Examples of improper integrals

Example 1. In class we considered an integral that was similar to

$$\int_0^\infty \frac{\mathrm{e}^{-x^3}}{1+\sqrt{x}}\,\mathrm{d}x\ . \tag{1}$$

(In the integral in class the lower limit of integration was 1.) We suspected that the integral converges, so that we tried to find a function $v : [0,\infty) \to \mathbb{R}$ such that

$$0 \le \frac{\mathrm{e}^{-x^3}}{1+\sqrt{x}} \le v(x) \quad \text{ and } \quad \int_0^\infty v(x)\,\mathrm{d}x \ \text{ converges}\ .$$

By the Comparison Test for Improper Integrals, this would imply that the integral (1) converges.

Let us try several candidates for v.

(a) By using that $\mathrm{e}^{-x^3} \le 1$ and that the function $\frac{1}{1+\sqrt{x}}$ is decreasing for $x \ge 0$, we have

$$0 \le \frac{\mathrm{e}^{-x^3}}{1+\sqrt{x}} \le \frac{1}{1+\sqrt{x}} \le \frac{1}{1+\sqrt{1}} = \frac{1}{2}\ ,$$

so we try to take $v(x) = \dfrac{1}{2}$, but this does not give us anything because $\displaystyle\int_0^\infty \frac{1}{2}\,\mathrm{d}x$ diverges:

$$\int_0^\infty \frac{1}{2}\,\mathrm{d}x = \lim_{a\to\infty} \int_0^a \frac{1}{2}\,\mathrm{d}x = \lim_{a\to\infty} \frac{a}{2} = \infty\ .$$

(b) One can try to be more sophisticated:

$$0 \le \frac{\mathrm{e}^{-x^3}}{1+\sqrt{x}} \le \frac{1}{1+\sqrt{x}} \le \frac{1}{\sqrt{x}}\ ,$$

so one can try $v(x) = \frac{1}{\sqrt{x}}$. This, however, introduces trouble at 0 because $\displaystyle\lim_{x\to 0^+} \frac{1}{\sqrt{x}} = \infty$, while the original integrand $\dfrac{\mathrm{e}^{-x^3}}{1+\sqrt{x}}$ had no trouble at 0. To avoid unnecessary trouble, we set

$$v(x) = \begin{cases} \dfrac{\mathrm{e}^{-x^3}}{1+\sqrt{x}} & \text{for } 0 \le x < 1, \\[2ex] \dfrac{1}{\sqrt{x}} & \text{for } x \ge 1\ . \end{cases} \tag{2}$$

This function has a finite jump at $x = 1$, but this does not do any harm. (*Exercise:* Can you choose $v(x)$ for $x \ge 1$ differently, so that the function $v(x)$ defined by (2) be continuous?) But even the choice (2) does not work because

$$\begin{aligned} \int_0^\infty \frac{\mathrm{e}^{-x^3}}{1+\sqrt{x}}\,\mathrm{d}x &= \int_0^1 \frac{\mathrm{e}^{-x^3}}{1+\sqrt{x}}\,\mathrm{d}x + \lim_{a\to\infty} \int_1^a \frac{\mathrm{e}^{-x^3}}{1+\sqrt{x}}\,\mathrm{d}x \\ &\le \int_0^1 1\,\mathrm{d}x + \lim_{a\to\infty} \int_1^a \frac{1}{\sqrt{x}}\,\mathrm{d}x = 1 + \lim_{a\to\infty} 2(\sqrt{a}-1) = \infty \end{aligned}$$

(here we used that $\frac{\mathrm{e}^{-x^3}}{1+\sqrt{x}} \le 1$ for $x \in [0,1]$).

(c) It looks like we have to keep the exponent: let's try

$$0 \leq \frac{\mathrm{e}^{-x^3}}{1+\sqrt{x}} \leq \mathrm{e}^{-x^3} \ , \tag{3}$$

so we set $v(x) = \mathrm{e}^{-x^3}$. We have to solve the integral $\displaystyle\int_0^\infty \mathrm{e}^{-x^3}\,\mathrm{d}x$, but we don't know how solve this integral...

