Solution of Problem 17.8/15

Let us parametrize the hemisphere by using polar coordinates (ρ, ϕ, θ). The radius ρ is 1, and we can use as parameters (u, v) the angular coordinates (ϕ, θ). Then the parametric equation of the surface S is

$$\mathbf{r}(\phi,\theta) = \sin\phi\cos\theta\,\mathbf{i} + \sin\phi\sin\theta\,\mathbf{j} + \cos\theta\,\mathbf{k}\ , \qquad \phi\in[0,\pi]\ , \quad \theta\in[0,\pi]\ .$$

The tangent vectors are

$$\begin{aligned}\mathbf{r}_\phi(\phi,\theta) &= \cos\phi\cos\theta\,\mathbf{i} + \cos\phi\sin\theta\,\mathbf{j} - \sin\phi\,\mathbf{k}\ ,\\ \mathbf{r}_\theta(\phi,\theta) &= -\sin\phi\sin\theta\,\mathbf{i} + \sin\phi\cos\theta\,\mathbf{j}\ ,\end{aligned}$$

and their cross product is

$$\mathbf{r}_\phi(\phi,\theta)\times\mathbf{r}_\theta(\phi,\theta) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\phi\cos\theta & \cos\phi\sin\theta & -\sin\phi \\ -\sin\phi\sin\theta & \sin\phi\cos\theta & 0 \end{vmatrix} = \sin^2\phi\cos\theta\,\mathbf{i} + \sin^2\phi\sin\theta\,\mathbf{j} + \sin\phi\cos\phi\,\mathbf{k}\ .$$

At this point we have to check if the orientation of the surface S, defined to be "upward" in the problem, matches the direction of the vector $\mathbf{r}_\phi\times\mathbf{r}_\theta$ – if it does, then the unit normal vector to S consistent with the orientation given in the statement of the problem will be $\mathbf{n} = \frac{\mathbf{r}_\phi\times\mathbf{r}_\theta}{|\mathbf{r}_\phi\times\mathbf{r}_\theta|}$; otherwise, the unit normal vector will be $\mathbf{n} = -\frac{\mathbf{r}_\phi\times\mathbf{r}_\theta}{|\mathbf{r}_\phi\times\mathbf{r}_\theta|}$; We see that the coefficient in front of $\mathbf{j}$ in $\mathbf{r}_\phi(\phi,\theta)\times\mathbf{r}_\theta(\phi,\theta)$ is $\sin^2\phi\sin\theta$, which is positive for $\phi\in[0,\pi]$ and $\theta\in[0,\pi]$, therefore the orientation given in the problem is consistent with the normal vector

$$\mathbf{r}_\phi(\phi,\theta)\times\mathbf{r}_\theta(\phi,\theta) = \sin^2\phi\cos\theta\,\mathbf{i} + \sin^2\phi\sin\theta\,\mathbf{j} + \sin\phi\cos\phi\,\mathbf{k}$$

and the unit normal

$$\mathbf{n} = \frac{\mathbf{r}_\phi\times\mathbf{r}_\theta}{|\mathbf{r}_\phi\times\mathbf{r}_\theta|}\ .$$

Now we compute the integral over the surface S:

$$\operatorname{curl}\mathbf{F} = (R_y - Q_z)\,\mathbf{i} + (P_z - R_x)\,\mathbf{j} + (Q_x - P_y)\,\mathbf{k} = -(\mathbf{i}+\mathbf{j}+\mathbf{k})\ .$$

$$\begin{aligned}\iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} &= \iint_S \operatorname{curl}\mathbf{F}\cdot\mathbf{n}\,dS\\ &= \iint_D \left(\operatorname{curl}\mathbf{F}\cdot\frac{\mathbf{r}_\phi\times\mathbf{r}_\theta}{|\mathbf{r}_\phi\times\mathbf{r}_\theta|}\right)|\mathbf{r}_\phi\times\mathbf{r}_\theta|\,dA\\ &= \iint_D \operatorname{curl}\mathbf{F}\cdot(\mathbf{r}_\phi\times\mathbf{r}_\theta)\,d\phi\,d\theta\\ &= -\iint_D (\mathbf{i}+\mathbf{j}+\mathbf{k})\cdot(\sin^2\phi\cos\theta\,\mathbf{i} + \sin^2\phi\sin\theta\,\mathbf{j} + \sin\phi\cos\phi\,\mathbf{k})\,d\phi\,d\theta\\ &= -\int_0^\pi\int_0^\pi (\sin^2\phi\cos\theta + \sin^2\phi\sin\theta + \sin\phi\cos\phi)\,d\phi\,d\theta\\ &= -(0+\pi+0) = -\pi\ .\end{aligned}$$

To find the integral over the boundary ∂S of the surface S, we first have to parametrize ∂S in such a way that the orientations of S and ∂S are consistent (this is explained on pages 1128–1129 of the book). Then the orientation of ∂S must be such that, if we look at the surface from, say, the point $(0, 10, 0)$, we should see the positive direction of ∂S as counterclockwise. One can parametrize ∂S as follows:

$$\mathbf{R}(t) = \sin t\,\mathbf{i} + \cos t\,\mathbf{k}\ , \qquad t \in [0, 2\pi]\ .$$

We will also need the tangent vector

$$\mathbf{R}'(t) = \cos t\,\mathbf{i} - \sin t\,\mathbf{k}\ ,$$

and the fact that

$$\mathbf{F}(\mathbf{R}(t)) = \mathbf{F}(X(t), Y(t), Z(t)) = Y(t)\,\mathbf{i} + Z(t)\,\mathbf{j} + X(t)\,\mathbf{k} = 0\,\mathbf{i} + \cos t\,\mathbf{j} + \sin t\,\mathbf{k}\ .$$

Computing the line integral over the boundary ∂S:

$$\begin{aligned}
\oint_{\partial S} \mathbf{F} \cdot \mathrm{d}\mathbf{R} &= \int_0^{2\pi} \mathbf{F}(\mathbf{R}(t)) \cdot \mathbf{R}'(t)\,\mathrm{d}t \\
&= \int_0^{2\pi} (\cos t\,\mathbf{j} + \sin t\,\mathbf{k}) \cdot (\cos t\,\mathbf{i} - \sin t\,\mathbf{k})\,\mathrm{d}t \\
&= -\int_0^{2\pi} \sin^2 t\,\mathrm{d}t \\
&= -\frac{1}{2} \cdot 2\pi \\
&= -\pi\ ,
\end{aligned}$$

where in the last step I used the fact that the average value of $\sin^2 t$ and $\cos^2 t$ over a period (i.e., over the interval $[0, 2\pi]$) is $\frac{1}{2}$, so that the integral of $\sin^2 t$ from 0 to 2π is equal to the average value of the integrand times the length of the integration interval. The fact that the averages of $\sin^2 t$ and $\cos^2 t$ over a period are $\frac{1}{2}$ are obvious from

$$\frac{1}{2\pi}\int_0^{2\pi} \sin^2 t\,\mathrm{d}t + \frac{1}{2\pi}\int_0^{2\pi} \sin^2 t\,\mathrm{d}t = \frac{1}{2\pi}\int_0^{2\pi} (\sin^2 t + \cos^2 t)\,\mathrm{d}t = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}t = 1\ ,$$

and from the fact that $\int_0^{2\pi} \sin^2 t\,\mathrm{d}t = \int_0^{2\pi} \sin^2 t\,\mathrm{d}t$, which is due to the identity $\sin t = \cos(\frac{\pi}{2} - t)$ (think of a right triangle with an angle $t < \frac{\pi}{2}$).