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RESTRICTION AND DIMENSIONAL REDUCTION OF DIFFERENTIAL OPERATORS

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We consider the restriction of a differential operator (DO) D acting on the sections $C^\infty(\xi)$ of a vector bundle ξ with base B, in the language of jet bundles. When the base of ξ is restricted to a submanifold $\widetilde{U} \subseteq B$, all information about derivatives in directions that are not tangent to $\widetilde{U}$ is lost. To restrict D to a DO $D_{\widetilde{U}}$ acting on sections $C^\infty(\xi_{\widetilde{U}})$ of the restricted bundle $\xi_{\widetilde{U}} = i^*\xi$ (with $i : \widetilde{U} \hookrightarrow B$ the natural embedding), one must choose an auxiliary DO M and express the derivatives non-tangent to $\widetilde{U}$ from the kernel of M. This is equivalent to choosing a splitting of certain short exact sequence of jet bundles. A property of M called *formal integrability* is crucial for restriction's self-consistency. We give an explicit example illustrating what can go wrong if M is not formally integrable.

As an important application of this methodology, we consider the dimensional reduction of DOs invariant with respect to the action of a connected Lie group G. The splitting relation comes from the Lie derivative of the action, which is formally integrable. The reduction of the action of another group is also considered.

Keywords: Differential operator; restriction to a submanifold; invariant field; invariant differential operator.

1. Introduction

In this paper we consider the geometric description of the process of restriction of a differential operator (DO) to a submanifold, and some of its applications. The essence of the problem can be understood from the following elementary example. Consider $\mathbb{R}^3$ with Cartesian coordinates (x^1, x^2, x^3) and let $\mathbf{e}_i$, $i = 1, 2, 3$, be the corresponding unit vectors. Let $\mathbf{A} : \mathbb{R}^3 \to \mathbb{R}^3 = \sum_{i=1}^{3} A^i(x^1, x^2, x^3)\, \mathbf{e}_i$ be a smooth vector field on $\mathbb{R}^3$. We can restrict the domain of $\mathbf{A}$ to the submanifold $\{x^3 = 0\}$, thus obtaining a mapping $\widetilde{\mathbf{A}} : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $\widetilde{\mathbf{A}}(x^1, x^2) := \mathbf{A}(x^1, x^2, 0) = \sum_{i=1}^{3} A^i(x^1, x^2, 0)\, \mathbf{e}_i$. Can we compute $\operatorname{div} \mathbf{A} = \sum_{i=1}^{3} \frac{\partial A^i}{\partial x^i}$ at $\{z = 0\}$ by using only the restricted field $\widetilde{\mathbf{A}}$? Clearly not, because in the process of restricting $\mathbf{A}$ to $\widetilde{\mathbf{A}}$ we lost all the information about the dependence of $\mathbf{A}$ on x^3, hence we cannot compute $\frac{\partial \widetilde{A}^3}{\partial x^3}$. Therefore, if we wanted to compute $\operatorname{div} \mathbf{A}$ at $\{x^3 = 0\}$ from $\widetilde{\mathbf{A}}$, we would need some auxiliary information.

The auxiliary information needed in the above example can come from some additional requirements on $\mathbf{A}$. For example, if each component A^i of $\mathbf{A}$ satisfies

$$x^1 \frac{\partial A^i}{\partial x^1} + 2x^2 \frac{\partial A^i}{\partial x^2} + \frac{\partial A^i}{\partial x^3} - \pi A^i = 0 , \qquad i = 1, 2, 3 , \tag{1}$$

then we can use this condition for $i = 3$ to express $\frac{\partial A^3}{\partial x^3}$ at $\{x^3 = 0\}$ as

$$\frac{\partial A^3}{\partial x^3}(x^1, x^2, 0) = -x^1 \frac{\partial \widetilde{A}^3}{\partial x^1}(x^1, x^2) - 2x^2 \frac{\partial \widetilde{A}^3}{\partial x^2}(x^1, x^2) + \pi \widetilde{A}^3(x^1, x^2) .$$

This allows us to obtain $\operatorname{div} \mathbf{A}$ at $\{x^3 = 0\}$ in terms of $\widetilde{\mathbf{A}}$ and its x^1 and x^2 derivatives only, as

$$\begin{aligned} \operatorname{div} \mathbf{A}(x^1, x^2, 0) =& \frac{\partial \widetilde{A}^1}{\partial x^1}(x^1, x^2) + \frac{\partial \widetilde{A}^2}{\partial x^2}(x^1, x^2) \\ & - x^1 \frac{\partial \widetilde{A}^3}{\partial x^1}(x^1, x^2) - 2x^2 \frac{\partial \widetilde{A}^3}{\partial x^2}(x^1, x^2) + \pi \widetilde{A}^3(x^1, x^2) . \end{aligned}$$

Therefore, for vector fields $\mathbf{A}$ satisfying the relation (1), we can define the restriction, $\widetilde{\operatorname{div}}$, of the DO div to $\{x^3 = 0\}$ by

$$\begin{aligned} \widetilde{\operatorname{div}}\, \widetilde{\mathbf{A}}(\widetilde{x}^1, \widetilde{x}^2) :=& \operatorname{div} \mathbf{A}(x^1, x^2, 0) \\ =& \frac{\partial \widetilde{A}^1}{\partial \widetilde{x}^1}(\widetilde{\mathbf{x}}) + \frac{\partial \widetilde{A}^2}{\partial \widetilde{x}^2}(\widetilde{\mathbf{x}}) - \widetilde{x}^1 \frac{\partial \widetilde{A}^3}{\partial \widetilde{x}^1}(\widetilde{\mathbf{x}}) - 2\widetilde{x}^2 \frac{\partial \widetilde{A}^3}{\partial \widetilde{x}^2}(\widetilde{\mathbf{x}}) + \pi \widetilde{A}^3(\widetilde{\mathbf{x}}) , \end{aligned}$$

where $\widetilde{\mathbf{x}} := (\widetilde{x}^1, \widetilde{x}^2) := (x^1, x^2)$ are the coordinates "internal" for the manifold $\{x^3 = 0\}$.

If instead of (1) we imposed the auxiliary condition

$$x^1 \frac{\partial A^i}{\partial x^1} + x^2 \frac{\partial A^i}{\partial x^2} + x^3 \frac{\partial A^i}{\partial x^3} - A^i = 0 , \qquad i = 1, 2, 3 , \tag{2}$$

then the restriction of the third component of (2) to $\{x^3 = 0\}$ does *not* contain the derivative $\frac{\partial A^3}{\partial x^3}$ needed to compute $\operatorname{div} \mathbf{A}$. The moral of this observation is that not every condition on $\mathbf{A}$ can be used as an auxiliary condition to compute the restricted DO.

There are many questions that one can ask at this point. If several derivatives are needed in order to compute the restricted DO, can one impose several auxiliary conditions? If several auxiliary conditions are needed, are they compatible? Can different auxiliary conditions lead to the same restricted DO? Can we obtain higher-order partial derivatives by differentiating the auxiliary condition(s) several times?

The answers to these questions are by no means straightforward. In particular, the answer to the last question is positive only if the DO from the auxiliary condition (like condition (1) in the example above) is *formally integrable* – a subtle concept discussed in Sect. 2.3 below. The dangers of using DOs that are not formally integrable is illustrated in Example 5 in Sect. 3.3.3.

The process of restriction of DOs to a submanifold is intimately related with the process of reduction of DOs that are invariant with respect to the action of a Lie group. Here we will sketch the basic ideas (we treat this in detail in Sect. 4). Let ξ and η be smooth vector bundles over the same base B, and G be a Lie group acting on ξ and η through vector bundle morphisms, with the same action of G on the common base B. This naturally induces an action of G on the sections $C^\infty(\xi)$ and $C^\infty(\eta)$ of the bundles ξ and η. Let $C^\infty(\xi)^G$ and $C^\infty(\eta)^G$ stand for the sets of the sections of ξ and η that are invariant with respect to this action (for brevity, we call these sections G-*invariant*). If some natural conditions on the actions of G on ξ and η are satisfied, one can construct the *reduced bundles* ξ^G and η^G – vector bundles over B/G such that the set $C^\infty(\xi^G)$ of all sections of ξ^G is in a bijective correspondence with the set $C^\infty(\xi)^G$ of all G-invariant sections of ξ (and similarly for η).

Let $D : C^\infty(\xi) \to C^\infty(\eta)$ be an order-k DO from ξ to η. The actions of G on $C^\infty(\xi)$ and $C^\infty(\eta)$ induce naturally an action of G on the set $\mathrm{Diff}_k(\xi, \eta)$ of such DOs. Each G-invariant DO D maps $C^\infty(\xi)^G$ to $C^\infty(\eta)^G$ and, hence, determines a *reduced* DO $D^G : C^\infty(\xi^G) \to C^\infty(\eta^G)$ between the sections of the reduced bundles.

The base of the reduced bundles is the set of orbits, B/G, of the action of G on the common base B of ξ and η. Since B/G is an abstractly defined object, it is preferable to work with some concrete realization of it. For such realization one can choose a collection of local submanifolds U_α ($\alpha \in \mathscr{A}$) of B that are transverse to the orbits of G in B. [a] The reduced DO D^G is a collection of DOs (one for each index $\alpha \in \mathscr{A}$) each of which is obtained as a restriction of the invariant DO D (acting on invariant sections) to the submanifold $\widetilde{U}_\alpha$. To perform this restriction, we need an auxiliary condition (like (1)) that allows us to express all derivatives that were "lost" in the restriction of the invariant section to $\widetilde{U}_\alpha$. Fortunately, in the case of invariant DOs, there is a natural DO which plays the role of the auxiliary relation (1) in the example above – this is the Lie derivative of the action of G. Intuitively, this is so because we can use the G-invariance of the section restricted to $\widetilde{U}_\alpha$ in order to extend it to an open neighborhood of $\widetilde{U}_\alpha$, and then compute all desired

[a] Here "transverse" means the following: let $\widetilde{U}_\alpha \subset B$ be one such local submanifold, let b be a point of $\widetilde{U}_\alpha$, and let $\mathcal{O}_b$ be the G-orbit of the point b (i.e., $\mathcal{O}_b = \{t_g(b) : g \in G\}$, where $t : G \times B \to B$ is the action of G on B); then "transversality" means that the tangent space $T_b B$ is a direct sum of the tangent spaces $T_b \widetilde{U}_\alpha$ and $T_b \mathcal{O}_b$.

derivatives of the extended section. In Sect. 4.3.5 we prove that the Lie derivative is a formally integrable DO, which guarantees that the procedure of dimensional reduction of an invariant DO is self-consistent.

The natural setting for considering DOs on vector bundles is the formalism of jet bundles. A section of the kth jet bundle of the vector bundle ξ is a coordinate-free notion for the kth Taylor polynomial of a section of ξ. In this language, a DO is interpreted as a purely algebraic object – the *total symbol* of the DO – which is a fiber-preserving mapping between jet bundles.

In the paper [1] we developed a geometric methodology for dimensional reduction of sections that are invariant with respect to the action of a connected (but not necessarily compact) Lie group G. In the present paper we apply the algorithm for restriction of DOs to a submanifold to the case of an invariant DO D. We obtain a systematic procedure for constructing the reduced DO D^G; our algorithm involves only computing derivatives and solving a linear system of algebraic equations.

The algorithm for computing the reduced DO D^G elucidates the geometric structures arising naturally in the process of reduction. These issues are important, for example, in Kaluza-Klein theories and in model building with a desired symmetry in elementary particle physics. An example of this approach is our construction [2] of DOs on Minkowski space that are invariant with respect to the (nonlinear) action of the conformal group, starting from the (linear) action of the orthogonal group on a bigger space.

The plan of the paper is the following. In Sect. 2 we introduce the jet bundle language for description of DOs, paying special attention to the concept of formal integrability (Sections 2.3 and 2.4). Sect. 3 is devoted to the problem of restriction of a DO to a submanifold. In Sect. 4 we develop an algorithm for reduction of an invariant DO, and resolve all related theoretical issues. We illustrate the concepts with several simple examples.

2. Differential operators on vector bundles

Here we introduce the coordinate-free definition of differential operators, following [3, Chapter IV].

2.1. *Jet bundles*

Let $\xi = (E, \pi, B)$ and η be finite-dimensional $\mathbb{K}$-vector bundles over the real finite-dimensional manifold B; the standard fiber $\xi_b := \pi^{-1}(b)$ of ξ over $b \in B$ is a vector space over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. As usual, $C^\infty(B)$ stands for the ring of all smooth $\mathbb{K}$-valued functions on B, and $C^\infty(\xi)$ denotes the set of all smooth sections of ξ (which is a module over $C^\infty(B)$). Throughout the paper it is always assumed that the manifolds, vector bundles, functions and sections of bundles are smooth (C^∞).

Let $I_b(B)$ be the ideal of the ring $C^\infty(B)$ consisting of all functions $f \in C^\infty(B)$ that vanish at the point $b \in B$:

$$I_b(B) := \{ f \in C^\infty(B) \,:\, f(b) = 0 \} \subseteq C^\infty(B) \ .$$

Let $I_b^k(B)$ stand for the ideal of the ring $C^\infty(B)$ consisting of all functions on B that can be represented as a product of k functions from $I_b(B)$.

Let $Z_b^k(B)$ stand for the subspace of $C^\infty(\xi)$ consisting of all sections of ξ that can be represented as products of a function from $I_b^{k+1}(B)$ and a section of ξ:

$$Z_b^k(\xi) := \{\, \psi \in C^\infty(\xi) \,:\, \psi = f \cdot \kappa \,,\ \text{where } f \in I_b^{k+1}(B)\,,\ \kappa \in C^\infty(\xi) \,\} \ .$$

Clearly, all partial derivatives of $\psi \in Z_b^k(\xi)$ (with respect to the coordinates in the base B) up to order k vanish at b.

For each $b \in B$ define $J^k(\xi)_b := C^\infty(\xi)/Z_b^k(\xi)$, and let $J^k(\psi)_b$ be the image of ψ under the canonical projection $C^\infty(\xi) \to J^k(\xi)_b : \psi \mapsto J^k(\psi)_b$. The *$k$-jet* $J^k(\psi)$ of the section $\psi \in C^\infty(\xi)$ is defined by $J^k(\psi)(b) := J^k(\psi)_b$ for any $b \in B$. The k-jet $J^k(\psi)_b$ is the coordinate-free concept for the section ("the field") ψ and its derivatives up to order k at b.

Let $J^k(\xi)$ stand for the disjoint union $\coprod_{b\in B} J^k(\xi)_b$ endowed with the natural vector bundle structure. In more detail, this means the following. Let (x^μ, z^a) be local coordinates in ξ, where $\mu = 1, \ldots, n$ (with $n = \dim B$), $a = 1, \ldots, \operatorname{rank} \xi$ ($\operatorname{rank} \xi$ is the dimension of the standard fiber of ξ). They generate local coordinates in the jet bundle $J^k(\xi)$,

$$\left(x^\mu, z^a, z^a_{\mu_1}, z^a_{\mu_1\mu_2}, \ldots, z^a_{\mu_1\cdots\mu_k}\right) \ , \quad \begin{cases} 1 \le \mu_1 \le \cdots \le \mu_i \le n \ , \ i = 1, \ldots, k \\ a = 1, \ldots, \dim \xi \ , \end{cases} \tag{3}$$

so that the coordinates of $J^k(\psi)(b)$ are

$$\Big(J^k(\psi)(b)\Big)^a_{\mu_1\cdots\mu_i} = \partial_{\mu_1\cdots\mu_i} \psi^a(b) := \frac{\partial^i \psi^a}{\partial x^{\mu_1} \cdots \partial x^{\mu_i}}(b) \ . \tag{4}$$

The transition functions gluing $J^k(\xi)$ come from the standard formulae for transformation of partial derivatives under a change of variables. From the definition, $J^0(\xi) = \xi$. The number of partial derivatives of order i of each component ψ^a is equal to $\binom{n+i-1}{i}$, which is the number of unordered selections of i objects, with repetition allowed, out of n distinct objects. The dimension of the fibers of the vector bundle $J^k(\xi)$ is, therefore,

$$\operatorname{rank} J^k(\xi) = \sum_{i=0}^{k} \binom{n+i-1}{i} \operatorname{rank} \xi = \binom{n+k}{k} \operatorname{rank} \xi \ . \tag{5}$$

For integers k and l satisfying $k \ge l \ge 0$, let

$$\pi^{k,l} : J^k(\xi) \to J^l(\xi) \tag{6}$$

stand for the natural projections ("cutting off" all derivatives of order $l+1, \ldots, k$).

2.2. *Differential operators and their symbols*

2.2.1. *Notations and natural isomorphisms*

We start by introducing some useful notations. Let $\mathbb{F}$ be a subfield of the field $\mathbb{K}$ (where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$), and B be a real finite dimensional manifold. Let γ be an $\mathbb{F}$-vector bundle over B, and ζ be a $\mathbb{K}$-vector bundle over B.

A vector bundle morphism F from γ to ζ over the identity in B is a mapping from γ to ζ whose restriction $F\restriction_{\gamma_b}$ to a fiber γ_b is an $\mathbb{F}$-linear mapping from γ_b to ζ_b. Denote by $\mathrm{Hom}\,(\gamma,\zeta)$ the set of all such morphisms; $\mathrm{Hom}\,(\gamma,\zeta)$ is naturally endowed with a structure of a $\mathbb{K}$-vector space. In Sections 2–3 – before we consider the action of a Lie group on the vector bundles – all vector bundle morphisms (and, more generally, all fiber-preserving mappings) are over the identity in B.

Let $L(\gamma,\zeta)$ be the $\mathbb{K}$-vector bundle over B whose fiber $L(\gamma,\zeta)_b$ over $b \in B$ consists of all $\mathbb{F}$-linear mappings from γ_b to ζ_b. Let $L^k(\gamma,\zeta)$ stand for the $\mathbb{K}$-vector bundle over B with $L^k(\gamma,\zeta)_b$ equal to the set of all k-linear (with respect to $\mathbb{F}$) mappings from γ_b to ζ_b. Denote by $L^k_s(\gamma,\zeta)$ the subbundle of $L^k(\gamma,\zeta)$ with $L^k_s(\gamma,\zeta)_b$ defined as the set of all symmetric k-linear (with respect to $\mathbb{F}$) mappings from γ_b to ζ_b.

