Homogeneous linear ODE of order n with constant coefficients

We consider the equation

$$a_n y^{(n)} + \dots + a_1 y' + a_0 y = 0, \qquad (H)$$

where $a_0, \dots, a_n$ are real constants, and $a_n \neq 0$.

Let (C) stand for the characteristic equ:

$$a_n r^n + \dots + a_1 r + a_0 = 0. \qquad (C)$$

The characteristic equation has exactly n roots (in general complex), if we count each root with its multiplicity. If all coefficients $a_0, \dots, a_n$ are real, the complex roots come in conjugate pairs, i.e., if $\alpha + i\beta$ is a root of (C) of multiplicity p, then $\alpha - i\beta$ is also a root of (C) of multiplicity p.

Example: Consider the 13^{th} degree equ

$$(r-3)^4 (r+1)^5 (r^2+4)^2 = 0:$$

- 3 is a root of multiplicity 4
- -1 is a root of multiplicity 5
- $\pm 2i$ are roots, each of mult. 2

Each root of the characteristic equation (C) contributes a term to the general solution of the homogeneous equation (H). Namely:

- Each real root $r_1 \in \mathbb{R}$ of (C) of multiplicity p contributes a term
$$Q_{p-1}(x)\, e^{r_1 x}$$
to the general solution of (H); here $Q_{p-1}(x)$ is a polynomial of degree $p-1$ (having exactly p coefficients:
$$Q_p(x) = C_1 + C_2 x + C_3 x^2 + \dots + C_p x^{p-1}.$$

- Each conjugate pair of complex roots $\alpha \pm i\beta$ of (C), each of the two roots with multiplicity p, contributes a term
$$e^{\alpha x}\left[R_{p-1}(x)\cos\beta x + S_{p-1}(x)\sin\beta x\right],$$
where $R_{p-1}(x)$ and $S_{p-1}(x)$ are arbitrary polynomials of degree $p-1$:
$$R_{p-1}(x) = A_1 + A_2 x + \dots + A_p x^{p-1},$$
$$S_{p-1}(x) = B_1 + B_2 x + \dots + B_p x^{p-1}.$$

Example:

$$9y^{(5)} - 6y^{(4)} + y^{(3)} = 0$$

$$9r^5 - 6r^4 + r^3 = 0$$

$$r^3(9r^2 - 6r + 1) = 0$$

$$9r^3\left(r - \tfrac{1}{3}\right)^2 = 0$$

$\Rightarrow \begin{cases} \bullet\ 0 \text{ is a root of mult. } 3 \\ \bullet\ \tfrac{1}{3} \text{ is a root of mult. } 2 \end{cases}$

$\Rightarrow$ the general solution of the ODE is

$$y(x) = \underbrace{C_1 + C_2 x + C_3 x^2}_{\text{coming from the root } 0} + \underbrace{(C_4 + C_5 x)\, e^{\frac{1}{3}x}}_{\text{coming from the root } \frac{1}{3}}$$

Example:

$$(D-2)^4 (D^2 - 6D + 25)^3 y = 0$$

$$(r-2)^4 (r^2 - 6r + 25)^3 = 0;$$

the roots of $r^2 - 6r + 25 = 0$ are $3 \pm 4i$ (each with multiplicity 1), so the charact. equ can be written as

$$(r-2)^4 [r - (3+4i)]^3 [r - (3-4i)]^3 = 0,$$

and the general solution of the ODE is

$$y(x) = \underbrace{(C_1 + C_2 x + C_3 x^2 + C_4 x^3)\, e^{2x}}_{\text{coming from the root } 2}$$

$$+ \underbrace{e^{3x}\left[(C_5 + C_6 x + C_7 x^2)\cos 4x + (C_8 + C_9 x + C_{10} x^2)\sin 4x\right]}_{\text{coming from the conjugate pair } 3 \pm 4i}$$

Example:

$$y^{(11)} + 12y^{(9)} + 48y^{(7)} + 64y^{(5)} = 0,$$

which can be written as

$$D^5 (D^2 + 4)^3 y = 0,$$

so the charact. equ. is

$$r^5 (r^2 + 4)^3 = 0,$$

and the general solution is

$$y(x) = C_1 + C_2 x + C_3 x^2 + C_4 x^3 + C_5 x^4 + (C_6 + C_7 x + C_8 x^2)\cos 2x + (C_9 + C_{10} x + C_{11} x^2)\sin 2x$$

(why?).