(d) So maybe we can try the idea from (c), but go one step further. Let us write

$$\int_0^\infty \frac{\mathrm{e}^{-x^3}}{1+\sqrt{x}}\,\mathrm{d}x = \int_0^1 \frac{\mathrm{e}^{-x^3}}{1+\sqrt{x}}\,\mathrm{d}x + \int_1^\infty \frac{\mathrm{e}^{-x^3}}{1+\sqrt{x}}\,\mathrm{d}x \tag{4}$$

and deal only with the second integral in the right-hand side of (4) because it is the troublesome one (the first integral in the right hand side is no greater than 1 – why?). We have

$$x \leq x^3 \quad \text{for } x \geq 1 \ ,$$

which implies that

$$-x \geq -x^3 \quad \text{for } x \geq 1 \ ,$$

which, in turn, yields

$$\mathrm{e}^{-x} \geq \mathrm{e}^{-x^3} \quad \text{for } x \geq 1 \ .$$

Note that in the last step we used the fact that if a function f is non-decreasing and $x \leq y$, then $f(x) \leq f(y)$! This allows us to extend (3) (for $x \geq 1$) as follows:

$$0 \leq \frac{\mathrm{e}^{-x^3}}{1+\sqrt{x}} \leq \mathrm{e}^{-x^3} \leq \mathrm{e}^{-x} \quad \text{for } x \geq 1 \ . \tag{5}$$

Now we use (4) and (5) to get

$$\begin{aligned}
\int_0^\infty \frac{\mathrm{e}^{-x^3}}{1+\sqrt{x}}\,\mathrm{d}x &= \int_0^1 \frac{\mathrm{e}^{-x^3}}{1+\sqrt{x}}\,\mathrm{d}x + \int_1^\infty \frac{\mathrm{e}^{-x^3}}{1+\sqrt{x}}\,\mathrm{d}x \\
&\leq \int_0^1 1\,\mathrm{d}x + \lim_{a\to\infty}\int_1^a \mathrm{e}^{-x}\,\mathrm{d}x \\
&= 1 + \lim_{a\to\infty}(1-\mathrm{e}^{-a}) = 2 \ .
\end{aligned}$$

This implies that the integral (1) converges. The splitting (4) and the bound (5) can be considered as choosing the function $v : [0,\infty) \to \mathbb{R}$ to be

$$v(x) = \begin{cases} \dfrac{\mathrm{e}^{-x^3}}{1+\sqrt{x}} & \text{for } 0 \leq x < 1, \\ \mathrm{e}^{-x} & \text{for } x \geq 1 \ . \end{cases}$$

Example 2. The integral

$$\int_0^\infty \frac{\mathrm{e}^{-x}}{\sqrt{x}}\,\mathrm{d}x$$

which is improper for two reasons – the integrand tends to ∞ when $x \to 0^+$, and the integration is over an infinitely long interval. To separate these two "bad" things, we write the integral as

$$\int_0^\infty \frac{\mathrm{e}^{-x}}{\sqrt{x}}\,\mathrm{d}x = \int_0^5 \frac{\mathrm{e}^{-x}}{\sqrt{x}}\,\mathrm{d}x + \int_5^\infty \frac{\mathrm{e}^{-x}}{\sqrt{x}}\,\mathrm{d}x \ , \tag{6}$$

and deal separately with each of the integrals in the right-hand side of (6). Of course, the number 5 was chosen arbitrarily.