Each fiber-preserving mapping $F : \gamma \to \zeta$ over the identity in B (in particular, each vector bundle morphism $F \in \mathrm{Hom}\,(\gamma,\zeta)$) induces a mapping F_* between the sections of γ and ζ defined by

$$F_* : C^\infty(\gamma) \to C^\infty(\zeta) : \psi \mapsto F_*\psi := F \circ \psi \ . \tag{7}$$

Let γ^* stand for the vector bundle dual to γ, i.e., each fiber $(\gamma^*)_b$ is the dual to the $\mathbb{F}$-vector space γ_b. Denote by $C^\infty(\gamma^* \otimes \zeta)$ the set of all $\mathbb{F}$-linear mappings from γ^* to ζ over the identity in B; clearly, $C^\infty(\gamma^* \otimes \zeta)$ is a $\mathbb{K}$-vector space. If $F \in \mathrm{Hom}\,(\gamma,\zeta)$, the mapping F_* from (7) can be considered as an element of $C^\infty(\gamma^* \otimes \zeta)$, and we obtain that the three $\mathbb{K}$-vector bundles introduced above are naturally isomorphic:

$$\mathrm{Hom}\,(\gamma,\zeta) \cong C^\infty(\gamma^* \otimes \zeta) \cong C^\infty(L(\gamma,\zeta)) \ . \tag{8}$$

2.2.2. *A differential operator and its total symbol*

Let ξ and η be $\mathbb{K}$-vector bundles over a common base B. A *differential operator* (DO) of order k from ξ to η is a mapping $D : C^\infty(\xi) \to C^\infty(\eta)$ such that, for any $b \in B$, if $\psi_1 \in C^\infty(\xi)$ and $\psi_2 \in C^\infty(\xi)$, then $J^k(\psi_1)(b) = J^k(\psi_2)(b)$ implies $D(\psi_1)(b) = D(\psi_2)(b)$. In other words, $D\psi(b)$ depends only on the values of ψ and its derivatives up to order k at the point b. The set of all DOs of order k from ξ to η will be denoted by $\mathrm{Diff}_k(\xi,\eta)$. A DO $D \in \mathrm{Diff}_k(\xi,\eta)$ can be identified with a fiber-preserving mapping $\widetilde{D}$ from $J^k(\xi)$ to η over the identity in B: in the notations of (7),

$$D = \widetilde{D}_* J^k \ . \tag{9}$$

The mapping $\widetilde{D}$ is called the *total symbol* of the DO D.

A DO $D \in \mathrm{Diff}_k(\xi, \eta)$ is said to be *linear* if it is a $\mathbb{K}$-linear mapping. Let $\mathrm{LDiff}_k(\xi, \eta)$ stand for the subset of $\mathrm{Diff}_k(\xi, \eta)$ that consists of all linear DOs. It is clear from the definition that $J^k \in \mathrm{LDiff}_k(\xi, J^k(\xi))$. A linear DO $D \in \mathrm{LDiff}_k(\xi, \eta)$ can be identified with a vector bundle morphism $\widetilde{D} \in \mathrm{Hom}\,(J^k(\xi), \eta)$ through (9). Using (8), we can write the following natural isomorphisms:

$$\mathrm{LDiff}_k(\xi, \eta) \cong \mathrm{Hom}\,(J^k(\xi), \eta) \cong C^\infty(J^k(\xi)^* \otimes \eta) \cong C^\infty(L(J^k(\xi), \eta))\ . \tag{10}$$

2.3. *Prolongation of a DO and formal integrability*

One can differentiate simultaneously both sides of a differential equation $D\psi = \phi$. In a coordinate-free language, the result of differentiating of a DO is called its *prolongation*. We will define this concept only for linear DOs (which is the only case that we will use in this paper).

Let $D = \widetilde{D}_* J^k \in \mathrm{LDiff}_k(\xi, \eta)$ be a linear DO, and $\widetilde{D} \in \mathrm{Hom}\,(J^k(\xi), \eta)$ be its total symbol. The *l-th prolongation* of D is a linear DO $P^l(D) \in \mathrm{LDiff}_{k+l}(\xi, J^l(\eta))$ is such that the following diagram commutes:

$$\begin{tikzcd} C^\infty(J^{k+l}(\xi)) \arrow[r, "\widetilde{P^l(D)}_*"] & C^\infty(J^l(\eta)) \\ C^\infty(\xi) \arrow[u, "J^{k+l}"] \arrow[ur, "P^l(D)"] \arrow[r, "D"] & C^\infty(\eta) \arrow[u, "J^l"'] \end{tikzcd} \tag{11}$$

The total symbol, $\widetilde{P^l(D)} \in \mathrm{Hom}\,\left(J^{k+l}(\xi), J^l(\eta)\right)$, of $P^l(D)$ is the only linear fiber-preserving mapping such that $P^l(D) = \widetilde{P^l(D)}_* J^{k+l}$ (in the notations of (7)).

For $D \in \mathrm{LDiff}_k(\xi, \eta)$, we set

$$R^{k,l} := \ker \widetilde{P^l(D)}\ , \tag{12}$$

which in general is a family of linear subspaces of the vector bundle $J^{k+l}(\xi)$.

A linear DO $D \in \mathrm{LDiff}_k(\xi, \eta)$ is said to be *formally integrable* if for each $l \geq 0$ the following conditions are satisfied:

(a) ("regularity", "constancy of rank") $R^{k,l}$ is a vector subbundle of $J^{k+l}(\xi)$;
(b) ("existence of formal solutions") the natural projection $\pi^{k+l+1,k+l}$: $R^{k,l+1} \to R^{k,l}$ is an epimorphism (i.e., a surjective linear mapping).

For a formally integrable DO $D \in \mathrm{LDiff}_k(\xi, \eta)$, the subbundle $R^{k,0} \subseteq J^k(\xi)$ is called its *equation*.

Remark 1. Clearly, if a DO is formally integrable, all its prolongations are also formally integrable DOs.

The formal integrability is crucially important for the dimensional reduction of invariant DOs considered in Sect. 4. Condition (b) from the definition of formal integrability deserves some discussion. Let $D \in \mathrm{LDiff}_k(\xi, \eta)$ and $\psi \in C^\infty(\xi)$. By differentiating the equation $D\psi = 0$, we obtain the equation $P^l(D)\psi = 0$ of order $(k + l)$. In general, it may happen that the equation $P^l(D)\psi = 0$ contains some condition only on derivatives of order $\leq (k + l - 1)$ that was not contained in the prolongation $P^{l-1}(D)\psi = 0$. If this is the case, then one can never be sure that all the conditions on the partial derivatives of some order are obtained by any finite number of prolongations. Condition (b) of the definition of formal integrability guarantees that this cannot happen for a formally integrable DO. The example below clarifies this important point.

Example 2. Here is an example of a DO that is not formally integrable, suggested in the classic work of Janet [4, pp. 76–77] and discussed in [5, Introduction] and [6, pp. 97–98]. Let ξ and η be globally trivial vector bundles over the manifold $B = \{(x^1, x^2, x^3) : x^2 > 1\} \subset \mathbb{R}^3$ with standard fibers $\mathbb{R}$ and $\mathbb{R}^2$, respectively. (The condition $x^2 > 1$ is not essential for this example, but it will be convenient in the example in Sect. 3.3.3, where the results of this example are used.) Let the DO $M \in \mathrm{LDiff}_2(\xi, \eta)$ be defined as $M := \begin{pmatrix} \partial_{11} - x^2\partial_{33} \\ \partial_{22} \end{pmatrix}$. Then one can easily check that $\pi^{3,2} : R^{2,1} \to R^{2,0}$ is an epimorphism, i.e., that the equation $P^1(M)(\psi) = 0$ does not impose any additional conditions on the first and second derivatives that were not present in the equation $M\psi = 0$. The second prolongation, however, contains the condition $\partial_{233}\psi = 0$, which was not present in $R^{2,1}$ – to see this, differentiate the first component of $M\psi = 0$ twice with respect to x^2, then note that the second component of $M\psi = 0$ (i.e., the equation $\partial_{22}\psi = 0$) implies that $\partial_{1122}\psi = 0$ and $\partial_{2233}\psi = 0$. In formal language, this means that the projection $\pi^{4,3} : R^{2,2} \to R^{2,1}$ is not an epimorphism. The differential equation $M\psi = 0$ is not difficult to solve explicitly, and its solution can be shown to be contain only 12 arbitrary constants, namely,

$$\begin{aligned} \psi(x^1, x^2, x^3) = {} & \alpha_1 x^1 \left[(x^1)^2 x^2 + (x^3)^2\right] x^3 + \alpha_2 \left[3(x^1)^2 x^2 + (x^3)^2\right] x^3 \\ & + \alpha_3 x^1 x^2 x^3 + \alpha_4 x^1 \left[(x^1)^2 x^2 + 3(x^3)^2\right] \\ & + \alpha_5 \left[(x^1)^2 x^2 + (x^3)^2\right] + \alpha_6 x^1 x^2 + \alpha_7 x^1 x^3 + \alpha_8 x^2 x^3 \\ & + \alpha_9 x^1 + \alpha_{10} x^2 + \alpha_{11} x^3 + \alpha_{12} \ . \end{aligned}$$

This happens because there are infinitely many conditions that the higher prolongations of $M\psi = 0$ imply on the lower-order derivatives (like the condition on $\partial_{112}\psi$ given above), which results in a general solution depending only on a finite number of parameters instead of depending on arbitrary functions.

2.4. *Involutivity and formal integrability*

A collection of vector fields on a manifold can be considered as a linear first order DO acting on the smooth scalar-valued functions defined on the manifold. It turns out that this DO is formally integrable exactly when the collection of vector fields is involutive. Since involutivity is a well-known property, below we give some motivating examples and state and prove a theorem clarifying this relation. The result is not needed for understanding the rest of the paper and is included here only to clarify the complicated concept of formal integrability of DOs. It is interesting to note, however, that Theorem 3 below is a particular case of Theorem 14 in Sect. 4.3.5 which establishes the formal integrability of the Lie derivative (66) of an action of a Lie group G on a vector bundle (because the fundamental vector fields (68) of the action G on B are in involution – see Remark 16).

We start with two motivating examples. Let ξ and η be globally trivial vector bundles over the manifold $B = \{(x^1, x^2, x^3) : x^1 > 0\} \subset \mathbb{R}^3$ with standard fibers $\mathbb{R}$ and $\mathbb{R}^2$, respectively. Let X and Y be vector fields on B, and $M := \begin{pmatrix} X \\ Y \end{pmatrix} \in \mathrm{LDiff}_1(\xi, \eta)$. Consider the following two cases:

- If $X = \partial_1$ and $Y = x^1\partial_1 + \partial_2 + \partial_3$, then one can easily check that the general solution of the equation $M\psi = 0$ (i.e., of the system $\partial_1\psi = 0$, $(x^1\partial_1 + \partial_2 + \partial_3)\psi = 0$) is $\psi(x^1, x^2, x^3) = g(x^2 - x^3)$, where g is an arbitrary differentiable function of one variable. Note that in this case $[X, Y] = X \in \operatorname{span}\{X, Y\}$.
- If $X = \partial_1$, $Y = \partial_2 + x^1\partial_3$, then the general solution of $M\psi = 0$ is $\psi(x^1, x^2, x^3) = \mathrm{const}$. Note that in this case $[X, Y] = \partial_3 \notin \operatorname{span}\{X, Y\}$.

The reason for the fact that in the latter case the general solution had smaller amount of "arbitrariness" (only one arbitrary constant instead of one arbitrary function of one variable) is due to the fact that the collection of vector fields $\{X, Y\}$ was not involutive. The involutivity of a set of vector fields is closely related to the formal integrability of the DO these vector fields define. We address this connection in the theorem below.

Theorem 3. *Let B be a real n-dimensional manifold, and (x^μ) be some local coordinates. Let $\xi = (B \times \mathbb{K}, \pi_1, B)$, $\eta = (B \times \mathbb{K}^d, \pi_1, B)$ be vector bundles over B, where $\mathbb{K}$ is some field. Let X_a $(a = 1, \ldots, d)$ be vector fields on B that are linearly independent at each point, and the DO $D \in \mathrm{LDiff}_1(\xi, \eta)$ be defined as*

$$D(\psi)(b) = \begin{pmatrix} X_1(b)^\mu\, \partial_\mu \psi(b) \\ \vdots \\ X_d(b)^\mu\, \partial_\mu \psi(b) \end{pmatrix}, \qquad \psi \in C^\infty(\xi) . \tag{13}$$

Then D is formally integrable if and only if the collection of vector fields $\{X_a\}_{a=1}^d$ is involutive.

Proof. The local coordinates (x^μ, z) in ξ and (x^μ, w^a) in η generate local jet bundle coordinates (x^μ, z, z_μ) in $J^1(\xi)$, $(x^\mu, z, z_\mu, z_{\mu\nu})$ in $J^2(\xi)$, and (x^μ, w^a, w^a_μ) in $J^1(\eta)$ (recall (3)). Let $b = (x^\mu) \in B$ be an arbitrary point.

The total symbols of D and its first prolongation are respectively

$$\widetilde{D} : J^1(\xi) \to \eta : (x^\mu, z, z_\mu) \mapsto (x^\mu, w^a) \quad \text{with} \quad w^a = X_a(b)^\mu z_\mu \ ,$$

and

$$\widetilde{P^1(D)} : J^2(\xi) \to J^1(\eta) : (x^\mu, z, z_\mu, z_{\mu\nu}) \mapsto (x^\mu, w^a, w^a_\nu)$$

with $w^a = X_a(b)^\mu z_\mu$, $w^a_\nu = \partial_\nu X_a(b)^\mu z_\mu + X_a(b)^\mu z_{\mu\nu}$. Let $R^{k,l}_b$ be the fiber over $b \in B$ of $R^{k,l} \subseteq J^{k+l}(\xi)$ (defined in (12)), and $\pi^{k+l,k} : J^{k+l}(\xi) \to J^k(\xi)$ be the canonical projections (6). In these notations,

$$(x^\mu, z, z_\mu) \in R^{1,0}_b \iff X_a(b)^\mu z_\mu = 0 \tag{14}$$

$$(x^\mu, z, z_\mu, z_{\mu\nu}) \in R^{1,1}_b \iff \begin{cases} X_a(b)^\mu z_\mu = 0 \\ \partial_\nu X_a(b)^\mu z_\mu + X_a(b)^\mu z_{\mu\nu} = 0 \end{cases} \tag{15}$$

(it is understood that the conditions in the right-hand side hold for all $a = 1, \dots, d$ and $\nu = 1, \dots, n$).

Multiplying $\partial_\nu X_a(b)^\mu z_\mu + X_a(b)^\mu z_{\mu\nu} = 0$ by $X_c(b)^\nu$, and $\partial_\nu X_c(b)^\mu z_\mu + X_c(b)^\mu z_{\mu\nu} = 0$ by $X_a(b)^\nu$ yields the system

$$X_c(b)^\nu \partial_\nu X_a(b)^\mu z_\mu + X_c(b)^\nu X_a(b)^\mu z_{\mu\nu} = 0$$
$$X_a(b)^\nu \partial_\nu X_c(b)^\mu z_\mu + X_a(b)^\nu X_c(b)^\mu z_{\mu\nu} = 0 \ .$$

Subtracting the first equation from the second, we obtain $[X_a, X_c](b)^\mu z_\mu = 0$, where $[X_a, X_c](b)^\mu$ is the μth component of the commutator of X_a and X_c at b. This implies that

$$(x^\mu, z, z_\mu) \in \pi^{2,1}(R^{1,1}_b) \implies \begin{cases} X_a(b)^\mu z_\mu = 0 \\ [X_a, X_c](b)^\mu z_\mu = 0 \quad \forall\, c = 1, \dots, d. \end{cases} \tag{16}$$

Assume that the DO D (13) is formally integrable. Then $\pi^{2,1} : R^{1,1} \to R^{1,0}$ is an epimorphism, so from (14) and (16) we obtain that $X_a(b)^\mu z_\mu = 0$ implies $[X_a, X_c](b)^\mu z_\mu = 0$ for any $c = 1, \dots, d$. Since this holds for every point $b \in B$, the formal integrability of D implies the involutivity of the collection of vector fields $\{X_a\}_{a=1}^d$.

Conversely, assume that $\{X_a\}_{a=1}^d$ is involutive. Then, by the Frobenius Theorem (see, e.g., [6, Sect. 2.4]), it is possible to choose local coordinates (x^μ) in B such that

$$\text{span}\, \{X_1, \cdots, X_d\} = \text{span} \left\{ \frac{\partial}{\partial x^{n-d+1}}, \dots, \frac{\partial}{\partial x^n} \right\} \ .$$

In these coordinates, the equation $D\psi = 0$ where D is defined in (13) is equivalent to the system

$$\frac{\partial \psi}{\partial x^{n-d+1}} = 0 \ , \quad \frac{\partial \psi}{\partial x^{n-d+2}} = 0 \ , \quad \dots \ , \quad \frac{\partial \psi}{\partial x^n} = 0 \ ,$$

hence the equations determining $R^{1,l}$ become

$$\begin{aligned}
R^{1,0} &= \{z_{\bar\beta} = 0\} \ ; \\
R^{1,1} &= \{z_{\bar\beta} = 0\,,\ z_{\bar\beta\mu} = 0\} \ ; \\
R^{1,2} &= \{z_{\bar\beta} = 0\,,\ z_{\bar\beta\mu} = 0\,,\ z_{\bar\beta\mu\nu} = 0\} \ ; \\
R^{1,3} &= \{z_{\bar\beta} = 0\,,\ z_{\bar\beta\mu} = 0\,,\ z_{\bar\beta\mu\nu} = 0\,,\ z_{\bar\beta\mu\nu\rho} = 0\} \ ;
\end{aligned}$$

etc.; here μ, ν, ρ, $\ldots$ take values from 1 to d, while $\bar\beta = n-d+1,\ldots,n$. This makes it obvious that all projections $\pi^{l+2,l+1} : R^{1,l+1} \to R^{1,l}$, $l = 0,1,2,\ldots$ are epimorphisms, i.e., D is formally integrable. □

The interested reader can find more about the relation between involutivity and formal integrability in [6, Ch. 4] and [7, Chapters IX and X].

3. Restriction of a DO to a submanifold

Let ξ and η be vector bundles over the manifold B. Let $C \subseteq B$ be a submanifold of B and $i : C \hookrightarrow B$ be the natural embedding. We will denote by ξ_C or, equivalently, by $\xi\restriction_C$, the bundle $i^*\xi$ induced by i. In other words, $\xi_C \equiv \xi\restriction_C = (E', \pi', C)$ where $E' := \pi^{-1}(C)$, π' is the restriction of π to E', and the fiber $(\xi_C)_c = (\pi')^{-1}(c)$ of ξ_C over a point $c \in C$ is the same as $\xi_c = \pi^{-1}(c)$.

In this section we will discuss the problems that occur in attempting to restrict a DO $D \in \mathrm{Diff}_k(\xi,\eta)$ to a DO $D_C \equiv D\restriction_C \in \mathrm{Diff}_{k'}(\xi_C,\eta_C)$ (in general, k' may not be equal to k). These problems are central in the process of dimensional reduction of invariant DOs which is considered in Sect. 4, so below we consider them in detail.

In what follows we adopt the following definition. We say that two linear subspaces, L_1 and L_2, of the the linear space L are *transversal* and write $L = L_1 \oplus L_2$ if

- they are complementary in the sense of linear algebra, i.e., each $v \in L$ can be decomposed as $v = v_1 + v_2$ with $v_1 \in L_1$, $v_2 \in L_2$;
- they intersect trivially: $L_1 \cap L_2 = \{0\}$.

