Expectation of a "smeared" derivative of a Wiener process

Let f be a *test function*, i.e., a smooth function which (together with all its derivatives) decays very quickly at infinity; the function f can also be chosen to be a smooth function that is nonzero only on a bounded interval of $\mathbb{R}$. (Different choices of classes of test functions yield mathematically different properties, but at the level of rigor of our course this is not going to matter.) Having introduced test functions, we can talk about *generalized functions* which are objects defined only when multiplied by a test functions and integrated. An example of a generalized function is the Dirac "δ-function" which can be defined by

$$\delta(f) := \int_{\mathbb{R}} \delta(t)\, f(t)\, \mathrm{d}t = f(0) \ . \tag{1}$$

It is often said that the "function" $\delta(t)$ has value 0 everywhere except at $t = 0$, where it has value ∞. Clearly, such a definition does not make much sense – what people really have in mind is the property (1).

Since the function f and all its derivatives decay very fast at infinity, when we integrate by parts, all boundary terms at $-\infty$ or ∞ are 0, i.e., for the nth derivative $\zeta^{(n)}$ of any generalized function ζ, we have after n integrations by parts

$$(\zeta^{(n)})(f) := \int_{\mathbb{R}} h'(t)\, f(t)\, \mathrm{d}t = (-1)^n \int_{\mathbb{R}} h(t)\, f^{(n)}(t)\, \mathrm{d}t \ .$$

This, in particular, implies that $(\delta^{(n)})(f) = (-1)^n f^{(n)}(0)$.

Similarly to (1), we can talk about the derivative of a Wiener process,

$$\xi(t) := \frac{\mathrm{d}B(t)}{\mathrm{d}t} \ ,$$

in the following generalized sense:

$$\xi(f) := \int_0^\infty \xi(t)\, f(t)\, \mathrm{d}t \ , \tag{2}$$

where f is an arbitrary test function. Because of the stochastic nature of the Wiener process, $\xi(f)$ is a random variable, not just a number.

Using that $\mathbb{E}[B(t)B(s)] = \min(t,s)$, we can find $\mathbb{E}[\xi(t)^2]$:

$$
\begin{aligned}
\mathbb{E}\left[\xi(f)^2\right] &= \mathbb{E}\left[\left(\int_0^\infty \frac{\mathrm{d}B(t)}{\mathrm{d}t}\, f(t)\,\mathrm{d}t\right)^2\right] = \mathbb{E}\left[\left(-\int_0^\infty B(t)\, f'(t)\,\mathrm{d}t\right)^2\right] \\
&\quad \text{(we integrated by parts; now we will interchange the integral and the expectation)} \\
&= \int_0^\infty \mathrm{d}t \int_0^\infty \mathrm{d}s\, \mathbb{E}[B(t)B(s)]\, f'(t)\, f'(s) = \int_0^\infty \mathrm{d}t \int_0^\infty \mathrm{d}s\, \min(t,s)\, f'(t)\, f'(s) \\
&= \int_0^\infty \mathrm{d}t\, f'(t) \left(\int_0^t s\, f'(s)\,\mathrm{d}s + \int_t^\infty t\, f'(s)\,\mathrm{d}s\right) = \int_0^\infty \mathrm{d}t\, f'(t) \left(\int_0^t s\,\mathrm{d}f(s) + t \int_t^\infty \mathrm{d}f(s)\right) \\
&\quad \text{(integrating by parts again)} \\
&= \int_0^\infty \mathrm{d}t\, f'(t) \left(t\, f(t) - \int_0^t f(s)\,\mathrm{d}s - t\, f(t)\right) = -\int_0^\infty \left(\int_0^t f(s)\,\mathrm{d}s\right) \mathrm{d}f(t) \\
&\quad \text{(integrating by parts again)} \\
&= -f(t) \int_0^t f(s)\,\mathrm{d}s \bigg|_{t=0}^{\infty} + \int_0^\infty f(t)^2\,\mathrm{d}t = \int_0^\infty f(t)^2\,\mathrm{d}t\ .
\end{aligned}
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