MATH 5103 Project Due by 11:59 pm on 5/2 (Sat)

Problem 1. [An iterative method for solving ODEs]
Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a C^1 function, let t_0, $T > t_0$, and a be given numbers, and consider the initial value problem (IVP)

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= f(x,t) \ , \\ x(t_0) &= a \ , \end{aligned} \tag{1}$$

for the unknown function $x : [t_0, T] \to \mathbb{R}$. In this problem you will develop an iterative method for finding a solution of the IVP (1), and will apply it to a particular example.

(a) Let the function $x(t)$ satisfy the relation

$$x(t) = a + \int_{t_0}^{t} f\big(x(s), s\big)\,\mathrm{d}s \ . \tag{2}$$

Show that $x(t)$ satisfies the IVP (1). Please specify what results you use in your derivation.

(b) Equation (2) is an integral equation for the function $x(t)$ that is equivalent to the IVP (1) One can use this fact to formulate an iterative procedure for solving the IVP (1). Namely, define the sequence of functions $x_0, x_1, x_2, \ldots$, with $x_n : [t_0, T] \to \mathbb{R}$ for every $n = 0, 1, 2, \ldots$, as follows:

$$\begin{aligned} x_0(t) &:= a \ , \\ x_n(t) &:= a + \int_{t_0}^{t} f\big(x_{n-1}(s), s\big)\,\mathrm{d}s \ , \qquad n \in \mathbb{N} \ . \end{aligned} \tag{3}$$

Apply this iterative procedure to the IVP

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= x - 5 \ , \\ x(t_0) &= 6 \ , \end{aligned} \tag{4}$$

to find the functions x_0, x_1, x_2, x_3, x_4, ...

(c) Solve the IVP (4) by using the standard methods from your ODE course. Find the Maclaurin series of your solution and compare it with the functions you obtained in part (b). Discuss briefly what you observe.

Problem 2. [Geometry and bifurcation analysis]
In Lectures 8 and 9 – see

`http://www.math.ou.edu/~npetrov/math4193_5103_s20.html`

– we considered the dynamics of the logistic equation with predation, namely,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = rx\left(1 - \frac{x}{k}\right) - \frac{x^2}{1+x^2} \ . \tag{5}$$

Here $x(t)$ is the population of budworms, the parameter $r > 0$ is related to their reproduction rate, and $k > 0$ is proportional to the carrying capacity of the system in an absence of predation. The term $-\dfrac{x^2}{1+x^2}$ takes into account the effect of predators on the population of budworms, assuming that the life span of the predators is much longer that the life span of the budworms. In this problem you will analyze the bifurcations in the system by using tools from multivariable calculus.

(a) What equation do the fixed points of equation (5) satisfy? Explain why the non-zero fixed points of (5) satisfy the equation

$$\Phi(k, r, x) = 0 \ , \tag{6}$$

where

$$\Phi(k, r, x) = r\left(1 - \frac{x}{k}\right) - \frac{x}{1+x^2} \ . \tag{7}$$

(b) Think of equations (6) and (7) as describing a surface Σ in the 3-dimensional space $\mathbb{R}^3_{(k,r,x)}$ of the variables $\mathbf{z} := (k, r, x)$, i.e.,

$$\begin{aligned} \Sigma &:= \left\{\mathbf{z} \in \mathbb{R}^3_{(k,r,x)} \ : \ \Phi(\mathbf{z}) = 0\right\} \\ &= \left\{(k, r, x) \in \mathbb{R}^3_{(k,r,x)} \ : \ r\left(1 - \frac{x}{k}\right) - \frac{x}{1+x^2} = 0\right\} \ . \end{aligned} \tag{8}$$

Let the point $\mathbf{z}_* := (k_*, r_*, x_*)$ belong to Σ. Explain why a point $\boldsymbol{\xi} := (\xi, \eta, \zeta)$ belongs to the tangent plane $T_{\mathbf{z}_*}\Sigma$ to the surface Σ at the point $\mathbf{z}_* \in \Sigma$ exactly when

$$(\boldsymbol{\xi} - \mathbf{z}_*) \cdot \nabla\Phi(\mathbf{z}_*) = 0 \ . \tag{9}$$

(c) In the space $\mathbb{R}^3_{(k,r,x)}$ of the variables (k, r, x), consider x as the vertical coordinate. Pick some value k_* of k and some value r_* of r. Depending on the values of k_* and r_*, the vertical line $\{k = k_*, r = r_*\}$ can intersect the surface Σ at one, two, or three points. In Figure 1 we show three intersections of the surface Σ with the vertical planes $\{k = 4.2\}$,