We say that two submanifolds B_1 and B_2 of the manifold B intersect *transversely* at $b \in B_1 \cap B_2$ if $T_bB = T_bB_1 \oplus T_bB_2$. We say that two subbundles ξ_1 and ξ_2 of the vector bundle ξ are *transversal* and write $\xi = \xi_1 \oplus \xi_2$ if $\xi_b = (\xi_1)_b \oplus (\xi_2)_b$ for each b in the base of ξ.

3.1. *Set-up and notations*

Let $\widetilde U$ be a submanifold of the common base B of ξ and η, and $\xi_{\widetilde U} \equiv \xi\restriction_{\widetilde U}$ and $\eta_{\widetilde U}$ be the corresponding restrictions. If $D \in \mathrm{Diff}_k(\xi,\eta)$, then in general the natural embedding $i : \widetilde U \hookrightarrow B$ does not provide us with enough information in order to construct a restricted DO $D_{\widetilde U}$. Indeed, let $\dim \widetilde U = \widetilde n$, and assume that the local coordinates $(x^1, \dots, x^n)$ in B are *adapted* to $\widetilde{U}$ in the sense that

$$\widetilde{U} = \{x^{\widetilde{n}+1} = \cdots = x^n = 0\} \; ; \tag{17}$$

in these coordinates, the natural embedding $i : \widetilde{U} \hookrightarrow B$ is given by

$$i\big((x^1, \dots, x^{\widetilde{n}})\big) = (x^1, \dots, x^{\widetilde{n}}, 0, \dots, 0) \; .$$

We will call $(x^1, \dots, x^{\widetilde{n}})$ *internal for* $\widetilde{U}$ and $(x^{\widetilde{n}+1}, \dots, x^n)$ *external for* $\widetilde{U}$ coordinates. Let $\psi \in C^\infty(\xi)$ and

$$\psi_{\widetilde{U}} \equiv \psi\restriction_{\widetilde{U}} := \psi \circ i \in C^\infty(\xi_{\widetilde{U}}) \tag{18}$$

be the restriction of its domain to $\widetilde{U}$. After we restrict the domain of ψ to $\widetilde{U}$, we lose all information about the dependence of ψ on the external for $\widetilde{U}$ coordinates, hence $D(\psi_{\widetilde{U}})$ cannot be computed if D contains partial derivatives with respect to the external coordinates.

Because of the bijective correspondence (9) between the DOs from $\mathrm{Diff}_k(\xi, \eta)$ and their total symbols, we will concentrate to the problem of restricting the domain of a jet of a section $\psi \in C^\infty(\xi)$. Taking the jet of a section does not "commute" with restricting the domain of the section. If we first find the k-jet $J^k(\psi)$ of the section $\psi \in C^\infty(\xi)$, and then restrict $J^k(\psi)$ to $\widetilde{U}$, we obtain $J^k(\psi)_{\widetilde{U}} \equiv J^k(\psi)\restriction_{\widetilde{U}} = J^k(\psi) \circ i$, which is a section of $J^k(\xi)_{\widetilde{U}}$. For any $b \in \widetilde{U}$, $J^k(\psi)_{\widetilde{U}}(b)$ (which is the same as $J^k(\psi)(b)$) contains derivatives with respect to all coordinates $x^1, \dots, x^n$, namely

$$\big(J^k(\psi)_{\widetilde{U}}(b)\big)^a_{\mu_1 \cdots \mu_i} = \partial_{\mu_1 \cdots \mu_i} \psi^a(b) \; , \qquad \begin{cases} 1 \le \mu_1 \le \cdots \le \mu_i \le n \; , \; i = 1, \dots, k \\ a = 1, \dots, \operatorname{rank} \xi \end{cases}$$

(in the notations introduced in (4)).

On the other hand, if we first restrict ψ to $\widetilde{U}$ and after that compute the kth jet of the restriction $\psi_{\widetilde{U}}$ (18), the result, $J^k_{\widetilde{U}}(\psi_{\widetilde{U}})$, is a section of $J^k_{\widetilde{U}}(\xi_{\widetilde{U}})$. Here we introduced the notations $J^k_{\widetilde{U}}(\xi_{\widetilde{U}})$ and $J^k_{\widetilde{U}} \in \mathrm{LDiff}_k(\xi_{\widetilde{U}}, J^k(\xi_{\widetilde{U}}))$, which simply mean that we work with the sections of the restricted bundle $\xi_{\widetilde{U}}$ (as in (18)). For any $b \in \widetilde{U}$, $J^k_{\widetilde{U}}(\psi_{\widetilde{U}})(b)$ contains only derivatives with respect to the internal for $\widetilde{U}$ coordinates $x^1, \dots, x^{\widetilde{n}}$:

$$\big(J^k_{\widetilde{U}}(\psi_{\widetilde{U}})(b)\big)^a_{\mu_1 \cdots \mu_i} = \partial_{\mu_1 \cdots \mu_i} \psi^a(b) \; , \qquad \begin{cases} 1 \le \mu_1 \le \cdots \le \mu_i \le \widetilde{n} \; , \; i = 1, \dots, k \\ a = 1, \dots, \operatorname{rank} \xi \; . \end{cases} \tag{19}$$

The dimensions of the fibers of $J^k(\xi)_{\widetilde{U}}$ and $J^k_{\widetilde{U}}(\xi_{\widetilde{U}})$ are

$$\operatorname{rank} J^k(\xi)_{\widetilde{U}} = \binom{n+k}{k} \operatorname{rank} \xi \; , \qquad \operatorname{rank} J^k_{\widetilde{U}}(\xi_{\widetilde{U}}) = \binom{\widetilde{n}+k}{k} \operatorname{rank} \xi \; . \tag{20}$$

Denote by $j^k : J^k(\xi)_{\widetilde{U}} \to J^k_{\widetilde{U}}(\xi_{\widetilde{U}})$ the natural projection given by "cutting off" all non-internal for $\widetilde{U}$ derivatives, i.e., derivatives containing at least one external for $\widetilde{U}$ partial derivative. In a coordinate-free language, for any $b \in \widetilde{U}$, the map j^k is given by

$$j^k\Big(J^k(\psi)_{\widetilde{U}}(b)\Big) = J^k_{\widetilde{U}}(\psi_{\widetilde{U}})(b)\ . \tag{21}$$

As usual, let

$$j^k_* \equiv (j^k)_* : C^\infty(J^k(\xi)_{\widetilde{U}}) \to C^\infty(J^k_{\widetilde{U}}(\xi_{\widetilde{U}})) : J^k(\psi)_{\widetilde{U}} \mapsto J^k_{\widetilde{U}}(\psi_{\widetilde{U}}) \tag{22}$$

be the map between the sections that is induced by j^k (as in (7)).

3.2. *Internal for $\widetilde{U}$ DOs and their restriction to $\widetilde{U}$*

There is a situation in which the restriction of a DO $D \in \mathrm{Diff}_k(\xi, \eta)$ to $\widetilde{U}$ is naturally and uniquely determined by the embedding $i : \widetilde{U} \to B$. This happens when, for an arbitrary $\psi \in C^\infty(\xi)$, the section $D\psi \in C^\infty(\eta)$ evaluated at the points of $\widetilde{U}$ (which, in formal notation, is $(D\psi)\restriction_{\widetilde{U}} \equiv (D\psi) \circ i$) contains only differentiations with respect to internal for $\widetilde{U}$ coordinates. In this case the DO D is said to be *internal (for $\widetilde{U}$)*.

Let $D = \widetilde{D}_* J^k \in \mathrm{Diff}_k(\xi, \eta)$ be internal for $\widetilde{U}$, and $b \in \widetilde{U}$. Then $(D\psi)(b)$ does not contain any non-internal for $\widetilde{U}$ derivatives, i.e., derivatives $\big(J^k(\psi)_{\widetilde{U}}(b)\big)^a_{\mu_1 \cdots \mu_i} = \partial_{\mu_1 \cdots \mu_i}\psi^a(b)$ for which at least one of the indices $\mu_1, \ldots, \mu_i$ exceeds $\widetilde{n}$. Clearly, in this case $(D\psi)(b)$ can be expressed only in terms of the coordinates $\big(J^k_{\widetilde{U}}(\psi_{\widetilde{U}})(b)\big)^a_{\mu_1 \cdots \mu_i}$ of the k-jet $J^k_{\widetilde{U}}(\psi_{\widetilde{U}})(b)$ of the restricted section $\psi_{\widetilde{U}}$ (see (19)). Therefore, for an internal for $\widetilde{U}$ DO D, we can define the restricted to $\widetilde{U}$ DO $D_{\widetilde{U}} \in \mathrm{Diff}_k(\xi_{\widetilde{U}}, \eta_{\widetilde{U}})$ as follows: given a section $\rho \in C^\infty(\xi_{\widetilde{U}})$, we can think of it as a restriction $\psi_{\widetilde{U}} = \psi \circ i$ of a section $\psi \in C^\infty(\xi)$ to $\widetilde{U}$, and then set

$$(D_{\widetilde{U}}\rho)(b) := (D\psi)(b)\ , \qquad b \in \widetilde{U}\ . \tag{23}$$

Since D is internal for $\widetilde{U}$, $(D\psi)(b)$ does not contain non-internal for $\widetilde{U}$ derivatives, so that the arbitrariness in the choice of $\psi \in C^\infty(\xi)$ such that $\psi_{\widetilde{U}} = \rho$ is immaterial. Since differentiation is a local operation, the section ψ does not need to be defined on B, but only on an open subset of B that contains $\widetilde{U}$. Clearly, if D is internal for $\widetilde{U}$, then the order of the restricted to $\widetilde{U}$ DO $D_{\widetilde{U}}$ is the same as the order of D.

In terms of the map j^k from (21), for an internal for $\widetilde{U}$ DO D we can naturally define the total symbol $\widetilde{D_{\widetilde{U}}}$ of the restricted DO $D_{\widetilde{U}}$ by

$$\widetilde{D}\restriction_{\widetilde{U}} = \widetilde{D_{\widetilde{U}}} \circ j^k\ , \tag{24}$$

where $\widetilde{D}\restriction_{\widetilde{U}} : J^k(\xi)_{\widetilde{U}} \to \eta_{\widetilde{U}}$ is the restriction of the total symbol $\widetilde{D} : J^k(\xi) \to \eta$ of D to the submanifold $\widetilde{U}$. In other words, $\widetilde{D_{\widetilde{U}}}$ is defined so that the diagram

$$\begin{tikzcd} C^\infty(J^k(\xi)_{\widetilde{U}}) \arrow[r, "j^k_*"] \arrow[d, "\left(\widetilde{D}\restriction_{\widetilde{U}}\right)_*"'] & C^\infty(J^k_{\widetilde{U}}(\xi_{\widetilde{U}})) \arrow[dl, "\left(\widetilde{D_{\widetilde{U}}}\right)_*"] \\ C^\infty(\eta_{\widetilde{U}}) & \end{tikzcd} \tag{25}$$

be commutative.

To show that the definitions (23) and (24) are consistent for an internal for $\widetilde{U}$ DO $D \in \mathrm{Diff}_k(\xi,\eta)$, we derive (23) from the definition (21), (22) of the map j^k and the definition (24) of $\widetilde{D_{\widetilde{U}}}$: if $\rho \in C^\infty(\xi_{\widetilde{U}})$ and $\psi \in C^\infty(\xi)$ is such that $\rho = \psi_{\widetilde{U}}$, then

$$\begin{aligned} D_{\widetilde{U}}\rho &= \big(\widetilde{D_{\widetilde{U}}}\big)_* J^k_{\widetilde{U}}(\psi_{\widetilde{U}}) = \big(\widetilde{D_{\widetilde{U}}}\big)_* j^k_*\big(J^k(\psi)_{\widetilde{U}}\big) \\ &= \big(\widetilde{D_{\widetilde{U}}} \circ j^k\big)_* \big(J^k(\psi)_{\widetilde{U}}\big) = \big(\widetilde{D}{\restriction_{\widetilde{U}}}\big)_* \big(J^k(\psi)_{\widetilde{U}}\big) \\ &= \big(\widetilde{D}_* J^k(\psi)\big)_{\widetilde{U}} = (D\psi)_{\widetilde{U}} \ . \end{aligned} \tag{26}$$

3.3. *Restriction of a non-internal DO*

3.3.1. *Natural geometric objects in the problem*

To restrict to $\widetilde{U}$ a DO that is not internal for $\widetilde{U}$, one needs information that does not come from the natural embedding $i : \widetilde{U} \hookrightarrow B$. A natural geometric object that plays a crucial role is the subbundle $I^k_{\widetilde{U}}$ of $J^k(\xi)_{\widetilde{U}}$ consisting of the k-jets of all vanishing on $\widetilde{U}$ sections of ξ, i.e., whose fiber over an arbitrary point $b \in \widetilde{U}$ is

$$(I^k_{\widetilde{U}})_b := \{\, J^k(\psi)(b) \,:\, \psi \in C^\infty(\xi) \text{ s.t. } \psi \circ i \equiv 0 \,\} \subseteq (J^k(\xi)_{\widetilde{U}})_b \ . \tag{27}$$

Let

$$i^k : I^k_{\widetilde{U}} \hookrightarrow J^k(\xi)_{\widetilde{U}} \tag{28}$$

stand for the natural embedding; for brevity, we write $I^k_{\widetilde{U}}$ instead of $i^k(I^k_{\widetilde{U}})$.

Let $(x^1,\dots,x^{\widetilde{n}})$ and $(x^{\widetilde{n}+1},\dots,x^n)$ be respectively the internal and the external for $\widetilde{U}$ local coordinates in B (recall (17)). For the partial derivatives of a section of ξ, we recall the terminology used in Sect. 3.1 and 3.2: *internal* derivatives are those that contain only differentiations with respect to the internal coordinates; all other derivatives are *non-internal*; by definition, the zeroth derivative (i.e., the section itself) is internal. In jet bundle coordinates (3) in $J^k(\xi)_{\widetilde{U}}$, the internal jet bundle coordinates in $J^k(\xi)_{\widetilde{U}}$ are z^a and those $z^a_{\mu_1\cdots\mu_i}$ for which $\mu_1 \le \widetilde{n}, \dots, \mu_i \le \widetilde{n}$, while the non-internal ones are $z^a_{\mu_1\cdots\mu_i}$ for which at least one of the μ's is strictly greater than $\widetilde{n}$. According to (20), the number of internal coordinates in $J^k(\xi)_{\widetilde{U}}$ is $\binom{\widetilde{n}+k}{k}\operatorname{rank}\xi$, while number of the non-internal ones is $\left[\binom{n+k}{k} - \binom{\widetilde{n}+k}{k}\right]\operatorname{rank}\xi$.

To simplify the notations, we temporarily write "int" for the set of all internal coordinates in $J^k(\xi)_{\widetilde{U}}$, and "non-int" for the set of all non-internal ones. Then $I^k_{\widetilde{U}}$ consists of those elements of $J^k(\xi)_{\widetilde{U}}$ all internal jet bundle coordinates of which are zero, while the non-internal ones are arbitrary; symbolically this can be written as $I^k_{\widetilde{U}} = \{(\mathrm{int} = 0, \text{non-int})\}$. The natural projection j^k eq:jk-explicit-expression maps the element $(\mathrm{int}, \text{non-int}) \in J^k(\xi)_{\widetilde{U}}$ to $(\mathrm{int}) \in J^k_{\widetilde{U}}(\xi_{\widetilde{U}})$, preserving the values of all the internal jet bundle coordinates (i.e., j^k simply "cuts off" all non-internal coordinates). Obviously, $j^k \circ i^k = 0$, so that we obtain the following short exact sequence:

$$0 \longrightarrow \{(\mathrm{int}=0, \text{non-int})\} \xrightarrow{i^k} \{(\mathrm{int}, \text{non-int})\} \xrightarrow{j^k} \{(\mathrm{int})\} \longrightarrow 0 \ . \tag{29}$$

In coordinate-free notations, the short exact sequence (29) can be written as the horizontal short exact sequence in the diagram

$$\begin{tikzcd} 0 \arrow[r] & I^k_{\widetilde{U}} \arrow[r, hook, shift left, "i^k"] & J^k(\xi)_{\widetilde{U}} \arrow[l, shift left, "\Pi^k"] \arrow[r, shift left, "j^k"] \arrow[d, "\widetilde{D}\restriction_{\widetilde{U}}"'] & J^k_{\widetilde{U}}(\xi_{\widetilde{U}}) \arrow[l, shift left, "\Sigma^k"] \arrow[r] \arrow[dl, "\widetilde{D_{\widetilde{U}}}"] & 0 \ , \\ & & \eta_{\widetilde{U}} & & \end{tikzcd} \tag{30}$$

in which we have also shown the maps from (24), as well as the maps Π^k and Σ^k which will be discussed below.

3.3.2. *Geometry of the restriction of a non-internal DO*

In the geometric language introduced above, the gist of the problem of restricting a DO $D \in \mathrm{Diff}_k(\xi, \eta)$ to a submanifold $\widetilde{U}$ of the base B is that while $J^k(\xi)_{\widetilde{U}}$ is isomorphic to the direct sum $I^k_{\widetilde{U}} \oplus J^k_{\widetilde{U}}(\xi_{\widetilde{U}})$, the bundle $J^k_{\widetilde{U}}(\xi_{\widetilde{U}})$ is not naturally embedded in $J^k(\xi)_{\widetilde{U}}$. One way to define the total symbol $\widetilde{D_{\widetilde{U}}}$ of the restricted DO $D_{\widetilde{U}}$ is to choose a splitting of the short exact sequence in (30), i.e., a vector bundle morphism $\Sigma^k \in \mathrm{Hom}\,(J^k_{\widetilde{U}}(\xi_{\widetilde{U}}), J^k(\xi)_{\widetilde{U}})$ over the identity in $\widetilde{U}$ that satisfies

$$j^k \circ \Sigma^k = \mathrm{Id}_{J^k_{\widetilde{U}}(\xi_{\widetilde{U}})} \ . \tag{31}$$

Equivalently, we can choose a vector bundle morphism $\Pi^k \in \mathrm{Hom}\,(J^k(\xi)_{\widetilde{U}}, I^k_{\widetilde{U}})$ over the identity in $\widetilde{U}$ such that

$$\Pi^k \circ i^k = \mathrm{Id}_{I^k_{\widetilde{U}}} \ , \qquad \ker \Pi^k = \Sigma^k(J^k_{\widetilde{U}}(\xi_{\widetilde{U}})) \ . \tag{32}$$

Then the total symbol of $D_{\widetilde{U}}$ is given by

$$\widetilde{D_{\widetilde{U}}} = \widetilde{D}\restriction_{\widetilde{U}} \circ \Sigma^k \ , \tag{33}$$

as shown in the diagram (30).

Conditions (31) and (32) imposed on Σ^k and Π^k guarantee that if the DO D is internal for $\widetilde{U}$, then $D_{\widetilde{U}}$ defined by (33) is the same as its natural restriction to $\widetilde{U}$ (discussed in Sect. 3.2).