Nonhomogeneous linear ODE of order n with constant coefficients

We are looking for the general solution of the nonhomogeneous equation

$$a_n y^{(n)} + \cdots + a_1 y' + a_0 y = f(x), \qquad (N)$$

where $a_0, \dots, a_n$ are constants, $a_n \neq 0$.

Let

$$a_n y^{(n)} + \cdots + a_1 y' + a_0 y = 0 \qquad (H)$$

be the corresponding homogeneous eqn, and

$$a_n r^n + \cdots + a_1 r + a_0 = 0 \qquad (C)$$

be the characteristic eqn of (H).

Notation:

$$L := a_n D^n + \cdots + a_1 D + a_0,$$

then (N) and (H) can be written as $Ly = f(x)$ and $Ly = 0$, respectively.

Two important rules:

1) $$\begin{pmatrix}\text{general sol'n}\\ y(x) \text{ of}\\ Ly = f(x)\end{pmatrix} = \begin{pmatrix}\text{general sol'n}\\ y_c(x) \text{ of}\\ Ly = 0\end{pmatrix} + \begin{pmatrix}\text{particular sol'n}\\ y_p(x) \text{ of}\\ Ly = f(x)\end{pmatrix}$$

2) If (N) reads

$$Ly = f_1(x) + \cdots + f_r(x),$$

then if $y_{p,1}(x), \dots, y_{p,r}(x)$ are solutions of

$$Ly = f_1(x), \dots, Ly = f_r(x),$$

resp., then

$$y_p(x) := y_{p,1}(x) + \cdots + y_{p,r}(x)$$

is a solution of

$$Ly = f_1(x) + \cdots + f_r(x).$$

Therefore, to find the general solution of $Ly = f(x)$, one has to find:

- the general solution of $Ly = 0$ (which is a standard procedure), and
- one particular solution of $Ly = f(x)$ (which is more complicated, and is explained below in some particular cases).

Method of undetermined coefficients for finding a particular sol'n of $Ly=f$

- Let
$$f(x) = e^{cx} P_m(x).$$
Then, if c is a root of (C) with multiplicity s, look for a particular solution of (N) of the form
$$y_p(x) = x^s e^{cx} Q_m(x)$$
(where $Q_m(x)$ is a polynomial of degree m)

- Let
$$f(x) = e^{cx}\left[P_{m_1}(x)\cos dx + P_{m_2}(x)\sin dx\right].$$
Then, if $c+id$ is a root of (C) with multiplicity s, define
$$m := \max(m_1, m_2),$$
and look for a particular solution of (N) of the form
$$y_p(x) = x^s e^{cx}\left[Q_m(x)\cos dx + \tilde{Q}_m(x)\sin dx\right],$$
where Q_m and $\tilde{Q}_m$ are polynomials of degree m.

Example:
$$y'' - 5y' + 6y = 2e^{3x}$$

- $r^2 - 5r + 6 = 0 \longrightarrow$ the roots of the characteristic equation are 2 & 3 (both simple), hence the general sol'n of the associated homogeneous eqn, $y'' - 5y' + 6y = 0$, is
$$y_c(x) = C_1 e^{2x} + C_2 e^{3x}.$$

- $f(x) = 2e^{3x}$ is of the form $e^{cx}P_m(x)$ with $c=3$, $m=0$. Since 3 is a root of (C) of multiplicity 1, we look for a particular solution of the nonhom. eqn. of the form
$$y_p(x) = x^1 e^{3x} P_0(x) = Axe^{3x}.$$
To determine the constant A, we plug $y_p(x)$ in the nonhomog. eqn:
$$y_p'(x) = A(1+3x)e^{3x}$$
$$y_p''(x) = 3A(2+3x)e^{3x}$$
(it is a great exercise to reproduce all these calculations), and substitute all these expressions in
$$y_p'' - 5y_p' + 6y_p = 2e^{3x};$$

$$3A(2+3x)e^{3x} - 5A(1+3x)e^{3x} + 6Axe^{3x} = 2e^{3x}$$

which, after expanding all expressions and simplifying, becomes

$$A + 0x = 2e^{3x} \Rightarrow A = 2,$$

so the function

$$y_p(x) = 2xe^{3x}$$

is a particular solution of (N). Therefore, the general solution of (N) is

$$y(x) = C_1 e^{2x} + C_2 e^{3x} + 2xe^{3x}.$$

<u>Example:</u>

$$y''' - y'' + y' - y = (x+1)e^{2x}$$

- $r^3 - r^2 + r - 1 = 0$

  $(r-1)(r^2+1) = 0$

  $\Rightarrow \begin{cases} 1 \text{ is a root of mult. } 1, \\ \pm i \text{ are roots of mult. } 1 \text{ each.} \end{cases}$

