)^n n!/x^{n+1}$, which gives $f^{(n)}(1) = (-1)^n n!$. It follows that the required Taylor series is

$$\sum_{n=0}^{\infty}(-1)^n(x-1)^n.$$

Notice that there is another way to reach this same conclusion,

$$\frac{1}{x} = \frac{1}{1-(1-x)} = \sum_{n=0}^{\infty}(1-x)^n = \sum{}_{n} = 0^{\infty}(-1)^n(x-1)^n.$$

It follows that $T_1(x) = 1-(x-1) = 2-x$, $T_2(x) = 1-(x-1)+(x-1)^2 = 3-3x+x^2$ and $T_3(x) = 1-(x-1)+(x-1)^2-(x-1)^3 = 4-6x+4x^2-x^3$. The graphs are shown below



22. We have $f^{(n)}(x) = (-1)^{n-1}(n-1)!/x^n$ if $n \geqslant 1$. It follows that

$$\ln x = \ln 4 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n4^n}(x-4)^n.$$

The third Taylor polynomial is

$$T_3(x) = \ln 4 + \frac{x-4}{4} - \frac{(x-4)^2}{32} + \frac{(x-4)^3}{192}.$$

The remainder term is $R_3(x) = -(x-4)^4/(4z^4)$ where z is between x and 4. We have $3 \leqslant x \leqslant 5$ and so $3 < z < 5$, hence $|R_3(x)| < 1/(4(3^4)) = 1/324 \simeq 0.0031$. The graph of f and T_3 are given below