We required that the maps Σ^k and Π^k be vector bundle morphisms for two reasons. Firstly, this is the case that occurs in dimensional reduction of invariant DOs considered in Sect. 4. Furthermore, if they are morphisms, the restriction $D_{\widetilde{U}}$ of a linear DO D will be linear as well. In Remark 7 we consider briefly the more general case when Σ^k is a nonlinear fiber-preserving map.

Conditions (31) and (32) imply that the maps Σ^k and Π^k are completely defined by $\Sigma^k(J^k_{\widetilde{U}}(\xi_{\widetilde{U}}))$, which is a subbundle of $J^k(\xi)_{\widetilde{U}}$ transversal to $I^k_{\widetilde{U}}$ in $J^k(\xi)_{\widetilde{U}}$:

$$J^k(\xi)_{\widetilde{U}} = I^k_{\widetilde{U}} \oplus \Sigma^k(J^k_{\widetilde{U}}(\xi_{\widetilde{U}})) \ . \tag{34}$$

Let us take an arbitrary point $b \in \widetilde{U}$ and consider in practical terms the meaning of the splitting $J^k(\xi)_b = (I^k_{\widetilde{U}})_b \oplus \Sigma^k(J^k_{\widetilde{U}}(\xi_{\widetilde{U}}))_b$ of the fiber of $J^k(\xi)_{\widetilde{U}}$ over b. In the notations "int" and "non-int" introduced above, $(I^k_{\widetilde{U}})_b = \{(\mathrm{int} = 0, \text{non-int})\}$ (recall the short exact sequence (29)). According to (20), the dimensions of the subspaces in the splitting are

$$\begin{aligned} \dim\bigl(I^k_{\widetilde{U}}\bigr)_b &= \left[\binom{n+k}{k} - \binom{\widetilde{n}+k}{k}\right] \operatorname{rank}\xi = \#\{\text{non-int}\}\ , \\ \dim \Sigma^k\bigl(J^k_{\widetilde{U}}(\xi_{\widetilde{U}})\bigr)_b &= \binom{\widetilde{n}+k}{k} \operatorname{rank}\xi = \#\{\mathrm{int}\}\ . \end{aligned}$$

To define a subspace $\Sigma^k(J^k_{\widetilde{U}}(\xi_{\widetilde{U}}))_b$ of dimension $\#\{\mathrm{int}\}$ in the linear space $J^k(\xi)_b$ of dimension $\dim J^k(\xi)_b = \#\{\text{non-int}\} + \#\{\mathrm{int}\}$, we can write a system of $\#\{\text{non-int}\}$ independent linear equations with $(\#\{\text{non-int}\} + \#\{\mathrm{int}\})$ unknowns. To make this explicit, we denote by $Z_{\mathcal{I}}$, for $\mathcal{I} = 1, \dots, \#\{\mathrm{int}\}$, the set of all internal for $\widetilde{U}$ coordintes in $J^k(\xi)_b$, and by $Z_{\mathcal{N}}$, for $\mathcal{N} = \#\{\mathrm{int}\} + 1, \dots, \#\{\mathrm{int}\} + \#\{\text{non-int}\}$, the set of all non-internal coordinates in $J^k(\xi)_b$. In coordinates $(Z_{\mathcal{I}}, Z_{\mathcal{N}})$, the maps Σ^k and Π^k have the form

$$\Sigma^k : (Z_{\mathcal{I}}) \mapsto \begin{pmatrix} Z_{\mathcal{I}} \\ \mathsf{M}(b) Z_{\mathcal{I}} \end{pmatrix}\ , \qquad \Pi^k : \begin{pmatrix} Z_{\mathcal{I}} \\ Z_{\mathcal{N}} \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ -\mathsf{M}(b) Z_{\mathcal{I}} + Z_{\mathcal{N}} \end{pmatrix}\ . \tag{35}$$

The matrix $\mathsf{M}(b)$ is of size $\#\{\text{non-int}\} \times \#\{\mathrm{int}\}$. This makes it clear that the set of all splittings of the horizontal short exact sequence in (30) is an affine space with a linear group $\operatorname{Hom}\bigl(J^k_{\widetilde{U}}(\xi_{\widetilde{U}}), I^k_{\widetilde{U}}\bigr)$.

The process of restriction of a DO $D \in \mathrm{Diff}_k(\xi, \eta)$ to a submanifold $\widetilde{U}$ of the base B is illustrated well by the diagram (36).

$$\begin{tikzcd}
 & & 0 \arrow[d] \\
C^\infty(\xi) \arrow[r, "J^k"] \arrow[d, "D"'] \arrow[ddr, dashed, "\restriction_{\widetilde{U}}"] & C^\infty(J^k(\xi)) \arrow[dl, "\widetilde{\mathrm{D}}_*"] \arrow[dr, dashed, "\restriction_{\widetilde{U}}"] & C^\infty(I^k_{\widetilde{U}}) \arrow[d, shift left, "i^k_*"] \\
C^\infty(\eta) \arrow[ddr, dashed, "\restriction_{\widetilde{U}}"] & & C^\infty(J^k(\xi)_{\widetilde{U}}) \arrow[u, shift left, "\Pi^k_*"] \arrow[d, shift left, "j^k_*"] \arrow[ddl, "(\widetilde{\mathrm{D}}\restriction_{\widetilde{U}})_*"] \\
 & C^\infty(\xi_{\widetilde{U}}) \arrow[r, "J^k_{\widetilde{U}}"] \arrow[d, "D_{\widetilde{U}}"'] & C^\infty(J^k_{\widetilde{U}}(\xi_{\widetilde{U}})) \arrow[u, shift left, "\Sigma^k_*"] \arrow[dl, "(\widetilde{\mathrm{D}_{\widetilde{U}}})_*"] \arrow[d] \\
 & C^\infty(\eta_{\widetilde{U}}) & 0
\end{tikzcd} \tag{36}$$

The triangle in the upper left corner of (36) represents the relation (9) between the DO D and its total symbol $\widetilde{D}$. The dashed arrows with label$\restriction_{\widetilde{U}}$ are the restrictions of the domains of the sections of ξ, $J^k(\xi)$, and η to $\widetilde{U}$, as in (18). The triangle involving $C^\infty(\xi_{\widetilde{U}})$, $C^\infty(J^k_{\widetilde{U}}(\xi_{\widetilde{U}}))$, and $C^\infty(\eta_{\widetilde{U}})$ represents the relation $D_{\widetilde{U}} = \big(\widetilde{D_{\widetilde{U}}}\big)_* J^k_{\widetilde{U}}$ between the desired restricted DO $D_{\widetilde{U}}$ and its total symbol $\widetilde{D_{\widetilde{U}}}$. Note that the vertical arrows between the five rightmost objects in the diagram (36) come from the horizontal short exact sequence in (30). The triangle involving $C^\infty(J^k(\xi)_{\widetilde{U}})$, $C^\infty(J^k_{\widetilde{U}}(\xi_{\widetilde{U}}))$, and $C^\infty(\eta_{\widetilde{U}})$ clarifies the role of the splitting Σ^k in the construction of the restricted DO $D_{\widetilde{U}}$ – namely,

$$D_{\widetilde{U}} = \big(\widetilde{D_{\widetilde{U}}}\big)_* J^k_{\widetilde{U}} = \big(\widetilde{D}\restriction_{\widetilde{U}}\big)_* \Sigma^k_* J^k_{\widetilde{U}} \ . \tag{37}$$

3.3.3. *Defining the splitting of* (30) *through an auxiliary DO*

Thanks to (31), (32), the splitting of the horizontal short exact sequence in (30) is defined completely by specifying a subbundle of $J^k(\xi)_{\widetilde{U}}$ transversal to $I^k_{\widetilde{U}}$ (recall (34)). Such a subbundle can sometimes be defined through an auxiliary linear DO $M \in \mathrm{LDiff}_k(\xi, \eta)$. If $\widetilde{M}\restriction_{\widetilde{U}} \in \mathrm{Hom}\,(J^k(\xi)_{\widetilde{U}}, \eta_{\widetilde{U}})$ is the restriction to $\widetilde{U}$ of the total symbol of M, then it may turn out that its kernel, $\ker\big(\widetilde{M}\restriction_{\widetilde{U}}\big) \subseteq J^k(\xi)_{\widetilde{U}}$, is a subbundle transversal to $I^k_{\widetilde{U}}$ in $J^k(\xi)_{\widetilde{U}}$. In this case we can set

$$\Sigma^k(J^k_{\widetilde{U}}(\xi_{\widetilde{U}})) \equiv \ker \Pi^k := \ker\big(\widetilde{M}\restriction_{\widetilde{U}}\big) \ . \tag{38}$$

In the light of the discussion before (36), equation (38) means that we can express all non-internal jet bundle coordinates in terms of the internal jet bundle coordinates from the equation $(M\psi)(b) = 0$, where b is an arbitrary point in $\widetilde{U}$. Having expressed all non-internal derivatives of ψ through the internal derivatives of ψ, we substitute them in $(D\psi)\restriction_{\widetilde{U}}$, and the result is $D_{\widetilde{U}}\psi_{\widetilde{U}}$ – an expression that contains only internal for $\widetilde{U}$ derivatives of the restricted section $\psi_{\widetilde{U}} \in C^\infty(\xi_{\widetilde{U}})$. From $D_{\widetilde{U}}\psi_{\widetilde{U}}$ we can read off the desired restricted DO $D_{\widetilde{U}} \in \mathrm{Diff}_k(\xi_{\widetilde{U}}, \eta_{\widetilde{U}})$.

To define the splitting (34), one can sometimes use a linear DO of order lower that k. Let $m < k$, and $M \in \mathrm{LDiff}_m(\xi, \eta)$. Then the $(k-m)$th prolongation $P^{k-m}(M)$ of M is a DO of order k, and one can hope that the restriction to $\widetilde{U}$ of its total symbol $\widetilde{P^{k-m}(M)}$ can be used to define $\Sigma^k(J^k_{\widetilde{U}}(\xi_{\widetilde{U}}))$, similarly to (38):

$$\Sigma^k(J^k_{\widetilde{U}}(\xi_{\widetilde{U}})) \equiv \ker \Pi^k := \ker\Big[\big(\widetilde{P^{k-m}(M)}\big)\restriction_{\widetilde{U}}\Big] \ . \tag{39}$$

While, in principle, it is possible to use an auxiliary linear DO M to define the splitting as in (38) or (39), finding such a DO may not be easy for several reasons.

- First of all, on what ground will one use certain DO M and not another one? In general, one can use some DO M if this would guarantee that certain properties are preserved. A fundamental example of this is the process of dimensional reduction of a DO invariant with respect to the action of a Lie group, in which case the needed auxiliary DO is the Lie derivative – see Sect. 4.

- A second problem is how to choose M so that the right-hand side of (38) or (39) indeed defines a subspace $\Sigma^k(J^k_{\widetilde{U}}(\xi_{\widetilde{U}}))$ transversal to $I^k_{\widetilde{U}}$ in $J^k(\xi)_{\widetilde{U}}$ as in (34). Taking care of this transversality requirement is highly non-trivial. For an example of finding such an auxiliary DO see Sect. 6 of our paper [2].
- Last but not least, the auxiliary DO must be formally integrable – otherwise, unexpected complications may occur, as illustrated in the Example 5 below.

Example 4. This is a simple example of restriction of a DO; we will use this example to discuss other issues in Examples 12 and 17.

Let $\xi = \eta = (E, \pi, B) := (\mathbb{R}^2_+ \times \mathbb{R}, \pi, \mathbb{R}^2_+)$ be the trivial line bundles over the upper half-plane $\mathbb{R}^2_+ := \{(x^1, x^2) : x^1 \in \mathbb{R}, x^2 > 0\}$, and $\pi\big((x^1, x^2), z\big) := (x^1, x^2)$. Let $D \in \mathrm{Diff}_2(\xi, \eta)$ be the Laplacian: $D = \partial_{11} + \partial_{22}$. The goal here is to restrict D to the straight line $\widetilde{U} := \{x^2 = 1\} \subset \mathbb{R}^2_+$; we parameterize this line by the first coordinate of its points, considering $\widetilde{U}$ as a copy of $\mathbb{R}$: $i_{\widetilde{U}} : \widetilde{U} \hookrightarrow \mathbb{R}^2_+ : \widetilde{x} \mapsto (\widetilde{x}, 1)$. The restriction to $\widetilde{U}$ of a section $\psi \in C^\infty(\xi)$ is given by $\psi_{\widetilde{U}} \in C^\infty(\xi_{\widetilde{U}})$, where $\psi_{\widetilde{U}}(\widetilde{x}) := \psi \circ i_{\widetilde{U}}(x) = \psi(\widetilde{x}, 1)$. We choose the auxiliary DO $M \in \mathrm{LDiff}_2(\xi, \eta)$ to be defined by $M\psi := x^1\partial_1\psi + x^2\partial_2\psi - 2\psi$; in Example 12 it will become clear that M is formally integrable. We will use the first prolongation of M as a splitting relation. The first prolongation of M is

$$\begin{aligned}
&x^1\partial_1\psi + x^2\partial_2\psi - 2\psi = 0\ ,\\
&x^1\partial_{11}\psi + x^2\partial_{12}\psi - \partial_1\psi = 0\ ,\\
&x^1\partial_{12}\psi + x^2\partial_{22}\psi - \partial_2\psi = 0\ .
\end{aligned} \tag{40}$$

Expressing $\partial_2\psi$, $\partial_{12}\psi$, and $\partial_{22}\psi$ from this system, and setting $(x^1, x^2) = (\widetilde{x}, 1)$, we obtain the splitting relations on $\widetilde{U}$:

$$\begin{aligned}
\partial_2\psi(\widetilde{x}, 1) &= -\widetilde{x}\,\partial_1\psi(\widetilde{x}, 1) + 2\psi(\widetilde{x}, 1)\ ,\\
\partial_{12}\psi(\widetilde{x}, 1) &= -\widetilde{x}\,\partial_{11}\psi(\widetilde{x}, 1) + \partial_1\psi(\widetilde{x}, 1)\ ,\\
\partial_{22}\psi(\widetilde{x}, 1) &= \widetilde{x}^2\,\partial_{11}\psi(\widetilde{x}, 1) - 2\widetilde{x}\,\partial_1\psi(\widetilde{x}, 1) + 2\psi(\widetilde{x}, 1)\ .
\end{aligned}$$

For the restricted to $\widetilde{U}$ Laplacian acting on $\psi_{\widetilde{U}}(\widetilde{x}) = \psi(\widetilde{x}, 1)$, we obtain

$$(D_{\widetilde{U}}\psi_{\widetilde{U}})(\widetilde{x}) = \big(1 + \widetilde{x}^2\big)\,\psi''_{\widetilde{U}}(\widetilde{x}) - 2\widetilde{x}\,\psi'_{\widetilde{U}}(\widetilde{x}) + 2\,\psi_{\widetilde{U}}(\widetilde{x})\ . \tag{41}$$

Example 5. This example shows the dangers of using an auxiliary DO M that is not formally integrable. We use the same notations as in Example 2 for B, ξ, η, and $M := \begin{pmatrix} \partial_{11} - x^2\partial_{33} \\ \partial_{22} \end{pmatrix} \in \mathrm{LDiff}_2(\xi, \eta)$. Let $D := \begin{pmatrix} \partial_{233} \\ 0 \end{pmatrix} \in \mathrm{Diff}_3(\xi, \eta)$, $\widetilde{U} = \{x^3 = 1\} \subset B$. Assume that we want to restrict D to $\widetilde{U}$ by using a splitting of the horizontal short exact sequence in (30) that comes from M as an auxiliary DO.

The kernel of the first prolongation $P^1(M)$ restricted to $\widetilde{U}$ – in other words, the solution of the equation $P^1(M)(\psi)\restriction_{\widetilde{U}} = 0$ – contains the equations $\partial_{11}\psi - x^2\partial_{33}\psi = 0$

and $\partial_{112}\psi - x^2\partial_{233}\psi - \partial_{33}\psi = 0$, from which we can express the non-internal derivative in D as

$$\partial_{233}\psi = \frac{1}{x^2}(\partial_{112}\psi - \partial_{33}\psi) = \frac{1}{x^2}\partial_{112}\psi - \frac{1}{(x^2)^2}\partial_{11}\psi \ .$$

Therefore, if one uses $\ker \widetilde{P^1(M)}$ as $\Sigma^3(J^3_{\widetilde{U}}(\xi_{\widetilde{U}}))$ in (38) to define the splitting, then the restriction of the DO D to $\widetilde{U}$ is $D_{\widetilde{U}} = \begin{pmatrix} \frac{1}{x^2}\partial_{112} - \frac{1}{(x^2)^2}\partial_{11} \\ 0 \end{pmatrix}$. If $\psi_{\widetilde{U}}(x^1,x^2) := \psi(x^1,x^2,1)$ is the restricted to $\widetilde{U}$ section, then it is easy to show that the general solution $\psi_{\widetilde{U}}$ of the restricted equation

$$(D_{\widetilde{U}}\psi_{\widetilde{U}})(x^1,x^2) = \begin{pmatrix} \frac{1}{x^2}\partial_{112}\psi_{\widetilde{U}}(x^1,x^2) - \frac{1}{(x^2)^2}\partial_{11}\psi_{\widetilde{U}}(x^1,x^2) \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

is $\psi_{\widetilde{U}}(x^1,x^2) = h_1(x^1)x^2 + x^1h_2(x^2) + h_3(x^2)$, where h_1, h_2, and h_3 are arbitrary smooth functions of one variable.

We can, however, treat D as a 4th-order DO and use $\Sigma^4(J^4_{\widetilde{U}}(\xi_{\widetilde{U}})) = \ker \widetilde{P^2(M)} \subseteq J^4(\xi)_{\widetilde{U}}$ to define the splitting, i.e., to express the non-internal derivatives in D from the equation $P^2(M)(\psi)\restriction_{\widetilde{U}} = 0$. As we showed in the example in Sect. 2.3, the second prolongation of the equation $M\psi = 0$ contains the condition $\partial_{233}\psi = 0$; clearly, this condition remains unchanged after restricting it to $\widetilde{U}$. Therefore the restricted DO for this choice of splitting becomes $D_{\widetilde{U}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, so that the general solution of the reduced equation $D_{\widetilde{U}}\psi_{\widetilde{U}}(x^1,x^2) = 0$ consists of all smooth functions of two variables.

Remark 6. We note that, even if the auxiliary DO M provides a splitting of the horizontal short exact sequence in the diagram (30) and is formally integrable, the order of the restricted DO $D_{\widetilde{U}}$ may be greater than the order of the original DO D. For a concrete example of this phenomenon we refer the reader to [2, Sect. 6].

Remark 7.

