Problem 2. Vehicles pass a pedestrian crossing at the instants of a Poisson process of intensity λ. You need a gap of duration a in order to cross the street. Let T be the first time at which you would succeed in crossing to the other side (including the time a during which you are crossing the street). The situation is shown pictorially in Figure 2.

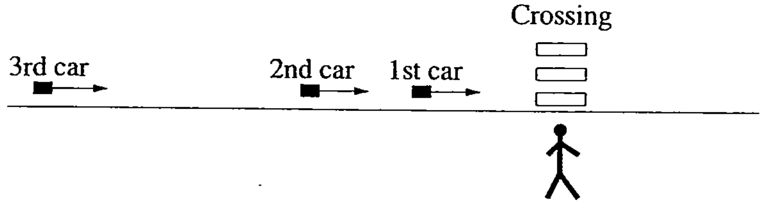


Figure 2: A pedestrian is waiting to cross the street.

In Figure 3, you see the time axis with the moments T_1, T_2, and T_3 of the first, second, and third cars passing the crossing, and the time interval a you need in order to cross. Clearly, a does not fit between 0 and T_1, as well as between T_1 and T_2; it, however, fits between T_2 and T_3, so you will start crossing at $t = T_2$, and reach the other side of the street at time $t = T_2 + a$, so that in this case the random variable T has value $T = T_2 + a$.

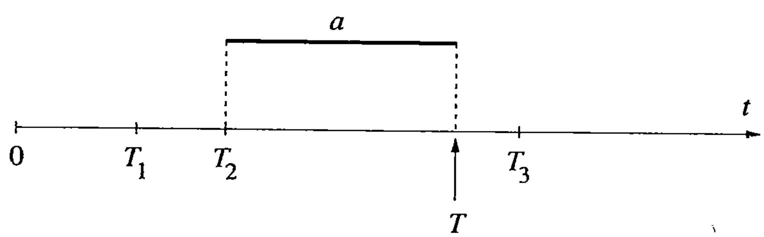


Figure 3: On the meaning of the random variable T.

(a) Show that the expectation $\mathbb{E}[T]$ satisfies the integral equation

$$\mathbb{E}[T] = \int_0^a (x + \mathbb{E}[T])\, \lambda \mathrm{e}^{-\lambda x}\, \mathrm{d}x + a\mathrm{e}^{-\lambda a}\,.$$

(b) Solve the integral equation for $\mathbb{E}[T]$ derived in (a).
Hint: This is very easy, you don't need to apply any sophisticated techniques! For your convenience, here are some integrals:

$$\int_0^a x\lambda \mathrm{e}^{-\lambda x}\, \mathrm{d}x = \frac{1}{\lambda} - \frac{1}{\lambda}\mathrm{e}^{-\lambda a} - a\mathrm{e}^{-\lambda a}$$

$$\int_0^a \lambda \mathrm{e}^{-\lambda x}\, \mathrm{d}x = 1 - \mathrm{e}^{-\lambda a}\,.$$

(c) Does the expression for $\mathbb{E}[T]$ you derived in part (b) behaves reasonably (i.e., matches your intuition) in the limit $\lambda \to 0$? Discuss briefly.

(d) Now suppose that there are two lines to cross, carrying independent Poissonian traffic with respective intensities λ and μ. You need time a to cross the first lane and time b to cross the second lane. Find the expected total time to cross the street in the following two cases:

(i) there is an island between the two lanes where you can stay safely;

(ii) you must cross both lanes in one go.

Hint: You can use your results from part (b) applied to cases (i) and (ii). Recall that the superposition of two Poisson processes is a Poisson process (see Problem 3(a) of Homework 4).

(e) Which of the total times found in (d) is greater – in case (i) or in case (ii)? Of course, you can answer this question by comparing your answers in the cases (i) and (ii) above, but don't do this here – just give a simple intuitive argument.

Problem 2:

(a) To derive the integral equation for $\mathbb{E}[T]$, we condition on the time T_1 when the first car passes the pedestrian crossing:

$$\mathbb{E}[T] = \mathbb{E}[\mathbb{E}[T|T_1]] = \int_{-\infty}^{\infty} \mathbb{E}[T|T_1=t_1] f_{T_1}(t_1)\, dt_1.$$

Since the traffic flow is Poissonian, $T_1 \sim Exp(\lambda)$

$$\Rightarrow f_{T_1}(t_1) = \begin{cases} \lambda e^{-\lambda t_1}, & t_1 > 0 \\ 0, & t_1 < 0. \end{cases}$$

To find the conditional expectation $\mathbb{E}[T|T_1=t_1]$, we reason as follows:

- if $t_1 > a$, then we can start crossing the street at time 0, so in this case $T = a$;

- if $t_1 < a$, then we will wait time t_1 for the first car to pass the crossing and at this moment (because the interevent times in a Poisson process are exponentially distributed, and exponential random variables are memoryless) we will start waiting "from scrath", so in this case the expected time will be $t_1 + \mathbb{E}[T]$.

