# Integration of rational functions – Partial fractions

Partial fraction decomposition of a rational function is a useful tool in integrating rational functions. The idea is to represent the rational function in the integral as a sum of terms of the form

$$\frac{\alpha}{(x-c)^k} \quad \text{and} \quad \frac{Ax+B}{(x^2+bx+c)^m} \ ,$$

which can be integrated relatively easily.

To integrate the rational function $\frac{P(x)}{Q(x)}$, where $P$ and $Q$ are polynomials, one first rewrites the quotient $\frac{P(x)}{Q(x)}$ in the form

$$\frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)} \ ,$$

where the degree of the polynomial $R$ (the numerator) is smaller than the degree of the polynomial $Q$ (the denominator). For example, as in Example 1 on page 509 of the book,

$$\frac{x^3+x}{x-1} = x^2+x+2+\frac{2}{x-1}$$

(check!). In this particular example, the function in the right-hand side is very easy to integrate:

$$\int \frac{x^3+x}{x-1}\,dx = \int \left(x^2+x+2+\frac{2}{x-1}\right)dx = \frac{x^3}{3}+\frac{x^2}{x}+2x+2\ln|x-1|+C \ .$$

From now on **assume that the degree of the numerator is smaller than the degree of the denominator!**

The method of partial fractions is a recipe to rewrite a rational function of the form as a sum of simpler rational functions. Assume that

$$Q(x) = (x-c_1)^{k_1}(x-c_2)^{k_2}\cdots(x-c_r)^{k_r}(x^2+a_1x+b_1)^{m_1}\cdots(x^2+a_sx+b_s)^{m_s} \ , \tag{1}$$

where $k_1, \ldots, k_r, m_1, \ldots, m_s$ are integers greater than or equal to 1, and each expression of the form $(x^2+ax+b)$ is irreducible, i.e., it cannot be represented as a product of the form $(x-\alpha)(x-\beta)$ (examples of irreducible expressions are $x^2+5$, $(x-7)^2+3$, etc.). Each factor in $Q(x)$ contributes several terms in the partial fraction expansion. The rule is the following: if the denominator $Q(x)$ has the form (1), then one should look for a partial fraction expansion of the form

$$\begin{aligned}
\frac{R(x)}{Q(x)} = {} & \frac{\alpha_1}{x-c_1}+\frac{\alpha_2}{(x-c_1)^2}+\frac{\alpha_3}{(x-c_1)^3}+\cdots+\frac{\alpha_{k_1}}{(x-c_1)^{k_1}} \\
& +\frac{\beta_1}{x-c_2}+\frac{\beta_2}{(x-c_2)^2}+\frac{\beta_3}{(x-c_2)^3}+\cdots+\frac{\beta_{k_2}}{(x-c_2)^{k_2}} \\
& +\text{(more terms for each factor of this type)} \\
& +\frac{A_1x+B_1}{x^2+a_1x+b_1}+\frac{A_2x+B_2}{(x^2+a_1x+b_1)^2}+\frac{A_3x+B_3}{(x^2+a_1x+b_1)^3}+\cdots+\frac{A_{m_1}x+B_{m_1}}{(x^2+a_1x+b_1)^{m_1}} \\
& +\frac{C_1x+D_1}{x^2+a_2x+b_2}+\frac{C_2x+D_2}{(x^2+a_2x+b_2)^2}+\frac{C_3x+D_3}{(x^2+a_2x+b_2)^3}+\cdots+\frac{C_{m_2}x+D_{m_2}}{(x^2+a_2x+b_2)^{m_2}} \\
& +\text{(more terms for each factor of this type)} \ .
\end{aligned}$$

Here $\alpha_1, \alpha_2, \ldots, \alpha_{k_1}, \beta_1, \beta_2, \ldots, \beta_{k_2}, \ldots, A_1, B_1, A_2, B_2, \ldots, A_{m_1}, B_{m_1}, C_1, D_1, C_2, D_2, \ldots, C_{m_2}, D_{m_2}, \ldots$ are constants that should be determined.

**Example.** If $Q(x) = x^3(x-2)^4(x^2+2x+5)^2$, we have to look for a partial fraction expansion of the form

$$\begin{aligned}\frac{\text{(numerator)}}{x^3(x-2)^4(x^2+2x+5)^2} &= \frac{\alpha_1}{x} + \frac{\alpha_2}{x^2} + \frac{\alpha_3}{x^3} + \frac{\beta_1}{x-2} + \frac{\beta_2}{(x-2)^2} + \frac{\beta_3}{(x-2)^3} + \frac{\beta_4}{(x-2)^4} \\ &\quad + \frac{A_1x+B_1}{x^2+2x+5} + \frac{A_2x+B_2}{(x^2+2x+5)^2} \ .\end{aligned} \tag{2}$$

Notice that here we have used that the expression $x^2+2x+5$ cannot be written as a product of two linear factors, $(x-p)(x-q)$; to establish this, try to solve the quadratic equation

$$x^2+2x+5=0$$

– clearly, $x^2+2x+5 = (x+1)^2+4 \geq 4 > 0$, so that this quadratic equations has no roots.