One can define the total symbol $\widetilde{D_{\widetilde{U}}}$ of the restricted DO $D_{\widetilde{U}}$ by choosing a nonlinear fiber-preserving mapping $\Sigma^k : J^k_{\widetilde{U}}(\xi_{\widetilde{U}}) \to J^k(\xi)_{\widetilde{U}}$ over the identity in $\widetilde{U}$ satisfying $j^k \circ \Sigma^k = \mathrm{Id}_{J^k(\xi_{\widetilde{U}})}$, and defining the total symbol of $D_{\widetilde{U}}$ by $\widetilde{D_{\widetilde{U}}} = \widetilde{D}\restriction_{\widetilde{U}} \circ \Sigma^k$ (cf. (37)). As before, the condition imposed on Σ^k guarantees that for an internal for $\widetilde{U}$ DO its restriction is the same as its natural restriction to $\widetilde{U}$.

4. Dimensional reduction of invariant DOs

If a Lie group G acts on a vector bundle ξ through vector bundle morphisms, one can naturally define an action of G on the sections $C^\infty(\xi)$ of ξ. We say that a section $\psi \in C^\infty(\xi)$ is G-invariant if it does not change under this action. Clearly, G-invariance is a restrictive condition, so that the set $C^\infty(\xi)^G$ of all G-invariant sections is smaller than $C^\infty(\xi)$. To avoid the redundancy in using $C^\infty(\xi)$ when working only with G-invariant sections, one can construct a reduced vector bundle ξ^G such that $C^\infty(\xi^G)$ is in bijective correspondence with $C^\infty(\xi)^G$. Let ξ and η be two vector bundles over the same base B, and let the group G act on both of them, with the same action on B. Then one can consider the DOs from $C^\infty(\xi)$ to $C^\infty(\eta)$, and define an action of G on the set $\mathrm{Diff}_k(\xi,\eta)$ of order-k DOs. Similarly to the case of invariant sections, from each invariant DO $D \in \mathrm{Diff}_k(\xi,\eta)^G$ one can construct a reduced DO $D^G \in \mathrm{Diff}_k(\xi^G,\eta^G)$.

In this section we study this construction in detail. In Sect. 4.1 and 4.2 we develope the geometric language, following [1, Sect. 2] (where the reader can find many more details). In Sect. 4.3 we define the action of a Lie group on DOs and give coordinate realizations of the reduced DOs, in which we use the methodology for restriction of a DO to a submanifold developed in Sect. 3. We describe the algorithm for reduction of DOs and give it a detailed theoretical justification. Finally, in Sect. 4.4 we consider G-reduction of a DO D that is invariant with respect to the action of two Lie groups, G and K. In this case the G-reduced DO D^G inherits some residual invariance with respect to the reduced action of K that comes from the original action of K on D.

4.1. *Group actions; reducible G-vector bundles*

Let G be a connected (possibly non-compact) Lie group that acts from the left on the vector bundle $\xi = (E,\pi,B)$ by vector bundle morphisms. If $\mathcal{T} := (t,T)$ is the action of G on ξ, where $t : G\times B \to B$ and $T : G\times E \to E$ are respectively the actions of G on the base B and on the total space E, then this means that $\pi\circ T_g = t_g\circ\pi$ for all $g\in G$, and that $T_g : \xi_b \to \xi_{t_g(b)}$ is a linear isomorphism for any $b\in B$ and $g\in G$.

The action $\mathcal{T}$ of G on ξ induces a natural action of G on the sections $C^\infty(\xi)$ of ξ:

$$g(\psi) = T_g\circ\psi\circ t_g^{-1}\ , \qquad g\in G\ , \quad \psi\in C^\infty(\xi)\ . \tag{42}$$

A section $\psi\in C^\infty(\xi)$ is said to be G-*invariant* (or G-*equivariant*) if $g(\psi)=\psi$, i.e.,

$$\psi(t_g(b)) = T_g(\psi(b)) \qquad \text{for all } g\in G \text{ and } b\in B\ . \tag{43}$$

Let $C^\infty(\xi)^G$ stand for the set of all G-invariant sections of ξ.

We impose two natural conditions on the action $\mathcal{T}$ (as in [1, Sect. 2.2]). The first one concerns the action of G on the base B of the bundle ξ.

Condition A. *All orbits of the action $t : G\times B\to B$ are of the same type, and these orbits form a locally trivial G-bundle $(B,p,B/G)$, where $p : B\to B/G$ is the natural projection.*

Remark 8. Several remarks are in order.

- The manifold B/G does not have a canonical realization, so that one of our tasks below will be to give a concrete realization of it.
- The Slice Theorem [8, Sect. 4.4] guarantees that Condition A is satisfied whenever G is compact.
- According to the Principal Orbits Theorem [8, Theorem 4.27], if G is compact and B/G is connected, then there exists a maximum orbit type, G/H, in B (where H is a closed subgroup of G) such that the union of all orbits of type G/H (called *principal orbits*) is open and dense in B.
- For a detailed discussion of Condition A see [1, Sect. 2.2.1].

Condition B from [1, Sect. 2.2.2] is a technically complicated requirement on the action T of G on the total E space of ξ. Instead of stating it in full generality, we formulate a simplified condition which is enough for our purposes. To state this simplified condition (called Condition B′ below), we give the following definitions.

For $b \in B$, let $G_b := \{\, g \in G \,:\, t_g(b) = b \,\} \subseteq G$ stand for the *stationary* (or *isotropy*) *group* of b with respect to the action t of G on B. Clearly, $T : G_b \times \xi_b \to \xi_b$ is a linear representation of the stationary group G_b in the linear space ξ_b. Define the *stationary subspace* of ξ_b (with respect to the representation T of G_b) as

$$\operatorname{st}\xi_b := \{\, u \in \xi_b \,:\, T_g(u) = u \;\; \forall\, g \in G_b \,\} \subseteq \xi_b \ . \tag{44}$$

Condition B′. *The family of vector spaces* $\operatorname{st}\xi_b \subseteq \xi_b$ *form a smooth vector subbundle of* ξ *which we denote by* $\operatorname{st}\xi$ *and call the* stationary subbundle *of* ξ *(with respect to the action* $\mathcal{T} = (t, T)$ *of* G *on* ξ*).*

The stationary subbundles are sometimes called *kinematic bundles* [9,10].

We say that the vector bundle ξ with action $\mathcal{T}$ of G on it is a *reducible G-vector bundle* if the action $\mathcal{T}$ of G on ξ satisfies Conditions A and B′.

4.2. *Reduced vector bundles*

Let ξ be a reducible G-vector bundle, i.e., the action $\mathcal{T} = (t, T)$ satisfies Conditions A and B′. Here we will construct the *reduced vector bundle* ξ^G such that there is a natural bijective correspondence

$$\theta : C^\infty(\xi^G) \to C^\infty(\xi)^G \tag{45}$$

between all sections of ξ^G and all G-invariant sections of ξ. Moreover, the construction will make it clear that θ is a homomorphism from the $C^\infty(B/G)$-module $C^\infty(\xi^G)$ to the $C^\infty(B)^G$-module $C^\infty(\xi)^G$ (cf. [1, Remark 2.10]).

We start by constructing a concrete realization of the set B/G of all orbits of the action t of G on B; B/G is the base of the reduced bundle ξ^G. Let $\{\widetilde{U}_\alpha\}_{\alpha \in \mathscr{A}}$ be a collection of submanifolds of B that intersect transversely the orbits of the action t of G on B (recall the definition of transverse intersection from the beginning of Sect. 3). We require that each G-orbit in B intersects at least one $\widetilde{U}_\alpha$, and that the intersection occurs at a single point. In Figure 1 the orbits of the action t of G on

B are drawn with dashed lines, and two transverse to them submanifolds, $\widetilde{U}_\alpha$ and $\widetilde{U}_\beta$, are shown with solid lines. Let $\widetilde{U}_{\alpha,\beta} \subseteq \widetilde{U}_\alpha$ be the set of those points $b \in \widetilde{U}_\alpha$ for

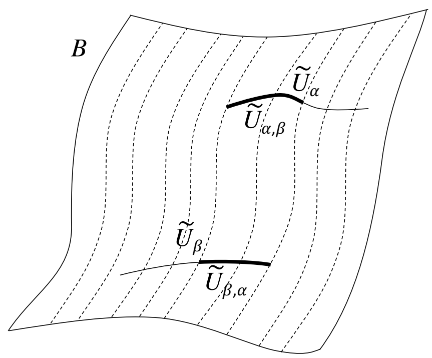

Fig. 1. On the construction of a coordinate realization of the base B/G of ξ^G.

which the orbit through b, $\mathcal{O}_b := \{t_g(b) : g \in G\}$, intersects the submanifold $\widetilde{U}_\beta$:

$$\widetilde{U}_{\alpha,\beta} := \{b \in \widetilde{U}_\alpha \,:\, \mathcal{O}_b \cap \widetilde{U}_\beta \neq \emptyset\} \subseteq \widetilde{U}_\alpha \; ; \tag{46}$$

the submanifolds $\widetilde{U}_{\alpha,\beta}$ and $\widetilde{U}_{\beta,\alpha} \subseteq \widetilde{U}_\beta$ are drawn with thick solid lines. The natural mappings $\widetilde{U}_{\beta,\alpha} \to \widetilde{U}_{\alpha,\beta}$ that take any $b \in \widetilde{U}_{\beta,\alpha}$ to the point $\mathcal{O}_b \cap \widetilde{U}_\alpha \in \widetilde{U}_{\alpha,\beta}$ "glue" the manifold B/G from the collection $\{\widetilde{U}_\alpha\}_{\alpha\in\mathscr{A}}$. The advantage of using this construction is that the abstract manifold B/G is realized locally as a concrete submanifold $\widetilde{U}_\alpha$ of B.

To construct the total space of the reduced bundle ξ^G, first note that if $\psi \in C^\infty(\xi)^G$ is a G-invariant section of ξ, then (43) and (44) imply that the value $\psi(b)$ belongs to the stationary subspace $\operatorname{st}\xi_b$: if $g \in G_b$, then $t_g(b) = b$, hence (43) becomes $\psi(b) = T_g(\psi(b))$. Therefore, we can consider $C^\infty(\xi)^G$ locally as a subset of $C^\infty(\operatorname{st}\xi)$. For any $\alpha \in \mathscr{A}$, we define the local realization ξ_α of ξ^G as the restriction of (the base of) $\operatorname{st}\xi$ to the submanifold $\widetilde{U}_\alpha$:

$$\xi_\alpha := \operatorname{st}\xi\restriction_{\widetilde{U}_\alpha} \; . \tag{47}$$

To "glue" ξ^G from its local coordinate realizations $\{\xi_\alpha\}_{\alpha\in\mathscr{A}}$, we define the restriction

$$\xi_{\alpha,\beta} := \xi_\alpha\restriction_{\widetilde{U}_{\alpha,\beta}} \left(= \operatorname{st}\xi\restriction_{\widetilde{U}_{\alpha,\beta}}\right) \tag{48}$$

of ξ_α to $\widetilde{U}_{\alpha,\beta}$ (46). For each $\alpha, \beta \in \mathscr{A}$ with $\widetilde{U}_{\alpha,\beta} \neq \emptyset$, one can define a natural isomorphism $\phi_{\alpha\beta} : \xi_{\beta,\alpha} \to \xi_{\alpha,\beta}$ as follows: let $b \in \widetilde{U}_{\beta,\alpha}$, and $g \in G$ be such that $t_g(b) \in \widetilde{U}_{\alpha,\beta}$, then for $e \in (\xi_{\beta,\alpha})_b = (\operatorname{st}\xi)_b$,

$$\phi_{\alpha\beta}(e) := T_g(e) \in \xi_{\alpha,\beta} \; ; \tag{49}$$

for more details, see [1, Eqns. (2.19)–(2.24)].

Now we can write the correspondence θ from (45) explicitly. A section $\psi^G \in C^\infty(\xi^G)$ of the reduced vector bundle ξ^G is realized as a collection $\{\psi_\alpha\}_{\alpha\in\mathscr{A}}$ of sections $\psi_\alpha \in C^\infty(\xi_\alpha)$ satisfying the compatibility conditions

$$\psi_\alpha = \phi_{\alpha\beta} \circ \psi_\beta \quad \text{whenever} \quad \widetilde{U}_{\alpha,\beta} \neq \emptyset \ . \tag{50}$$

The G-invariant section $\psi = \theta(\psi^G) \in C^\infty(\xi)^G$ can be constructed by extending the values of the coordinate realizations ψ_α from $\widetilde{U}_\alpha$ to B by G-invariance; the compatibility conditions (50) guarantee the consistency of this procedure. Explicitly, $\psi(b)$ for an arbitrary $b \in B$ is constructed as follows. Let $\alpha \in \mathscr{A}$ be such an index that the orbit $\mathcal{O}_b$ of t through b has a non-empty intersection with $\widetilde{U}_\alpha$; let $\mathcal{O}_b \cap \widetilde{U}_\alpha = \{b_\alpha\}$. Let $g \in G$ be such that $t_g(b_\alpha) = b$, then

$$\psi(b) = T_g \circ \psi_\alpha \circ t_g^{-1}(b) = T_g \circ \psi_\alpha(b_\alpha) \ . \tag{51}$$

One can show that $\psi(b)$ does not depend of the arbitrariness in the choices of α and g.

Conversely, a G-invariant section $\psi \in C^\infty(\xi)^G$ naturally belongs to $C^\infty(\operatorname{st}\xi)^G$ and, hence, determines uniquely sections $\psi_\alpha \in C^\infty(\xi_\alpha)$ by restriction of the domain of ψ to $\widetilde{U}_\alpha$:

$$\psi_\alpha := \psi{\restriction_{\widetilde{U}_\alpha}} \in C^\infty(\operatorname{st}\xi{\restriction_{\widetilde{U}_\alpha}}) = C^\infty(\xi_\alpha) \ , \qquad \alpha \in \mathscr{A} \ . \tag{52}$$

The family $\{\psi_\alpha\}_{\alpha\in\mathscr{A}}$ thus constructed satisfies the compatibility conditions (50) and, therefore, determines a section $\psi^G = \theta^{-1}(\psi) \in C^\infty(\xi^G)$.

4.3. *Invariant DOs and their reduction*

4.3.1. *Lie group action on a DO; invariant DO; reduced DO*

Let ξ and η be reducible G-vector bundles over B with the same action t of G on the common base B, and let $\mathcal{T}^\xi = (t, T^\xi)$ and $\mathcal{T}^\eta = (t, T^\eta)$ be the actions of G on the corresponding bundles. The actions of $g \in G$ on $C^\infty(\xi)$ and $C^\infty(\eta)$,

$$\begin{aligned} g^\xi &: C^\infty(\xi) \to C^\infty(\xi) : \psi \mapsto g^\xi(\psi) \ := \ T_g^\xi \circ \psi \circ t_g^{-1} \ , \\ g^\eta &: C^\infty(\eta) \to C^\infty(\eta) : \chi \mapsto g^\eta(\chi) \ := \ T_g^\eta \circ \chi \circ t_g^{-1} \ , \end{aligned} \tag{53}$$

define an action of g on $\operatorname{Diff}_k(\xi,\eta)$:

$$g : \operatorname{Diff}_k(\xi,\eta) \to \operatorname{Diff}_k(\xi,\eta) : D \mapsto g(D) \ := \ g^\eta \circ D \circ (g^\xi)^{-1} \ . \tag{54}$$

We say that a DO $D \in \operatorname{Diff}_k(\xi,\eta)$ is *G-invariant* if $g(D) = D$, i.e.,

$$g^\eta \circ D = D \circ g^\xi \ , \qquad \text{for all } g \in G \ . \tag{55}$$

Let $\operatorname{Diff}_k(\xi,\eta)^G$ stand for the set of all G-invariant order-k DOs from ξ to η.

If $\psi \in C^\infty(\xi)^G$ is a G-invariant section of ξ and $D \in \operatorname{Diff}_k(\xi,\eta)^G$, then $D\psi$ is a G-invariant section of η – indeed, (55) yields $g^\eta(D\psi) = D(g^\xi(\psi)) = D\psi$. Let

$$\theta^\xi : C^\infty(\xi^G) \to C^\infty(\xi)^G \ , \qquad \theta^\eta : C^\infty(\eta^G) \to C^\infty(\eta)^G \tag{56}$$

be the two bijections (45) for ξ and η, respectively. Then each G-invariant DO $D \in \mathrm{Diff}_k(\xi,\eta)^G$ maps $C^\infty(\xi)^G$ to $C^\infty(\eta)^G$ and, therefore, generates a reduced DO

$$D^G := (\theta^\eta)^{-1} \circ D \circ \theta^\xi : C^\infty(\xi^G) \to C^\infty(\eta^G) \tag{57}$$

between the sections of the reduced bundles ξ^G and η^G.

Remark 9. If $\psi \in C^\infty(\xi)^G$, then $\psi \in C^\infty(\mathrm{st}\,\xi)^G$ (see the text before (47)). Thus, the above reasoning implies that $D \in \mathrm{Diff}_k(\xi,\eta)^G$ can be naturally considered as a DO in $\mathrm{Diff}_k(\mathrm{st}\,\xi, \mathrm{st}\,\eta)^G$.

4.3.2. *Coordinate realizations of the reduced DO*

To describe explicitly the reduced DO $D^G \in \mathrm{Diff}_k(\xi^G,\eta^G)$, we use the coordinate realizations, $\{\xi_\alpha\}_{\alpha\in\mathscr{A}}$ and $\{\eta_\alpha\}_{\alpha\in\mathscr{A}}$, of the reduced bundles ξ^G and η^G constructed as in Sect. 4.2.

Namely, we choose a family $\{\widetilde{U}_\alpha\}_{\alpha\in\mathscr{A}}$ of submanifolds of B transversal to the orbits of the action t of G on B; we use the same family for ξ^G and for η^G. The coordinate realizations $\xi_\alpha = \mathrm{st}\,\xi\restriction_{\widetilde{U}_\alpha}$ and $\eta_\alpha = \mathrm{st}\,\eta\restriction_{\widetilde{U}_\alpha}$ are constructed as in (47). The transition isomorphisms $\phi^\xi_{\alpha\beta} : \xi_{\beta,\alpha} \to \xi_{\alpha,\beta}$ and $\phi^\eta_{\alpha\beta} : \eta_{\beta,\alpha} \to \eta_{\alpha,\beta}$ (recall (48) and (49)) glue the reduced bundles ξ^G and η^G from their coordinate realizations $\{\xi_\alpha\}_{\alpha\in\mathscr{A}}$ and $\{\eta_\alpha\}_{\alpha\in\mathscr{A}}$. A section $\psi^G \in C^\infty(\xi^G)$ is realized as a collection $\{\psi_\alpha\}_{\alpha\in\mathscr{A}}$ of sections $\psi_\alpha \in C^\infty(\xi_\alpha)$; the corresponding G-invariant section $\psi = \theta^\xi(\psi^G) \in C^\infty(\xi)^G$ is given by (51).