- In the integral $\displaystyle\int_0^5 \frac{\mathrm{e}^{-x}}{\sqrt{x}}\,\mathrm{d}x$, the exponent e^{-x} in the integrand is a "tame" function – it is bounded above and below: $\mathrm{e}^{-5} \le \mathrm{e}^{-x} \le 1$ for $x \in [0,5]$. So to simplify the integrand and retain the "bad" function in it, we write

$$0 \le \frac{\mathrm{e}^{-x}}{\sqrt{x}} \le \frac{1}{\sqrt{x}} \qquad \text{for } x \in [0,5] \ . \tag{7}$$

  We have

$$\int_0^5 \frac{1}{\sqrt{x}}\,\mathrm{d}x = \lim_{a\to 0^+} \int_a^5 \frac{1}{\sqrt{x}}\,\mathrm{d}x = \lim_{a\to 0^+} 2(\sqrt{x} - \sqrt{a}) = 2\sqrt{5} \ . \tag{8}$$

- In the integral $\displaystyle\int_5^\infty \frac{\mathrm{e}^{-x}}{\sqrt{x}}\,\mathrm{d}x$, the function that decays faster is e^{-x} (the function $\frac{1}{\sqrt{x}}$ decays at a much slower pace), so we expect that we should keep e^{-x}, and write

$$0 \le \frac{\mathrm{e}^{-x}}{\sqrt{x}} \le \frac{\mathrm{e}^{-x}}{\sqrt{5}} \qquad \text{for } x \in [5,\infty) \ . \tag{9}$$

  Now we have

$$\int_5^\infty \frac{\mathrm{e}^{-x}}{\sqrt{5}}\,\mathrm{d}x = \frac{1}{\sqrt{5}} \int_5^\infty \mathrm{e}^{-x}\,\mathrm{d}x = \frac{1}{\sqrt{5}} \left(-\mathrm{e}^{-x}\right)\Big|_{x=5}^{\infty} = \frac{\mathrm{e}^{-5}}{\sqrt{5}} \ . \tag{10}$$

  Note that in this calculation I used a shorthand notation – I wrote

$$\left(-\mathrm{e}^{-x}\right)\Big|_{x=5}^{\infty} = \mathrm{e}^{-5}$$

  instead of

$$\lim_{a\to\infty} \left(\mathrm{e}^{-5} - \mathrm{e}^{-a}\right) = \mathrm{e}^{-5} \ .$$

  Using such shorthand notations in practice is OK, as long as you understand that they should be understood as limits. But for this class we will try to use the detailed notations with limits.

Putting everything together, with the numbers above the (in)equalities standing for the number of the equation(s) that were used to derive them:

$$\begin{aligned}\int_0^\infty \frac{\mathrm{e}^{-x}}{\sqrt{x}}\,\mathrm{d}x &\overset{(6)}{\leq} \int_0^5 \frac{\mathrm{e}^{-x}}{\sqrt{x}}\,\mathrm{d}x + \int_5^\infty \frac{\mathrm{e}^{-x}}{\sqrt{x}}\,\mathrm{d}x \\ &\overset{(7),(9)}{=} \int_0^5 \frac{1}{\sqrt{x}}\,\mathrm{d}x + \int_5^\infty \frac{\mathrm{e}^{-x}}{\sqrt{5}}\,\mathrm{d}x \\ &\overset{(8),(10)}{=} 2\sqrt{5} + \frac{\mathrm{e}^{-5}}{\sqrt{5}} = 4.4721359\ldots + 0.0150665\ldots = 4.4872024\ldots\ .\end{aligned}$$

The above computations not only prove that the integral (1) exists, but also give us an upper bound on its value, namely,

$$\int_0^\infty \frac{\mathrm{e}^{-x}}{\sqrt{x}}\,\mathrm{d}x \leq 4.4872024\ldots\ . \tag{11}$$

The true value of the integral is

$$\int_0^\infty \frac{\mathrm{e}^{-x}}{\sqrt{x}}\,\mathrm{d}x = \sqrt{\pi} = 1.7724538\ldots\ . \tag{12}$$

We see that the true value (12) is indeed less than the upper bound (11).