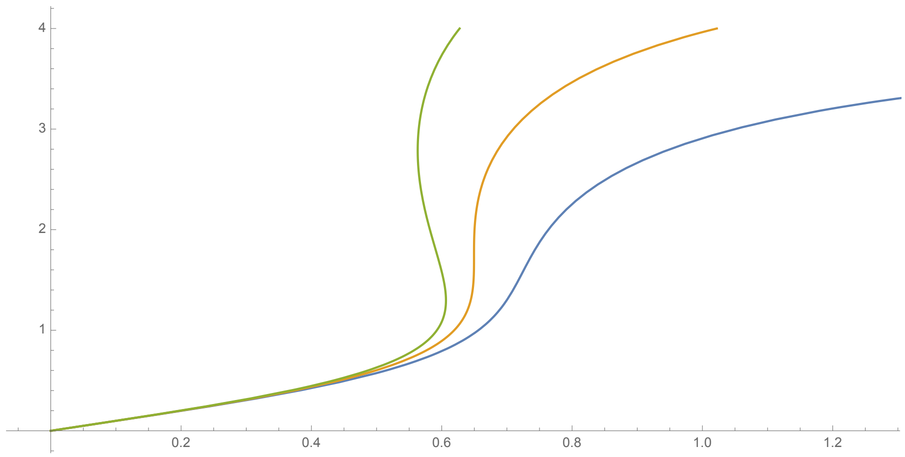

Figure 1: Plots of the projections onto the (r, x)-plane (r on the horizontal axis, x on the vertical) of the intersections of the surface Σ and the planes $\{k = 4.2\}$ (bottom curve), $\{k = 3\sqrt{3} \approx 5.2\}$ (middle curve, with one point with a vertical tangent), and $\{k = 6.4\}$ (top curve, with two points with vertical tangents).

$\{k = 3\sqrt{3}\}$, and $\{k = 6.4\}$; these intersections are then projected onto the (r, x)-plane – this is what is plotted in the figure, with r on the horizontal and x on the vertical axis.

Look at the intersection of the surface Σ with the vertical plane $\{k = 6.4\}$ – this intersection is the top curve in Figure 1. There are two points on this curve where the tangent line to the curve is vertical. If you think of the surface Σ, at points such as these two points, the tangent plane to Σ at such points is vertical. The locus of the points on Σ at which the tangent plane to Σ is vertical is called a *fold* of the surface Σ.

Use (8) to show that the fold of the surface Σ is a line that can be written in a parametric form as

$$\Gamma = \left\{ \big(k(v), r(v), x(v)\big) = \left(\frac{2v^3}{v^2 - 1}, \frac{2v^3}{(1 + v^2)^2}, v \right) \in \mathbb{R}^3_{(k,r,x)} \, : \, v \in (1, \infty) \, . \right\} \qquad (10)$$

The condition $v > 1$ comes from concrete expression for $k(v)$ and the fact that k must be positive. For your convenience, here is the expression for the gradient of the function Φ:

$$\nabla\Phi(k, r, x) = \left\langle \frac{rx}{k^2}, 1 - \frac{x}{k}, -\frac{r}{k} + \frac{x^2 - 1}{(1 + x^2)^2} \right\rangle \, .$$

(d) Now let us approach the problem differently. First show that the surface Σ given by (8) can be written in a parametric form by

$$\mathbf{R} : D \to \mathbb{R}^3_{(k,r,x)} : (u, v) \mapsto \mathbf{R}(u, v) = \left(u, \frac{v}{(1 + v^2)\left(1 - \frac{v}{u}\right)}, v \right) , \qquad (11)$$

where $D := \{(u, v) \in \mathbb{R}^2 \, : \, u > 0, \, v > 1\}$.

(e) The tangent plane $T_{\mathbf{R}(u,v)}\Sigma$ to the surface Σ at the point $\mathbf{R}(u,v) \in \Sigma$ can be written as

$$T_{\mathbf{R}(u,v)}\Sigma = \operatorname{span}\left\{\frac{\partial \mathbf{R}}{\partial u}(u,v),\ \frac{\partial \mathbf{R}}{\partial v}(u,v)\right\} . \tag{12}$$

Let

$$\Pi : \mathbb{R}^3_{(k,r,x)} \to \mathbb{R}^2_{(k,r)} : (k,r,x) \mapsto (k,r)$$

be the natural projection, and consider the map

$$\Pi \circ \mathbf{R} : D \to \mathbb{R}^2_{(k,r)} . \tag{13}$$

Let $\mathbf{R}(u_*, v_*) \in \Sigma$ be a given point of Σ. We say that the surface Σ is *projectable* to the (k,r)-plane in a small neighborhood of $\mathbf{R}(u_*, v_*)$ if there exists a small neighborhood $U \in \mathbb{R}^2_{(u,v)}$ of (u_*, v_*) such that the mapping

$$\Pi \circ \mathbf{R}|_U : U \subseteq \mathbb{R}^2_{(u,v)} \to \Pi \circ \mathbf{R}(U) \subseteq \mathbb{R}^2_{(k,r)}$$

is invertible, and both $\Pi \circ \mathbf{R}|_U$ and its inverse are differentiable.