Now we describe the coordinate realization of the reduced DO $D^G \in \mathrm{Diff}_k(\xi^G,\eta^G)$ corresponding to a G-invariant DO $D \in \mathrm{Diff}_k(\xi,\eta)^G$. Since $\psi^G \in C^\infty(\xi^G)$ is realized as a family of sections of $\xi_\alpha = \mathrm{st}\,\xi\restriction_{\widetilde{U}_\alpha}$, the reduced DO D^G can be defined as a family $\{D_\alpha\}_{\alpha\in\mathscr{A}}$ of DOs

$$D_\alpha \in \mathrm{Diff}_k(\xi_\alpha,\eta_\alpha) = \mathrm{Diff}_k(\mathrm{st}\,\xi\restriction_{\widetilde{U}_\alpha}, \mathrm{st}\,\eta\restriction_{\widetilde{U}_\alpha}) \ . \tag{58}$$

Since the coordinate realization ψ_α of $\psi^G \in C^\infty(\xi^G)$ is the restriction (52) of the G-invariant section $\psi = \theta^\xi(\psi^G) \in C^\infty(\xi)^G$ to $\widetilde{U}_\alpha \subseteq B$, we set

$$D_\alpha\psi_\alpha := (D\psi)\restriction_{\widetilde{U}_\alpha} \in C^\infty(\eta_\alpha) \ . \tag{59}$$

It is easy to check that this family $\{D_\alpha\}_{\alpha\in\mathscr{A}}$ is compatible with the transition isomorphisms $\phi^\xi_{\alpha\beta}$ and $\phi^\eta_{\alpha\beta}$ (recall (50)) – namely, for a pair of indices α and β such that $\widetilde{U}_{\alpha,\beta} \neq \emptyset$, the following compatibility property holds:

$$D_\alpha\psi_\alpha = \phi^\eta_{\alpha\beta} \circ D_\beta(\phi^\xi_{\beta\alpha} \circ \psi_\alpha) \qquad \text{for each } \psi_\alpha \in C^\infty(\xi_\alpha) \ . \tag{60}$$

Since an invariant DO $D \in \mathrm{Diff}_k(\xi,\eta)^G$ can be considered as a DO in $\mathrm{Diff}_k(\mathrm{st}\,\xi, \mathrm{st}\,\eta)^G$ (recall Remark 9), the construction of the coordinate realizations D_α (58) of the reduced DO D^G is closely related to the restriction of D to the submanifolds $\widetilde{U}_\alpha$ of the base B – a process discussed in Sect. 3. According to the discussion there, to define the restriction D_α of D to $\widetilde{U}_\alpha$, we have to choose a splitting Σ^k_α of short exact sequences of jet bundles for each $\alpha \in \mathscr{A}$ (recall (30)–(33)).

Moreover, because of (60), the family of splittings $\{\Sigma^k_\alpha\}_{\alpha\in\mathscr{A}}$ must satisfy appropriate compatibility conditions. In the reduction of invariant DOs, the family $\{\Sigma^k_\alpha\}_{\alpha\in\mathscr{A}}$ comes naturally for a simple reason: if we know the value of a G-invariant section $\psi \in C^\infty(\xi)^G$ at $\widetilde{U}_\alpha$, we can reconstruct the values of ψ in an open neighborhood of $\widetilde{U}_\alpha$ and find all derivatives of ψ, and then express the non-internal derivatives in terms on the internal ones, thus obtaining the desired splitting Σ^k_α. It is easy to show that the coordinate realizations $\{D_\alpha\}_{\alpha\in\mathscr{A}}$ of D^G coming from $\{\Sigma^k_\alpha\}_{\alpha\in\mathscr{A}}$ satisfies the compatibility conditions (60).

The implementation of the algorithm just described, however, is hampered by practical difficulties. Extending a section of $\xi_\alpha = \operatorname{st}\xi\restriction_{\widetilde{U}_\alpha}$ to a neighborhood of $\widetilde{U}_\alpha$ in B is generally a difficult procedure that involves solving systems of nonlinear equations. To avoid the need of doing this, below we reformulate the problem of dimensional reduction of an invariant DO in the geometric language developed in Sect. 3, and in the following sections we develop the algorithm in detail and deal with issues related to its consistency.

We start by defining objects like the ones introduced in Sect. 3.3, but replacing $\widetilde{U}$ by $\widetilde{U}_\alpha$, and ξ and η by their stationary subbundles. Let $i_\alpha : \widetilde{U}_\alpha \hookrightarrow B$ be the natural embedding, I^k_α be the subbundle of $J^k(\operatorname{st}\xi)_{\widetilde{U}_\alpha}$ with fiber over $b \in \widetilde{U}_\alpha$ given by

$$(I^k_\alpha)_b := \{\, J^k(\psi)(b) \,:\, \psi \in C^\infty(\operatorname{st}\xi) \text{ s.t. } \psi\circ i_\alpha \equiv 0 \,\} \subseteq (J^k(\operatorname{st}\xi)_{\widetilde{U}_\alpha})_b \tag{61}$$

(cf. (27)). Let $i^k_\alpha : I^k_\alpha \hookrightarrow J^k(\operatorname{st}\xi)_{\widetilde{U}_\alpha}$ and $j^k_\alpha : J^k(\operatorname{st}\xi)_{\widetilde{U}_\alpha} \to J^k_{\widetilde{U}_\alpha}(\operatorname{st}\xi\restriction_{\widetilde{U}_\alpha}) = J^k_{\widetilde{U}_\alpha}(\xi_\alpha)$ be respectively the natural embedding and the natural projection as in (28) and (21); as before, we will write I^k_α instead of $i^k_\alpha(I^k_\alpha)$. Here we used the notation $J^k_{\widetilde{U}_\alpha}$ introduced in Sect. 3.1 that reminds us that this is the k-jet of a vector bundle with base $\widetilde{U}_\alpha$ (recall equation (19) and the text preceding it). Consider the invariant DO D as an element of $\operatorname{Diff}_k(\operatorname{st}\xi, \operatorname{st}\eta)^G$, and let $\widetilde{D} : J^k(\operatorname{st}\xi) \to \operatorname{st}\eta$ be its total symbol. Let $D_\alpha \in \operatorname{Diff}_k(\xi_\alpha, \eta_\alpha)$ be the restriction of D to $\widetilde{U}_\alpha$, and $\widetilde{D_\alpha} : J^k(\xi_\alpha) \to \eta_\alpha$ be its total symbol, so that $D_\alpha = \big(\widetilde{D_\alpha}\big)_* J^k_{\widetilde{U}_\alpha}$. Then the diagram (30) becomes

$$\begin{tikzcd}
0 \arrow[r] & I^k_\alpha \arrow[r, hook, shift left, "i^k_\alpha"] & J^k(\operatorname{st}\xi)_{\widetilde{U}_\alpha} \arrow[l, shift left, "\Pi^k_\alpha"] \arrow[r, shift left, "j^k_\alpha"] \arrow[d, "\widetilde{D}\restriction_{\widetilde{U}_\alpha}"'] & J^k_{\widetilde{U}_\alpha}(\xi_\alpha) \arrow[l, shift left, "\Sigma^k_\alpha"] \arrow[r] \arrow[dl, "\widetilde{D_\alpha}"] & 0\,. \\
& & \eta_\alpha & &
\end{tikzcd} \tag{62}$$

The total symbol $\widetilde{D_\alpha}$ of the coordinate realization D_α of the reduced DO D^G is then given as the composition $\widetilde{D_\alpha} = \widetilde{D}\restriction_{\widetilde{U}_\alpha} \circ \Sigma^k_\alpha$, as in (33).

The G-invariance of the DO D naturally determines a vector bundle morphism $\Sigma^k_\alpha \in \operatorname{Hom}\big(J^k_{\widetilde{U}_\alpha}(\xi_\alpha), J^k(\operatorname{st}\xi)_{\widetilde{U}_\alpha}\big)$ (over the identity in $\widetilde{U}_\alpha$) that splits the horizontal short exact sequence in (62), i.e., such that $j^k_\alpha \circ \Sigma^k_\alpha = \operatorname{Id}_{J^k_{\widetilde{U}_\alpha}(\xi_\alpha)}$ (cf. (31)). As explained in the beginning of Sect. 3.3.2, this condition guarantees that Σ^k_α provides a representation of $J^k(\mathrm{st}\,\xi)_{\widetilde{U}_\alpha}$ as a direct sum,

$$J^k(\mathrm{st}\,\xi)_{\widetilde{U}_\alpha} = I^k_\alpha \oplus \Sigma^k_\alpha(J^k_{\widetilde{U}_\alpha}(\xi_\alpha)) \tag{63}$$

(recall (34)), and that Σ^k_α is completely defined by its image, $\Sigma^k_\alpha(J^k_{\widetilde{U}_\alpha}(\xi_\alpha))$. To describe the explicit construction of $\Sigma^k_\alpha(J^k_{\widetilde{U}_\alpha}(\xi_\alpha))$ (which will be done in Sect. 4.3.4), we will need the concept of Lie derivatives which we define below.

4.3.3. *Lie derivatives*

Assume that the action $\mathcal{T} = (t, T)$ of G on ξ is such that ξ is a reducible G-vector bundle. Let $x = (x^\mu)$ stand for some local coordinates in the base B; we will often identify a point $b \in B$ with its coordinates x. Assume that a basis $(e_a(x))$ has been chosen in each fiber $\xi_x = \pi^{-1}(x)$, and let $z = (z^a)$ be the coordinates in the fibers in this basis. If the action $\mathcal{T}$ preserves the fibers of ξ, then its general form is $\mathcal{T}_g(x^\mu, z^a) = \big(t_g(x)^\mu,\, T_g(x,z)^a\big)$ for $g \in G$. If, in addition, the action is through vector bundle morphisms (i.e., if T is linear in the fibers), then the general form of $\mathcal{T}$ is

$$\mathcal{T}_g(x^\mu, z^a) = \big(t_g(x)^\mu,\, T_g(x)^a{}_c\, z^c\big)\ , \qquad g \in G\ . \tag{64}$$

By assumption, we consider only actions $\mathcal{T}$ of the form (64).

In the local coordinates (x^μ, z^a), the action (42) of G on the sections of ξ has the form

$$g(\psi)^a(b) = T_g(t_g^{-1}(b))^a{}_c\ \psi^c(t_g^{-1}(b))\ , \qquad g \in G\ , \quad \psi \in C^\infty(\xi)\ . \tag{65}$$

Let $\mathfrak{g}$ be the Lie algebra of G with generators λ_i, $\mathfrak{g}^*$ be its dual, and $\mathrm{e}^{s\lambda}$ (with s in an open interval in $\mathbb{R}$ containing 0) be the local 1-parameter subgroup of G generated by $\lambda \in \mathfrak{g}$. Let $\mathfrak{g}^* \otimes \xi$ be a vector bundle whose sections are of the form $\Lambda \otimes \psi$, where $\Lambda \in \mathfrak{g}^*$ is an element of $\mathfrak{g}^*$ independent on the point in the base, and $\psi \in C^\infty(\xi)$. In other words, for any $b \in B$, $(\Lambda \otimes \psi)(b) = \Lambda \otimes \psi(b)$ takes an element $\lambda \in \mathfrak{g}$ and produces $\langle \Lambda, \lambda\rangle\, \psi(b) \in \xi_b$ (where $\langle\cdot,\cdot\rangle$ is the natural pairing between $\mathfrak{g}^*$ and $\mathfrak{g}$). To emphasize this peculiarity of $\mathfrak{g}^* \otimes \xi$, for the space of its sections we will use the notation $\mathfrak{g}^* \otimes C^\infty(\xi)$ instead of $C^\infty(\mathfrak{g}^* \otimes \xi)$.

The *Lie derivative* of $\mathcal{T}$ is the DO $L \in \mathrm{LDiff}_1(\xi, \mathfrak{g}^* \otimes \xi)$ defined by

$$(L\psi)(\lambda) := \frac{\mathrm{d}}{\mathrm{d}s}\mathrm{e}^{s\lambda}(\psi)\bigg|_{s=0} \in C^\infty(\xi)\ , \qquad \psi \in C^\infty(\xi)\ , \quad \lambda \in \mathfrak{g} \tag{66}$$

(where we used the notation (42)). In local coordinates, if $b = (x^\mu)$, and λ_i is a generator of $\mathfrak{g}$, then

$$\begin{aligned}(L\psi)(\lambda_i)^a(b) &= \frac{\mathrm{d}}{\mathrm{d}s}\left[T_{\exp(s\lambda_i)}\left(t_{\exp(-s\lambda_i)}(b)\right)^a{}_c\,\psi^c\left(t_{\exp(-s\lambda_i)}(b)\right)\right]\bigg|_{s=0} \\ &=: -X_i(b)^\mu\, \partial_\mu \psi^a(b) + Z_i(b)^a{}_c\, \psi^c(b)\ .\end{aligned} \tag{67}$$

Here $X_i = X_i{}^{\mu}\partial_{\mu} \in C^{\infty}(\tau(B))$ (where $\tau(B)$ is the tangent bundle of B) with

$$X_i(b)^{\mu} := \frac{\mathrm{d}}{\mathrm{d}s} t_{\exp(s\lambda_i)}(b)^{\mu}\Big|_{s=0} \tag{68}$$

are the fundamental vector fields of the action t of G on B, and

$$Z_i(b)^a{}_c := \frac{\mathrm{d}}{\mathrm{d}s} T_{\exp(s\lambda_i)}\left(t_{\exp(-s\lambda_i)}(b)\right)^a{}_c\Big|_{s=0} \ . \tag{69}$$

Remark 10. The stationary subbundle $\mathrm{st}\,\xi \subseteq \xi$ (recall (42) and Condition B′) is invariant under the action of G, so the Lie derivative can also be considered as a DO in $\mathrm{LDiff}_1(\mathrm{st}\,\xi, \mathfrak{g}^* \otimes \mathrm{st}\,\xi)$.

4.3.4. *Lie derivatives and natural splittings Σ^k_α: the algorithm*

To formalize the idea of obtaining the splittings Σ^k_α – and, hence, the coordinate realizations D_α of the reduced DO D^G – we employ the Lie derivative $L \in \mathrm{LDiff}_1(\xi,\, \mathfrak{g}^* \otimes \xi)$ of the action $\mathcal{T}$ of G on ξ. According to Remark 10, we can consider L as an element of $\mathrm{LDiff}_1(\mathrm{st}\,\xi,\, \mathfrak{g}^* \otimes \mathrm{st}\,\xi)$, as we will do here.

Let $L = \widetilde{L}_* \, J^1 \in \mathrm{LDiff}_1(\mathrm{st}\,\xi,\, \mathfrak{g}^* \otimes \mathrm{st}\,\xi)$ be the Lie derivative, and $\widetilde{L} \in \mathrm{Hom}\,(J^1(\mathrm{st}\,\xi),\, \mathfrak{g}^* \otimes \mathrm{st}\,\xi)$ be its total symbol. The $(k-1)$st prolongation of L is a linear order-k DO

$$P^{k-1}(L) = \left(\widetilde{P^{k-1}(L)}\right)_* J^k \in \mathrm{LDiff}_k\big(\mathrm{st}\,\xi,\, \mathfrak{g}^* \otimes J^{k-1}(\mathrm{st}\,\xi)\big)$$

with total symbol $\widetilde{P^{k-1}(L)} \in \mathrm{Hom}\left(J^k(\mathrm{st}\,\xi),\, \mathfrak{g}^* \otimes J^{k-1}(\mathrm{st}\,\xi)\right)$. Here we again use the notation $\mathfrak{g}^* \otimes J^{k-1}(\mathrm{st}\,\xi)$ instead of $J^{k-1}(\mathfrak{g}^* \otimes \mathrm{st}\,\xi)$ because the element of $\mathfrak{g}^*$ does not depend on the point in the base – for the same reason we introduced the notation $\mathfrak{g}^* \otimes C^{\infty}(\xi)$ instead of $C^{\infty}(\mathfrak{g}^* \otimes \xi)$ in Sect. 4.3.3.

We introduce a special notation for the subbundle $\ker \widetilde{P^{k-1}(L)} \subseteq J^k(\mathrm{st}\,\xi)$ with base restricted to $\widetilde{U}_\alpha$:

$$R^k_\alpha := \left(\ker \widetilde{P^{k-1}(L)}\right)\restriction_{\widetilde{U}_\alpha} \subseteq J^k(\mathrm{st}\,\xi)_{\widetilde{U}_\alpha} \ . \tag{70}$$

We define the splitting Σ^k_α by setting

$$\Sigma^k_\alpha(J^k_{\widetilde{U}_\alpha}(\xi_\alpha)) := R^k_\alpha \ . \tag{71}$$

Below we give the algorithm for dimensional reduction of an invariant DO that implements the choice (71), before proceeding with its theoretical justification in Sect. 4.3.5. Let $\psi \in C^{\infty}(\mathrm{st}\,\xi)^G$, and assume that we work in local coordinates adapted to $\widetilde{U}_\alpha$ (17).

Algorithm for dimensional reduction of a G-invariant DO $D \in \mathrm{Diff}_k(\xi,\eta)^G$:

Step 1 Let $\lambda_1, \ldots, \lambda_{\dim G}$ be a basis of $\mathfrak{g}$. Write down the system

$$L\psi(\lambda_i) = 0 \ , \quad i = 1, \ldots, \dim G \ . \tag{72}$$

Step 2 Find the $(k-1)$st prolongation of (72), i.e., compute all partial derivatives of each of the equations in (72) up to order $(k-1)$, thus obtaining the system

$$\begin{aligned} L\psi(\lambda_i) &= 0 \ , \\ \partial_{\mu_1} L\psi(\lambda_i) &= 0 \ , \quad 1 \le \mu_1 \le n \ \big(= \dim B\big) \ , \\ \partial_{\mu_1\mu_2} L\psi(\lambda_i) &= 0 \ , \quad 1 \le \mu_1 \le \mu_2 \le n \ , \\ &\vdots \\ \partial_{\mu_1\cdots\mu_{k-1}} L\psi(\lambda_i) &= 0 \ , \quad 1 \le \mu_1 \le \mu_2 \le \cdots \le \mu_{k-1} \le n \ ; \end{aligned} \tag{73}$$

here i takes values $1, \dots, \dim G$.

Step 3 Restrict all the equations from (73) to the submanifold $\widetilde{U}_\alpha$. From the restricted equations express all non-internal for $\widetilde{U}_\alpha$ partial derivatives of ψ in terms of the internal for $\widetilde{U}_\alpha$ partial derivatives.

Step 4 Substitute all non-internal partial derivatives in the DO D with the expressions obtained for them in Step 3, thus obtaining the coordinate realization $D_\alpha \in \mathrm{Diff}_k(\xi_\alpha, \eta_\alpha)$ (59).

Step 5 Repeat steps 3 and 4 for any $\alpha \in \mathscr{A}$ to obtain the coordinate realization $\{D_\alpha\}_{\alpha\in\mathscr{A}}$ of the reduced DO $D^G \in \mathrm{Diff}_k(\xi^G, \eta^G)$ (57).

Remark 11. Note that the above algorithm for computing D^G involves only elementary operations – computing derivatives and solving a system of linear algebraic equations. Moreover, the results in Sect. 4.3.5 guarantee that the linear system has constant rank and allows us to express all non-internal derivatives in a unique way.