Therefore,

$$\mathbb{E}[T \mid T_1 = t_1] = \begin{cases} a, & t_1 > a \\ t_1 + \mathbb{E}[T], & t_1 < a. \end{cases}$$

Plug this in the integral above:

$$\mathbb{E}[T] = \int_0^\infty \mathbb{E}[T \mid T_1 = t_1]\, \lambda e^{-\lambda t_1}\, dt_1$$

$$= \int_0^a \left(t_1 + \mathbb{E}[T]\right) \lambda e^{-\lambda t_1}\, dt_1$$

$$+ \int_a^\infty a \lambda e^{-\lambda t_1}\, dt_1,$$

which, after solving the last integral, yields the desired equation

$$\mathbb{E}[T] = \int_0^a \left(t_1 + \mathbb{E}[T]\right) \lambda e^{-\lambda t_1}\, dt_1 + a e^{-\lambda a}.$$

(b) We have

$$\mathbb{E}[T] = \int_0^a t_1 \lambda e^{-\lambda t_1} dt_1 + \mathbb{E}[T] \int_0^a \lambda e^{-\lambda t_1} dt_1 + a e^{-\lambda a}$$

and, using the expressions for the integral given in the hint, we obtain

$$\cancel{\mathbb{E}[T]} = \frac{1}{\lambda} - \frac{1}{\lambda} e^{-\lambda a} - \cancel{a e^{-\lambda a}} + \mathbb{E}[T]\left(\cancel{1} - e^{-\lambda a}\right) + \cancel{a e^{-\lambda a}}$$

$$\Rightarrow \quad \mathbb{E}[T] = \frac{e^{\lambda a} - 1}{\lambda}.$$

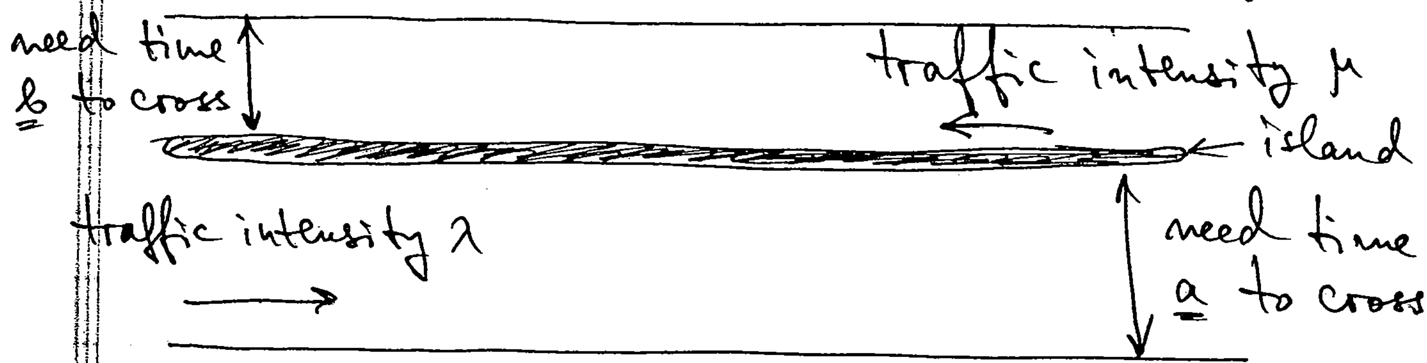
(c) Clearly, $\lambda \to 0$ means the limit of no traffic at all, in which the time T will be equal to a. Indeed, we have

$$\frac{e^{\lambda a} - 1}{\lambda} = \frac{\cancel{1} + \lambda a + \frac{(\lambda a)^2}{2!} + \dots - \cancel{1}}{\lambda} \xrightarrow[\lambda \to 0]{} a\,;$$

another way to get this limit is to use L'Hospital's rule:

$$\lim_{\lambda \to 0} \frac{e^{\lambda a} - 1}{\lambda} = \lim_{\lambda \to 0} \frac{\frac{d}{d\lambda}(e^{\lambda a} - 1)}{\frac{d}{d\lambda}\lambda} = a.$$

(d) If there is an island between the lanes, then the total time of crossing



the two lanes will be equal to the sum of the two times:

$$\frac{e^{\lambda a}-1}{\lambda} + \frac{e^{\mu b}-1}{\mu}.$$

If there is no island between the lanes, then the problem is the same as crossing a street with traffic intensity $\lambda+\mu$ (because the "sum" of two Poisson processes is a Poisson process is a Poisson process with intensity equal to the sum of the two intensities). Also, the total time needed to cross the street will be $a+b$ therefore the expectation of T in this case will be

$$\frac{e^{(\lambda+\mu)(a+b)}-1}{\lambda+\mu}.$$