  $\Rightarrow y_c(x) = C_1 e^{x} + C_2 \cos x + C_3 \sin x.$

- $f(x) = (x+1)e^{2x} = e^{2x} P_1(x)$, and 2 is <u>not</u> a root of the char. eqn,

so we are looking for $y_p(x)$ of the form

$$y_p(x) = x^0 e^{2x} Q_1(x) = e^{2x}(Ax+B).$$

To determine the values of A and B, we compute:

$$y_p'(x) = e^{2x}(2Ax + 2B + A),$$

$$y_p''(x) = e^{2x}(4Ax + 4B + 4A)$$

$$y_p'''(x) = e^{2x}(8Ax + 8B + 12A),$$

and substitute in the nonhom. eqn:

$$e^{2x}(8Ax+8B+12A) - e^{2x}(4Ax+4B+4A) + e^{2x}(2Ax+2B+A) - e^{2x}(Ax+B) = (x+1)e^{2x}$$

$$\Rightarrow 5Ax + 9A + 5B = x + 1;$$

equating the coefficients in front of x^1 and x^0, we obtain the system

$$\left|\begin{aligned} 5A &= 1 \\ 9A + 5B &= 1, \end{aligned}\right.$$

whose solution is

$$A = \frac{1}{5}, \quad B = -\frac{4}{25}.$$

Therefore, a particular solution of the nonhom. eqn is

$$y_p(x) = e^{2x}\left(\frac{x}{5} - \frac{4}{25}\right),$$

and the general solution of the nonhom. eqn is

$$y(x) = C_1 e^x + C_2 \cos x + C_3 \sin x + e^{2x}\left(\frac{x}{5} - \frac{4}{25}\right).$$

Example:

$$y^{(3)} + y' = 2 - \sin x$$

- $r^3 + r = 0 \Rightarrow r(r^2+1) = 0$

$$\Rightarrow \begin{cases} 0 \text{ is a root of mult. } 1, \\ \pm i \text{ are roots of mult. } 1 \text{ each.} \end{cases}$$

$$\Rightarrow y_c(x) = C_1 + C_2 \cos x + C_3 \sin x.$$

- look for a particular solution, $y_{p,1}(x)$, of

$$y^{(3)} + y' = 2 = e^{0x} P_0(x).$$

Since 0 is a root of the char. eqn. of mult. 1, look for $y_{p,1}(x)$ of the form

$$y_{p,1}(x) = x^1 e^{0x} P_0(x) = Ax.$$

Substituting $y_{p,1}(x)$ in $y^{(3)} + y' = 2$, we obtain $A = 2$

$$\Longrightarrow y_{p,1}(x) = 2x.$$

- Look for a particular solution, $y_{p,2}(x)$, of

$$y^{(3)} + y' = -\sin x.$$

Since

$$\sin x = e^{0x}\left[P_0(x)\cos x + P_0(x)\sin x\right],$$

and $0 + i$ is a root of the char. eqn. of multiplicity 1, we look for $y_{p,2}(x)$ of the form

$$y_{p,2}(x) = x^1 e^{0x}\left[Q_0(x)\cos(1 \cdot x) + \tilde{Q}_0(x)\sin(1 \cdot x)\right] = x\left(B\cos x + C\sin x\right).$$

Plug $y_{p,2}(x)$ in

$$y^{(3)} + y' = -\sin x$$

to find that $B = 0$, $C = \frac{1}{2}$

$$\Rightarrow y_{p,2}(x) = \frac{1}{2} x \sin x.$$

Hence, the general solution of the complete nonhomogeneous equation is

$$y(x) = C_1 + C_2 \cos x + C_3 \sin x + 2x + \frac{1}{2} x \sin x.$$