Example 12. This example is a continuation of Example 4 and uses the same notations. Let the group $G := \mathbb{R}_+$ be the multiplicative group of positive real numbers acting on ξ and η as follows:

$$\mathcal{T}^\xi_g\big((x^1, x^2), z\big) = \mathcal{T}^\eta_g\big((x^1, x^2), z\big) := \big((gx^1, gx^2), g^2 z\big) \ . \tag{74}$$

The action of $\mathbb{R}_+$ on $C^\infty(\xi)$ is

$$g^\xi(\psi)(x^1, x^2) = g^2\, \psi(g^{-1}x^1, g^{-1}x^2) \ , \tag{75}$$

and similarly for $g^\eta : C^\infty(\eta) \to C^\infty(\eta)$. In other words, a $\mathbb{R}_+$-invariant section $\psi \in C^\infty(\xi)^{\mathbb{R}_+}$ is a function on $\mathbb{R}^2_+$ satisfying the homogeneity property $\psi(gx^1, gx^2) = g^2\, \psi(x, y)$ for all $g > 0$. The orbits of $\mathbb{R}_+$ in $\mathbb{R}^2_+$ are rays through the origin, so as a realization of $B/G = \mathbb{R}^2_+/\mathbb{R}_+$ we choose the the manifold $\widetilde{U}$ defined in Example 4 which intersects each orbit at one point transversely. Since the stationary group of the action (74) is trivial, the reduced bundle is simply ξ with base restricted to $\widetilde{U}$: $\xi_\alpha = \mathrm{st}\,\xi|_{\widetilde{U}} = \xi_{\widetilde{U}}$. If $\psi \in C^\infty(\xi)^{\mathbb{R}_+}$ is $\mathbb{R}_+$-invariant section of ξ, then the corresponding reduced section $\psi^{\mathbb{R}_+} = (\theta^\xi)^{-1}(\psi) \in C^\infty(\xi^{\mathbb{R}_+})$ (recall (56)) is simply the restriction to $\widetilde{U}$:

$$\psi^{\mathbb{R}_+}(\widetilde{x}) = \big((\theta^\xi)^{-1}(\psi)\big)(\widetilde{x}) = \psi(\widetilde{x}, 1) \ . \tag{76}$$

Conversely, if $\psi^{\mathbb{R}_+} \in C^\infty(\xi^{\mathbb{R}_+})$ is a section of the reduced bundle, then the corresponding $\mathbb{R}_+$-invariant section of ξ can be computed from the action (75) to be

$$\psi(x^1, x^2) = \big(\theta^\xi(\psi^{\mathbb{R}_+})\big)(x^1, x^2) = (x^2)^2\, \psi^{\mathbb{R}_+}\!\left(\frac{x^1}{x^2}\right) . \tag{77}$$

The Lie derivative of the action (75) of $\mathbb{R}_+$ on $C^\infty(\xi)$ is

$$\frac{\mathrm{d}}{\mathrm{d}g}\bigg|_{g=1} g^\xi(\psi)(x^1, x^2) = \frac{\mathrm{d}}{\mathrm{d}g}\bigg|_{g=1} \left[g^2\, \psi(g^{-1}x^1, g^{-1}x^2)\right] = -x^1\partial_1\psi - x^2\partial_2\psi + 2\psi .$$

It turns out that this Lie derivative is equal (up to an overall minus sign) to the operator M from Example 4, hence M is formally integrable (which was mentioned there without proof). Reusing the calculations from Example 4 (recall (40)), we obtain the same expression for the $\mathbb{R}_+$-reduced Laplacian $D^{\mathbb{R}_+}$ as the restricted DO $D_{\widetilde{U}}$ in (41): for $\psi^{\mathbb{R}_+} \in C^\infty(\xi^{\mathbb{R}_+})$,

$$(D^{\mathbb{R}_+}\psi^{\mathbb{R}_+})(\widetilde{x}) = \left(1 + \widetilde{x}^2\right) (\psi^{\mathbb{R}_+})''(\widetilde{x}) - 2\widetilde{x}(\psi^{\mathbb{R}_+})'(\widetilde{x}) + 2\psi^{\mathbb{R}_+}(\widetilde{x}) . \tag{78}$$

Because of the simplicity of this example, one can obtain directly the general form of the set of $\mathbb{R}_+$-invariant functions – the vanishing of the Lie derivative results in the quasilinear first-order PDE $-x^1\partial_1\psi - x^2\partial_2\psi + 2\psi = 0$ with general solution $\psi(x^1, x^2) = (x^2)^2\, \Psi\big(\frac{x^1}{x^2}\big)$, where Ψ is an arbitrary smooth function of one variable, and the corresponding reduced section is $\psi^{\mathbb{R}_+}(\widetilde{x}) = \psi(\widetilde{x}, 1) = \Psi(\widetilde{x})$. Applying the Laplacian D to $\psi(x^1, x^2)$ gives us

$$D\left[(x^2)^2\, \Psi\!\left(\frac{x^1}{x^2}\right)\right] = \left(\frac{x^1}{x^2}\right)^2 \Psi''\!\left(\frac{x^1}{x^2}\right) - \frac{2x^1}{x^2}\, \Psi'\!\left(\frac{x^1}{x^2}\right) + 2\, \Psi\!\left(\frac{x^1}{x^2}\right) ,$$

which, after setting $(x^1, x^2) = (\widetilde{x}, 1)$, gives the expression (78) for $D^{\mathbb{R}_+}$.

4.3.5. *Lie derivatives and natural splittings Σ^k_α: theoretical justification*

Now we turn to the theoretical issues that remain to be resolved. We need to prove that the subbundle R^k_α (70) is indeed transversal to I^k_α in $J^k(\mathrm{st}\,\xi)_{\widetilde{U}_\alpha}$ – this is the content of Theorem 13 below. This transversality allows us to choose the image of Σ^k_α to be equal to R^k_α (71), which provides us with a splitting (63) of the short exact sequence in (62). In the course of the proof we also show that the Lie derivative is a formally integrable DO (Theorem 14 below).

We start by introducing some notations. Let $b \in \widetilde{U}_\alpha$ be an arbitrary point in the base $\widetilde{U}_\alpha$ of the coordinate realization ξ_α of ξ^G. Let $\dim \widetilde{U}_\alpha = \widetilde{n}$; clearly, $\dim G - \dim G_b = n - \widetilde{n}$. Let $(x^\mu) = (x^1, \ldots, x^n)$ be local coordinates in B in a neighborhood of b that are adapted to the submanifold $\widetilde{U}_\alpha$ (recall (17)). Split the indices $\mu = 1, \ldots, n$ into two groups:

$$\mu = (\widetilde{\mu}, \underline{\mu}) , \qquad \widetilde{\mu} := (1, \ldots, \widetilde{n}) , \quad \underline{\mu} := (\widetilde{n} + 1, \ldots, n) ,$$

so that $x^{\widetilde{\mu}}$ are coordinates in $\widetilde{U}_\alpha$, and

$$\widetilde{U}_\alpha = \{x^{\underline{\mu}} = 0\} := \{x^{\widetilde{n}+1} = 0, \ldots, x^n = 0\} . \tag{79}$$

Similarly, split the indices $i = 1, \dots, \dim G$ into two groups:

$$i = (\bar{i}, \underline{i}) \ , \qquad \bar{i} := (1, \dots, \dim G_b) \ , \quad \underline{i} := (\dim G_b + 1, \dots, \dim G)$$

in such a way that the elements $\lambda_{\bar{i}} \in \mathfrak{g}$ from the basis of $\mathfrak{g}$ span the Lie algebra of G_b: $\operatorname*{span}_{\bar{i}=1,\dots,\dim G_b} \{\lambda_{\bar{i}}\} = \mathrm{Lie}\,(G_b)$. Clearly, $\underline{\mu}$ and $\underline{i}$ represent the same number of indices.

Let $X_i = X_i{}^{\mu}\,\partial_\mu \in C^\infty(\tau(B))$ and Z_i be the objects defined in (68) and (69). In the new notations, the fields X_i can be split into two groups: $X_i = (X_{\bar{i}}, X_{\underline{i}})$, and their components,

$$X_{\bar{i}} = X_{\bar{i}}{}^{\widetilde{\mu}}\,\partial_{\widetilde{\mu}} + X_{\bar{i}}{}^{\underline{\mu}}\,\partial_{\underline{\mu}} \ , \qquad X_{\underline{i}} = X_{\underline{i}}{}^{\widetilde{\mu}}\,\partial_{\widetilde{\mu}} + X_{\underline{i}}{}^{\underline{\mu}}\,\partial_{\underline{\mu}} \ ,$$

can be arranged in a block matrix form:

$$(X_i{}^{\mu}) = \begin{pmatrix} X_{\bar{i}}{}^{\widetilde{\mu}} & X_{\bar{i}}{}^{\underline{\mu}} \\ X_{\underline{i}}{}^{\widetilde{\mu}} & X_{\underline{i}}{}^{\underline{\mu}} \end{pmatrix} \ . \tag{80}$$

Because of the special choices of local coordinates (79) and basis of $\mathfrak{g}$, the following holds:

$$(X_i(b)^{\mu}) = \begin{pmatrix} X_{\bar{i}}(b)^{\widetilde{\mu}} & X_{\bar{i}}(b)^{\underline{\mu}} \\ X_{\underline{i}}(b)^{\widetilde{\mu}} & X_{\underline{i}}(b)^{\underline{\mu}} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & X_{\underline{i}}(b)^{\underline{\mu}} \end{pmatrix} \ . \tag{81}$$

The square matrix $\big(X_{\underline{i}}{}^{\underline{\mu}}(b)\big)$ has full rank:

$$\mathrm{rank}\,\big(X_{\underline{i}}(b)^{\underline{\mu}}\big) = n - \widetilde{n} \tag{82}$$

(which, by continuity, implies that $\mathrm{rank}\,\big(X_{\underline{i}}{}^{\underline{\mu}}\big) = n - \widetilde{n}$ in an open neighborhood of b). Note that $\widetilde{U}_\alpha$ was not chosen in any special way (in particular, different points from $\widetilde{U}_\alpha$ may have different stationary subgroups), so that the vanishing of the components of X_i as in (81) occurs only at the point b.

For brevity, do not write explicitly the values that the indices take; the notation for an index indicates the range of its values as follows: $\mu = 1, \dots, n$; $\widetilde{\mu} = 1, \dots, \widetilde{n}$; $\underline{\mu} = \widetilde{n} + 1, \dots, n$; $i = 1, \dots, \dim G$; $\bar{i} = 1, \dots, \dim G_b$; $\underline{i} = \dim G_b + 1, \dots, \dim G$. We denote by $(R^l_\alpha)_b$ the fiber over $b \in \widetilde{U}_\alpha$ of $R^l_\alpha \subseteq J^l(\mathrm{st}\,\xi)_{\widetilde{U}_\alpha}$ (70), for $l = 1, \dots, k$. We omit the coordinates (x^μ) of the base point from the notations of the jet bundle coordinates (3), and the index α from R^l_α.

According to (67), an element $(z^a, z^a_\mu) \in J^1(\xi)_b$ belongs to R^1_b exactly when

$$-X_i(b)^{\mu} z^a_\mu + Z_i(b)^a{}_c z^c = 0 \ . \tag{83}$$

Because of (81), these equations can be rewritten as

$$Z_{\bar{i}}(b)^a{}_c z^c = 0 \ , \tag{84}$$

$$-X_{\underline{i}}(b)^{\underline{\mu}}\, z^a_{\underline{\mu}} + Z_{\underline{i}}(b)^a{}_c z^c = 0 \ . \tag{85}$$

The equations (84) mean that if $(z^a, z^a_\mu) \in R^1_b$, then (z^a) must belong to the stationary subbundle $\mathrm{st}\,\xi_b$; clearly,

$$\mathrm{rank}\,\mathrm{st}\,\xi = \mathrm{rank}\,\xi - (\text{the number of independent equations in (84)}) \ .$$

Since by (82) the matrix $\big(X_{\underline{i}}(b)^{\underline{\mu}}\big)$ is invertible, from (85) we can express all non-internal first derivatives $z^a_{\underline{\mu}}$ as functions of z^a. Therefore the dimension of R^1_b is equal to the sum of $\operatorname{rank}\operatorname{st}\xi$ and the number of the internal derivatives $z^a_{\widetilde{\mu}}$ of the fields from $\operatorname{st}\xi$:

$$\dim R^1_b = (1+\widetilde{n}) \operatorname{rank}\operatorname{st}\xi \ .$$

Note that this is equal to $\operatorname{rank} J^1_{\widetilde{U}_\alpha}(\xi_\alpha) = \binom{\widetilde{n}+1}{1}\operatorname{rank}\operatorname{st}\xi$ (recall (20)).

An element $(z^a, z^a_\mu, z^a_{\mu\rho}) \in J^2(\xi)_b$ belongs to R^2_b if, in addition to (84) and (85), the following equations hold:

$$-\partial_\rho X_{\bar{i}}(b)^\mu z^a_\mu + \partial_\rho Z_{\bar{i}}(b)^a{}_c z^c + Z_{\bar{i}}(b)^a{}_c z^c_\rho = 0 \ , \tag{86}$$

$$-\partial_\rho X_{\underline{i}}(b)^\mu z^a_\mu - X_{\underline{i}}(b)^{\underline{\mu}} z^a_{\underline{\mu}\rho} + \partial_\rho Z_{\underline{i}}(b)^a{}_c z^c + Z_{\underline{i}}(b)^a{}_c z^c_\rho = 0 \tag{87}$$

(obtained by differentiating (83) with respect to x^ρ and using (81)). Note that in (86) $z^a_{\mu\rho}$ do not appear, while (87) contains all non-internal second derivatives $z^a_{\underline{\mu}\rho}$ (and no internal second derivatives). The invertibility of the matrix $\big(X_{\underline{i}}(b)^{\underline{\mu}}\big)$ (82) guarantees that all non-internal second derivatives $z^a_{\underline{\mu}\rho}$ can be expressed in terms of lower-order derivatives. From this we conclude that $(\dim R^2_b - \dim R^1_b)$ is no greater than the number of internal second derivatives $z^a_{\widetilde{\mu}\widetilde{\rho}}$, i.e., $\dim R^2_b - \dim R^1_b \leq \binom{\widetilde{n}+2-1}{2}\operatorname{rank}\operatorname{st}\xi$ (see the text preceding (5)).

An element $(z^a, z^a_\mu, z^a_{\mu\rho}, z^a_{\mu\rho\sigma}) \in J^3(\xi)_b$ belongs to R^3_b when it satisfies (84), (85), (86), (87), and

$$\begin{aligned}&-\partial_{\rho\sigma} X_{\bar{i}}(b)^\mu z^a_\mu - \partial_\rho X_{\bar{i}}(b)^\mu z^a_{\mu\sigma} - \partial_\sigma X_{\bar{i}}(b)^\mu z^a_{\mu\rho} \\ &\quad + \partial_{\rho\sigma} Z_{\bar{i}}(b)^a{}_c z^c + \partial_\rho Z_{\bar{i}}(b)^a{}_c z^c_\sigma + \partial_\sigma Z_{\bar{i}}(b)^a{}_c z^c_\rho + Z_{\bar{i}}(b)^a{}_c z^c_{\rho\sigma} = 0 \ ,\end{aligned} \tag{88}$$

$$\begin{aligned}&-\partial_{\rho\sigma} X_{\underline{i}}(b)^\mu z^a_\mu - \partial_\rho X_{\underline{i}}(b)^\mu z^a_{\mu\sigma} - \partial_\sigma X_{\underline{i}}(b)^\mu z^a_{\mu\rho} - X_{\underline{i}}(b)^{\underline{\mu}} z^a_{\underline{\mu}\rho\sigma} \\ &\quad + \partial_{\rho\sigma} Z_{\underline{i}}(b)^a{}_c z^c + \partial_\rho Z_{\underline{i}}(b)^a{}_c z^c_\sigma + \partial_\sigma Z_{\underline{i}}(b)^a{}_c z^c_\rho + Z_{\underline{i}}(b)^a{}_c z^c_{\rho\sigma} = 0 \ .\end{aligned} \tag{89}$$

Again, (88) does not contain any highest-order derivatives, while (89) contains all non-internal highest-order derivatives $z^a_{\underline{\mu}\rho\sigma}$ (and no other highest-order derivatives). Since all non-internal highest-order derivatives $z^a_{\underline{\mu}\rho\sigma}$ can be expressed from (89) due to (82), we obtain $\dim R^3_b - \dim R^2_b \leq \binom{\widetilde{n}+3-1}{3}\operatorname{rank}\operatorname{st}\xi$.

Continuing in this manner, we find $\dim R^l_b - \dim R^{l-1}_b \leq \binom{\widetilde{n}+l-1}{l}\operatorname{rank}\operatorname{st}\xi$, which, together with (5), yields

$$\dim R^l_b \leq \binom{\widetilde{n}+l}{l}\operatorname{rank}\operatorname{st}\xi = \operatorname{rank} J^l_{\widetilde{U}_\alpha}(\xi_\alpha) \ . \tag{90}$$

On the other hand, if $\psi \in C^\infty(\xi)^G$, then its l-jet at b, $J^l(\psi)_b$, belongs to R^l_b because $L(\psi) = 0$ and, therefore, $P^{l-1}(L)(\psi) = 0$:

$$\{J^l(\psi)_b \,:\, \psi \in C^\infty(\xi)^G\} \subseteq R^l_b \qquad \text{for any } b \in \widetilde{U}_\alpha \ . \tag{91}$$

The bijective correspondence $\theta : C^\infty(\xi^G) \to C^\infty(\xi)^G$ (45) implies a bijective correspondence

$$\{J^l(\rho)_b \,:\, \rho \in C^\infty(\xi_\alpha)\} \longrightarrow \{J^l(\psi)_b \,:\, \psi \in C^\infty(\xi)^G\} \ , \qquad b \in \widetilde{U}_\alpha \tag{92}$$

– the function $\psi \in C^\infty(\xi)^G$ corresponding to $\rho \in C^\infty(\xi^G)$ is the extension of ρ to a neighborhood of $\widetilde{U}_\alpha$ by the action of G. Therefore the dimension of R^l_b is no smaller than the dimension of the two linear spaces in (92), i.e.,

$$\dim R^l_b \geq \operatorname{rank} J^l_{\widetilde{U}_\alpha}(\xi_\alpha) = \binom{\widetilde{n}+l}{l} \operatorname{rank} \operatorname{st}\xi \ . \tag{93}$$

From the opposite inequalities (90) and (93) it is clear that R^l_b and the two linear spaces in (92) have the same dimension. This fact together with (91) also imply that

$$R^l_b = \{J^l(\psi)_b \,:\, \psi \in C^\infty(\xi)^G\} \subseteq J^k(\operatorname{st}\xi)_b \ , \qquad b \in \widetilde{U}_\alpha \ .$$

This allows us to define the map $\Sigma^k_\alpha : J^k_{\widetilde{U}_\alpha}(\xi_\alpha) \to J^k(\operatorname{st}\xi)_{\widetilde{U}_\alpha}$ in a way that satisfies all required properties, namely

$$\Sigma^k_\alpha\big(J^k_{\widetilde{U}_\alpha}(\rho)_b\big) := J^k(\psi)_b \,\in\, R^k_b \,\subseteq\, J^k(\operatorname{st}\xi)_b \ , \tag{94}$$

where ψ is the extension of ρ by G-invariance.