**Example.** Here is an example of a denominator that requires some preliminary work. Let

$$Q(x) = (x^2-9)(x^2-x-6)^2(x^2+6x+16) \ .$$

One has to first try to write each factor as a product of linear factors (a "linear factor" is a factor of the form $(x-p)$): we have

$$\begin{aligned} x^2-9 &= (x+3)(x-3) \ , \\ x^2-x-6 &= (x-3)(x+2) \ , \\ x^2+6x+16 &= (x+3)^2+7 \geq 7 > 0 \ , \quad \text{so this expression is not a product of linear factors} \ . \end{aligned}$$

Therefore

$$\begin{aligned} Q(x) &= (x^2-9)(x^2-x-6)^2(x^2+6x+16) \\ &= (x+3)(x-3)\cdot(x-3)^2(x+2)^2\cdot(x^2+6x+16) \\ &= (x+3)\,(x-3)^3\,(x+2)^2\,(x^2+6x+16) \ , \end{aligned}$$

so we have to look for a partial fraction expansion of the form

$$\begin{aligned}\frac{\text{(numerator)}}{Q(x)} &= \frac{\text{(numerator)}}{(x+3)\,(x-3)^2\,(x+2)\,(x^2+6x+16)} \\ &= \frac{\alpha}{x+3} + \frac{\beta_1}{x-3} + \frac{\beta_2}{(x-3)^2} + \frac{\beta_3}{(x-3)^3} + \frac{\gamma_1}{x+2} + \frac{\gamma_2}{(x+2)^2} + \frac{Ax+B}{x^2+6x+16} \ .\end{aligned} \tag{3}$$

To determine the unknown constants, one has to multiply both sides of the partial fraction expansion (i.e., both sides of the equalities (2) or (3)) by $Q(x)$, and then equate the coefficients of the like powers. The simple example below shows in detail how this works.

**Example.** Solve the integral $\displaystyle\int \frac{x}{(x-3)(x^2+1)}\,dx$.

**Solution.** In this case the denominator, $Q(x) = (x-3)(x^2+1)$, is already as simple as possible (why?), so we can directly look for a partial fraction expansion of the form

$$\frac{x}{(x-3)(x^2+1)} = \frac{\alpha}{x-3} + \frac{Ax+B}{x^2+1} \ .$$

Multiply both sides by $Q(x) = (x-3)(x^2+1)$:

$$\frac{x}{(x-3)(x^2+1)} \cdot (x-3)(x^2+1) = \frac{\alpha}{x-3} \cdot (x-3)(x^2+1) + \frac{Ax+B}{x^2+1} \cdot (x-3)(x^2+1) \ ,$$

which simplifies to

$$x = \alpha(x^2+1) + (Ax+B)(x-3) = \alpha x^2 + \alpha + Ax^2 + Bx - 3Ax - 3B = (\alpha + A)x^2 + (B-3A)x + \alpha - 3B \ .$$

Equating the coefficients of the like powers of $x$ in the left- and right-hand side of this equality,

$$0 \cdot x^2 + 1 \cdot x + 0 \cdot x^0 = (\alpha + A)x^2 + (B-3A)x^1 + (\alpha - 3B)x^0 \ ,$$

we obtain the system of linear equatlions

$$\alpha + A = 0 \, , \quad B - 3A = 1 \, , \quad \alpha - 3B = 0 \, .$$

From the third equation we express $\alpha = 3B$, plug this in the first equation to obtain $3B + A = 0$, so $A = -3B$ which, substituted in the second equation, yields $B - 3(-3B) = 1$, from which $B = \frac{1}{10}$, $A = -3B = -\frac{3}{10}$, $\alpha = 3B = \frac{3}{10}$. Therefore,

$$\frac{x}{(x-3)(x^2+1)} = \frac{\alpha}{x-3} + \frac{Ax+B}{x^2+1} = \frac{3}{10}\frac{1}{x-3} + \frac{1}{10}\frac{-3x+1}{x^2+1} \ .$$

Finally, we use this to compute the integral:

$$\begin{aligned}\int \frac{x}{(x-3)(x^2+1)}\,dx &= \frac{3}{10}\int \frac{dx}{x-3} - \frac{3}{10}\int \frac{x}{x^2+1}\,dx + \frac{1}{10}\int \frac{dx}{x^2+1} \\ &= \frac{3}{10}\ln|x-3| - \frac{3}{20}\ln(x^2+1) + \frac{1}{10}\arctan x + C \ .\end{aligned}$$

**Remark.** Finally, a couple of words on integrals of the form $\displaystyle\int \frac{Ax+B}{x^2+ax+b}\,dx$. Let us illustrate the idea on the particular example $\displaystyle\int \frac{x+5}{x^2+6x+25}\,dx$. First, complete the square in the denominator:

$$x^2 + 6x + 25 = x^2 + 2 \cdot 3 \cdot x \ \underline{+3^2} \ \underline{-3^2} \ + 25 = (x+3)^2 + 16 = (x+3)^2 + 4^2$$

(which clearly implies that the denominator is not a product of linear factors). We have

$$\begin{aligned}\int \frac{x+5}{x^2+6x+25}\,dx &= \int \frac{x+5}{(x+3)^2+4^2}\,dx = \int \frac{(x+3)+2}{(x+3)^2+4^2}\,dx \\ &= \int \frac{u}{u^2+4^2}\,du + 2\int \frac{du}{u^2+4^2} = \frac{1}{2}\int \frac{d(u^2+4)}{u^2+4^2} + 2 \cdot \frac{1}{16}\int \frac{du}{\left(\frac{u}{4}\right)^2+1} \\ &= \frac{1}{2}\ln(u^2+4^2) + \frac{1}{2}\int \frac{dv}{v^2+1} = \ln\sqrt{(x+3)^2+16} + \frac{1}{2}\arctan v + C \\ &= \ln\sqrt{x^2+6x+25} + \frac{1}{2}\arctan\frac{x+3}{4} + C\end{aligned}$$

(here we set $u = x+3$, $v = \frac{u}{4}$). Checking that the derivative of the result is $\displaystyle\frac{x+5}{x^2+6x+25}$ is a good exercise.