Express the condition for projectability of the surface Σ onto the (k,r)-plane in terms of the rank of the derivative $D(\Pi \circ \mathbf{R})(u,v)$ of the mapping $\Pi \circ \mathbf{R}$,

$$D(\Pi \circ \mathbf{R})(u,v) = \begin{pmatrix} \dfrac{\partial R_1}{\partial u}(u,v) & \dfrac{\partial R_1}{\partial v}(u,v) \\ \dfrac{\partial R_2}{\partial u}(u,v) & \dfrac{\partial R_2}{\partial v}(u,v) \end{pmatrix} .$$

Hint: Think about the relation between the entries of the matrix $D(\Pi \circ \mathbf{R})(u,v)$ and the expressions for the partial derivatives of $\mathbf{R}$ in (12).

(f) Write down the mapping $\Pi \circ \mathbf{R}$ explicitly using (11) and (13). For your convenience, here is the explicit expression for its derivative:

$$D(\Pi \circ \mathbf{R})(u,v) = \begin{pmatrix} 1 & 0 \\ -\dfrac{v^2}{(u-v)^2(1+v^2)} & \dfrac{u(u-uv^2+2v^3)}{(u-v)^2(1+v^2)^2} \end{pmatrix}$$

Use this to rederive the equation for the fold of the surface Σ.

(g) Now consider the projection of the fold of Σ onto the (k,r)-plane – you can obtain it easily as a parameterized curve from (10). This projection is drawn in Figure 2. There is a point where it looks like this curve is not differentiable. For which value of the parameter v does this happen, and what are the values of k and r at this point?

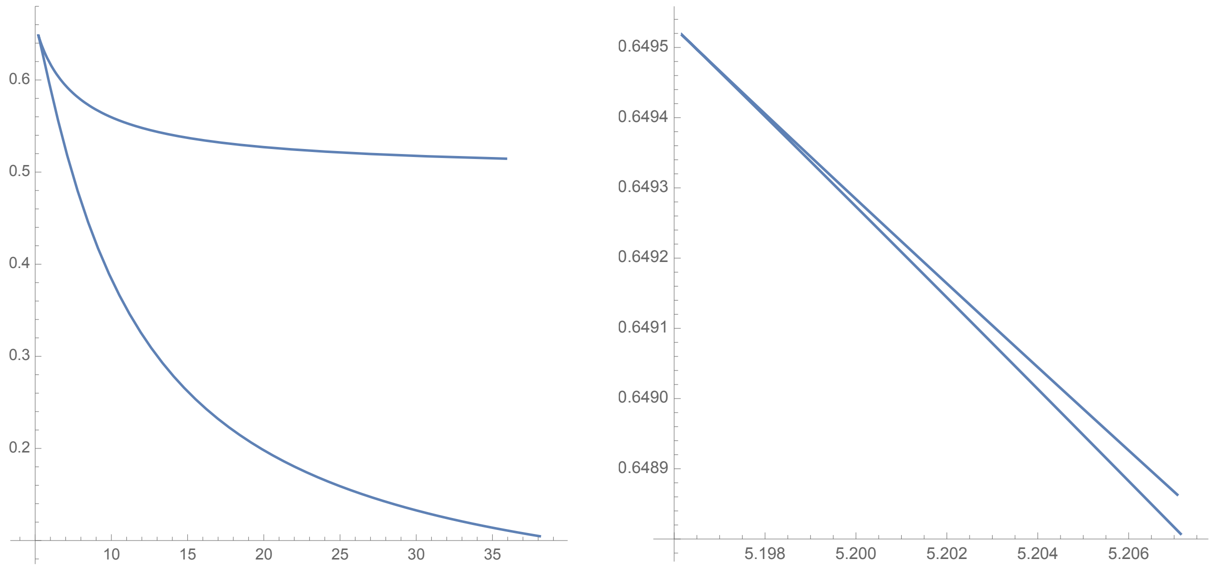

Figure 2: Plots of the projection of the fold of the surface Σ onto the (k, r)-plane (k on the horizontal axis, r on the vertical); the figure on the right is a magnification of the tip of the region.

(h) Finally, you have to find out whether the point where the projection in Figure 2 is non-differentiable is a cusp. We say that it is a *cusp* if the angle between the two smooth branches of the curve at the point where these two branches meet is zero.

Warning: If the projection in Figure 2 is written as $\mathbf{r}(v) = (k(v), r(v))$, and if the value of v where this curve is not differentiable is v_*, then you may be tempted to find whether the angle between the two branches at the point where they meet is zero by checking whether $\mathbf{r}'(v_*^-) \cdot \mathbf{r}'(v_*^+)$ is zero. This dot product, however, may be zero even if the angle between the two branches is not zero, or even if the curve is smooth (i.e., there is no point where it is non-differentiable), so that you have to perform a more subtle analysis.