We summarize the above reasoning in the following two theorems.

Theorem 13. *The subbundle R^k_α (70) is transversal to I^k_α (61) in $J^k(\operatorname{st}\xi)_{\widetilde{U}_\alpha}$: $J^k(\operatorname{st}\xi)_{\widetilde{U}_\alpha} = I^k_\alpha \oplus R^k_\alpha$. Therefore (71) defines a morphism Σ^k_α (94) splitting the horizontal short exact sequence in (62).*

Theorem 14. *Under the assumptions for existence of reduced bundles, the Lie derivative $L \in \operatorname{LDiff}_1(\xi, \mathfrak{g}^* \otimes \xi)$ (and its restriction $L \in \operatorname{LDiff}_1(\operatorname{st}\xi, \mathfrak{g}^* \otimes \operatorname{st}\xi)$) is formally integrable.*

The morphisms Σ^k_α determine the total symbols of the coordinate representations D_α (58) of the reduced DO D^G (57) as in (33): $\widetilde{D_\alpha} = \widetilde{D}\restriction_{\widetilde{U}_\alpha} \circ \Sigma^k_\alpha : J^k_{\widetilde{U}_\alpha}(\xi_\alpha) \to \eta_\alpha$.

Remark 15. It is clear that the order of D^G cannot exceed the order of the original DO D. This is in contrast with the process of restricting a DO to a submanifold by using an auxiliary DO, as explained in Remark 6.

Remark 16. As noted in the beginning of Sect. 2.4, Theorem 14 is a far-reaching generalization of Theorem 3. Clearly, when ξ is the trivial bundle $B \times \mathbb{R} \to B$ and G acts trivially on the fibers, the formal integrability of the Lie derivative follows directly from the involutivity of the fundamental vector fields of the action t of G on B.

4.4. *Dimensional reduction of group action and invariant DOs*

If there are additional geometric structures in the vector bundles that are compatible with the action of the group G, they induce analogous structures in the reduced bundles. An important example of this is the reduction of a group action and the preservation of this action in the process of reduction. Namely, if another group K

acts on the G-reducible bundles ξ and η, and the actions of G and K commute, then after the G-reduction, the group K still acts on the sections of ξ^G and η^G, and these actions can naturally be computed from the original actions of K. The situation is similar for a DO D that is invariant with respect to the actions of both G and K – after the reduction, the reduced DO D^G is invariant with respect to the reduced action of K on ξ^G and η^G. In [1, Sect. 2.5] we treated this problem for invariant sections; below we discuss it briefly for invariant DOs.

Let G and K be connected Lie groups acting through vector bundle morphisms on the vector bundles ξ and η over the same base B. For $g \in G$ and $k \in K$ we denote these actions by $\mathcal{T}^\xi_g = (t_g, T^\xi_g)$, $\mathcal{T}^\eta_g = (t_g, T^\eta_g)$, $\mathcal{F}^\xi_k = (f_k, F^\xi_k)$, $\mathcal{F}^\eta_k = (f_k, F^\eta_k)$; note that the actions of $\mathcal{T}^\xi$ and $\mathcal{T}^\eta$ on B are the same (namely, t), similarly for $\mathcal{F}^\xi$ and $\mathcal{F}^\eta$. Let the actions of G and K commute:

$$\mathcal{T}^\xi_g \circ \mathcal{F}^\xi_k = \mathcal{F}^\xi_k \circ \mathcal{T}^\xi_g \ , \quad \mathcal{T}^\eta_g \circ \mathcal{F}^\eta_k = \mathcal{F}^\eta_k \circ \mathcal{T}^\eta_g \qquad \text{for all } g \in G\,, k \in K \ . \tag{95}$$

These actions induce actions of G and K on $C^\infty(\xi)$ and $C^\infty(\eta)$ as in (53) and

$$\begin{aligned} k^\xi : C^\infty(\xi) \to C^\infty(\xi) : \psi \mapsto k^\xi(\psi) \ &:= \ F^\xi_k \circ \psi \circ f_k^{-1} \ , \\ k^\eta : C^\infty(\eta) \to C^\infty(\eta) : \chi \mapsto k^\eta(\chi) \ &:= \ F^\eta_k \circ \chi \circ f_k^{-1} \ . \end{aligned} \tag{96}$$

We define the G-reduced actions, $\mathcal{F}^{\xi,G} = \left(f^G, F^{\xi,G}\right)$ and $\mathcal{F}^{\eta,G}$, through the actions on the coordinate realizations $\{\xi_\alpha\}_{\alpha\in\mathscr{A}}$ and $\{\eta_\alpha\}_{\alpha\in\mathscr{A}}$ of ξ^G and η^G. Let $k \in K$, $b \in \widetilde{U}_\alpha$, and $\sigma \in (\xi_\alpha)_b = \operatorname{st}\xi_b$. If we use some local coordinates $\widetilde{x}$ in $\widetilde{U}_\alpha$ as coordinates in base of the reduced bundle, then $b = i_{\widetilde{U}_\alpha}(\widetilde{x})$, where $i_{\widetilde{U}_\alpha} : \widetilde{U}_\alpha \hookrightarrow B$ is the natural embedding. Generally, $f_k(b)$ does not belong to $\widetilde{U}_\alpha$, so let $g(k,\widetilde{x}) \in G$ be such that $t_{g(k,\widetilde{x})} \circ f_k(b) \in \widetilde{U}_\alpha$, and define the G-reduced action of K on ξ^G by

$$\begin{aligned} f^G_k(\widetilde{x}) &:= t_{g(k,\widetilde{x})} \circ f_k \circ i_{\widetilde{U}_\alpha}(\widetilde{x}) \in \widetilde{U}_\alpha \ , \\ F^{\xi,G}_k(\sigma) &:= T^\xi_{g(k,\widetilde{x})} \circ F^\xi_k(\sigma) \in (\xi_\alpha)_{f^G_k(\widetilde{x})} \ ; \end{aligned} \tag{97}$$

similarly for the G-reduced action of K on η^G. It is easy to show that the non-uniqueness in the choice of $g(k,\widetilde{x})$ is immaterial. The action $k^{\xi,G}$ of K on the reduced section $\psi^G \in C^\infty(\xi^G)$ is then given by

$$k^{\xi,G}(\psi^G)(\widetilde{x}) := F^{\xi,G}_k \circ \psi^G \circ (f^G_k)^{-1}(i_{\widetilde{U}_\alpha}(\widetilde{x})) \ . \tag{98}$$

Note that in the right-hand side of this expression the value of $F^{\xi,G}$ must be evaluated at the point $(f^G_k)^{-1}(\widetilde{x}) = (t_{g(k,\widetilde{x})} \circ f_k)^{-1}(i_{\widetilde{U}_\alpha}(\widetilde{x}))$ (as is done in (106) in Example 17 below).

It can be easily checked that concrete expression (98) for the action $k^{\xi,G}$ can be expressed in a more abstract way in terms of the bijection θ^ξ (56) as

$$k^{\xi,G} : C^\infty(\xi^G) \to C^\infty(\xi^G) : \psi^G \mapsto k^{\xi,G}(\psi^G) := (\theta^\xi)^{-1} \circ k^\xi \circ \theta^\xi \circ \psi^G \ ; \tag{99}$$

similarly for $k^{\eta,G}$.

We define the actions of G and K on $\mathrm{Diff}_l(\xi,\eta)$ as in (54) and

$$k : \mathrm{Diff}_l(\xi,\eta) \to \mathrm{Diff}_l(\xi,\eta) : D \mapsto k(D) \ := \ k^\eta \circ D \circ (k^\xi)^{-1} \ . \tag{100}$$

Let $D \in \mathrm{Diff}_l(\xi, \eta)$ be simultaneous G- and K-invariant. Then, thanks to the commutativity (95), the reduced DO $D^G \in \mathrm{Diff}_l(\xi^G, \eta^G)$ will be invariant with respect to the reduced action

$$\begin{aligned} k^G &: \mathrm{Diff}_l(\xi^G, \eta^G) \to \mathrm{Diff}_l(\xi^G, \eta^G) \\ &: D^G \mapsto k^G(D^G) := k^{\eta,G} \circ D^G \circ (k^{\xi,G})^{-1} \ . \end{aligned} \tag{101}$$

Example 17. This example is a continuation of Examples 4 and 12 and uses the notations introduced there. Consider the local action of $K = SO_0(2)$ on the base $B = \mathbb{R}^2_+$ of the base of the vector bundles ξ and η, and let the actions $\mathcal{F}^\xi$ and $\mathcal{F}^\eta$ be given by

$$\mathcal{F}^\xi_k\big((x^1, x^2), z\big) = \mathcal{F}^\eta_k\big((x^1, x^2), z\big) := \big(f_k(x^1, x^2), z\big) \ , \tag{102}$$

where $f_k(x^1, x^2)$ is the rotation by k radians (for small enough k):

$$f_k(x^1, x^2) = (x^1 \cos k - x^2 \sin k, x^1 \sin k + x^2 \cos k) \ . \tag{103}$$

Since the action of $SO_0(2)$ on the fibers of ξ is trivial, the action of $SO_0(2)$ on $C^\infty(\xi)$ is

$$\begin{aligned} k^\xi(\psi)(x^1, x^2) &= F^\xi_k \circ \psi \circ (f_k)^{-1}(x^1, x^2) \\ &= \psi(x^1 \cos k + x^2 \sin k, -x^1 \sin k + x^2 \cos k) \ , \end{aligned} \tag{104}$$

so $C^\infty(\xi)^{SO_0(2)}$ consists of all functions whose value depends only on the distance to the origin in $\mathbb{R}^2$. The explicit calculation of the action (100) of $SO_0(2)$ on the Laplacian D,

$$\begin{aligned} k(D)(\psi)(x^1, x^2) &= k^\eta \circ D \circ (k^\xi)^{-1} \circ \psi(x^1, x^2) \\ &= F^\eta_k \circ \big[D\big((F^\xi_k)^{-1} \circ \psi \circ f_k\big)\big] \circ f_k^{-1}(x^1, x^2) = \big[D(\psi \circ f_k)\big]\left(f_k^{-1}(x^1, x^2)\right) \\ &= \left[\left(\tfrac{\partial^2}{\partial (y^1)^2} + \tfrac{\partial^2}{\partial (y^2)^2}\right)\psi(y^1 \cos k + y^2 \sin k, -y^1 \sin k + y^2 \cos k)\right]\Big|_{(y^1,y^2) = f_k^{-1}(x^1,x^2)} \\ &= \left(\tfrac{\partial^2}{\partial (x^1)^2} + \tfrac{\partial^2}{\partial (x^2)^2}\right)\psi(x^1, x^2) = D\psi(x^1, x^2) \ , \end{aligned}$$

confirms the well-known fact that the Laplacian is rotationally invariant.

It is easy to check that the actions $\mathcal{T}^\xi$ and $\mathcal{F}^\xi$ of $\mathbb{R}_+$ and $SO_0(2)$ on ξ commute; the same holds for $\mathcal{T}^\eta$ and $\mathcal{F}^\eta$. This implies that the actions of $\mathbb{R}_+$ and $SO_0(2)$ on the sections (recall (53) and (96)) commute as well. Therefore, after $\mathbb{R}_+$-reduction, the sections of $\xi^{\mathbb{R}_+}$ and $\eta^{\mathbb{R}_+}$ will be invariant with respect to the reduced actions of $SO_0(2)$. To compute the reduced actions, we parameterize $\widetilde{U}$ by $\widetilde{x} \in \mathbb{R}$ by $i_{\widetilde{U}}(\widetilde{x}) = (\widetilde{x}, 1)$. Following the recipe above, for a given $k \in SO_0(2)$, we find an element of $g(k, \widetilde{x}) \in \mathbb{R}_+$ such that $t_{g(k,\widetilde{x})} \circ f_k(i_{\widetilde{U}}(\widetilde{x})) \in \widetilde{U}$. In this particular example $g(k, \widetilde{x}) = (\widetilde{x} \sin k + \cos k)^{-1}$, so (74), (102), and (103) give the following expression for the reduced action $f^{\mathbb{R}_+}$ of $SO_0(2)$ on $\widetilde{U}$:

$$f_k^{\mathbb{R}_+}(\widetilde{x}) = \frac{\widetilde{x} \cos k - \sin k}{\widetilde{x} \sin k + \cos k} \ . \tag{105}$$

By (97) and (98), the reduced action of $SO_0(2)$ on $C^\infty(\xi^{\mathbb{R}_+})$ is

$$\begin{aligned} k^{\xi,\mathbb{R}_+}(\psi^{\mathbb{R}_+})(\widetilde{x}) &= F_k^{\xi,\mathbb{R}_+} \circ \psi^{\mathbb{R}_+} \circ (f_k^{\mathbb{R}_+})^{-1}(\widetilde{x}) \\ &= g\big(k,\widetilde{y}\big)^2\Big|_{\widetilde{y}=(f_k^{\mathbb{R}_+})^{-1}(\widetilde{x})}\ \psi^{\mathbb{R}_+}\Big((f_k^{\mathbb{R}_+})^{-1}(\widetilde{x})\Big) \\ &= \frac{1}{(\widetilde{y}\sin k+\cos k)^2}\Bigg|_{\widetilde{y}=(f_k^{\mathbb{R}_+})^{-1}(\widetilde{x})}\ \psi^{\mathbb{R}_+}\bigg(\frac{\widetilde{x}\cos k+\sin k}{-\widetilde{x}\sin k+\cos k}\bigg) \\ &= (-\widetilde{x}\sin k+\cos k)^2\,\psi^{\mathbb{R}_+}\bigg(\frac{\widetilde{x}\cos k+\sin k}{-\widetilde{x}\sin k+\cos k}\bigg)\ . \end{aligned} \tag{106}$$

We use this expression and the explicit form (78) of $D^{\mathbb{R}_+}$ to calculate

$$\begin{aligned} &D^{\mathbb{R}_+}\big[(k^{\xi,\mathbb{R}_+})^{-1}(\psi^{\mathbb{R}_+})\big](\widetilde{x}) \\ &= \bigg[(1+\widetilde{x}^2)\frac{\mathrm{d}^2}{\mathrm{d}\widetilde{x}^2} - 2\widetilde{x}\frac{\mathrm{d}}{\mathrm{d}\widetilde{x}} + 2\bigg]\bigg[(\widetilde{x}\sin k+\cos k)^2\,\psi^{\mathbb{R}_+}\bigg(\frac{\widetilde{x}\cos k-\sin k}{\widetilde{x}\sin k+\cos k}\bigg)\bigg] \\ &= \frac{1+\widetilde{x}^2}{(\widetilde{x}\sin k+\cos k)^2}\,(\psi^{\mathbb{R}_+})''\bigg(\frac{\widetilde{x}\cos k-\sin k}{\widetilde{x}\sin k+\cos k}\bigg) \\ &\quad - \frac{2\widetilde{x}\cos 2k+(\widetilde{x}^2-1)\sin 2k}{(\widetilde{x}\sin k+\cos k)^2}\,(\psi^{\mathbb{R}_+})'\bigg(\frac{\widetilde{x}\cos k-\sin k}{\widetilde{x}\sin k+\cos k}\bigg) + 2\psi^{\mathbb{R}_+}\bigg(\frac{\widetilde{x}\cos k-\sin k}{\widetilde{x}\sin k+\cos k}\bigg)\ . \end{aligned}$$

Finally, (101), (102), (105), and the above calculation yield the following expression for the action of $SO_0(2)$ on the reduced operator $D^{\mathbb{R}_+}$:

$$\begin{aligned} &\big[k^{\mathbb{R}_+}(D^{\mathbb{R}_+})(\psi^{\mathbb{R}_+})\big](\widetilde{x}) = \big[(k^{\eta,\mathbb{R}_+}\circ D^{\mathbb{R}_+}\circ(k^{\xi,\mathbb{R}_+})^{-1}\circ(\psi^{\mathbb{R}_+})\big](\widetilde{x}) \\ &\quad = \big\{D^{\mathbb{R}_+}\big[(k^{\xi,\mathbb{R}_+})^{-1}(\psi^{\mathbb{R}_+})\big]\big\}\circ(f^{\mathbb{R}_+})_k^{-1}(\widetilde{x}) \\ &\quad = \text{the right-hand side of the previous equation with } \widetilde{x} \text{ replaced by } (f^{\mathbb{R}_+})_k^{-1}(\widetilde{x}) \\ &\quad = \bigg[(1+\widetilde{x}^2)\frac{\mathrm{d}^2}{\mathrm{d}\widetilde{x}^2} - 2\widetilde{x}\frac{\mathrm{d}}{\mathrm{d}\widetilde{x}} + 2\bigg]\psi^{\mathbb{R}_+}(\widetilde{x}) = D^{\mathbb{R}_+}\psi^{\mathbb{R}_+}(\widetilde{x})\ . \end{aligned}$$

Not surprisingly, the $SO_0(2)$-invariant Laplace operator D resulted in a reduced operator $D^{\mathbb{R}_+}$ that is invariant with respect to the reduced action of $SO_0(2)$.

Remark 18. Dimensional reduction can be used to construct DOs invariant with respect to a complicated group action. We have employed this idea in [2] to construct DOs invariant with respect to the (nonlinear) action of the connected component of the conformal group $C_0(1,3)$ on Minkowski space $\mathbb{R}^{1,3}$. The idea, due to Dirac [11], is that $C_0(1,3)$ is locally isomorphic to the orthogonal group $O(2,4)$ in $\mathbb{R}^{2,4}$. It is easy to construct $O_0(2,4)$-invariant DOs acting on fields defined on $\mathbb{R}^{2,4}$. The action on $\mathbb{R}^{2,4}$ of the multiplicative group $\mathbb{R}^*$ of nonzero real numbers commutes with the action of $O_0(2,4)$. Starting with Maxwell's equations on $\mathbb{R}^{2,4}$, which are naturally $O_0(2,4)$- and $\mathbb{R}^*$-invariant, we reduce them to $O_0(2,4)^{\mathbb{R}^*}$-invariant equations on the projectization $\mathbb{RP}^5$ of $\mathbb{R}^{2,4}$. Minkowski space $\mathbb{R}^{1,3}$ is realized as the projectivized light cone $Q\mathbb{P}^5$ in $\mathbb{R}^{2,4}$. The $\mathbb{R}^*$-reduction is standard, while the restriction to $Q\mathbb{P}^5$ is related to restriction of DOs, discussed in Sect. 3. Using these techniques, we were able to reproduce in a systematic way many results derived previously in the literature. In [2] the reader can find many details (invariant subbundles, reduced gauge transformations, "universal" splitting relations for this construction, etc.) illustrating many of the issues discussed in the present paper. For some recent developments see, e.g., [12,13].

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