Dimensional Reduction of Invariant Fields and Differential Operators.

II. Reduction of Invariant Differential Operators

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Abstract. In the present paper, which is a sequel of [1], we consider the dimensional reduction of differential operators (DOs) that are invariant with respect to the action of a connected Lie group $G$. The action of $G$ on vector bundles induces naturally actions of $G$ on their sections and on the DOs between them. In [1] we constructed explicitly the reduced bundle $\xi^G$, such that the set of all its sections, $C^\infty(\xi^G)$, is in a bijective correspondence with the set $C^\infty(\xi)$ of all $G$-invariant sections of the original vector bundle $\xi$. The main goal of the present paper is, given a $G$-invariant DO $D : C^\infty(\xi) \to C^\infty(\eta)$ to construct the reduced DO $D^G : C^\infty(\xi^G) \to C^\infty(\eta^G)$. Our construction of $D^G$ uses the geometrically natural language of jet bundles which best reveals the geometry of the DOs and reduces the manipulations with DOs to simple algebraic operations. Since $\xi^G$ was constructed in [1] by restricting a certain bundle to a submanifold of its base, an essential ingredient of the dimensional reduction of a DO is the restriction of the DO to a submanifold of the base. To perform such a restriction, one has to find splittings of certain short exact sequences of jet bundles, which in practice can be achieved by choosing an appropriate auxiliary DO – a highly non-trivial procedure involving arbitrary choices. However, in the case of a $G$-invariant DO $D$, this splitting is provided automatically by the $G$-invariance of $D$ (this uses the Lie derivative of the action of $G$). Certain properties of the DOs – in particular, their formal integrability – turn out to be crucial in our construction. We discuss this in detail and give an explicit example showing what can go wrong if one uses an auxiliary DO that is not formally integrable. Finally, we discuss the structure of the set of all $G$-invariant DOs that lead to the same reduced DO.

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1. Introduction

In this paper we continue the presentation of the method for dimensional reduction initiated in [1]. In the present paper we consider the reduction of differential operators invariant with respect to the actions of a connected (but not necessarily compact) Lie group on vector bundles.

The natural setting for considering differential operators on vector bundles is the language of jet bundles. It interprets a differential operator as a purely algebraic object – the total symbol of the differential operator – which is a fiber-preserving mapping between jet bundles. This is convenient for treating the symmetry properties of differential operators and their dimensional reduction.

We briefly recall the set-up described in [1]. Let $\xi$ and $\eta$ be vector bundles over the same base $B$, and $G$ be a a Lie group acting on $\xi$ and $\eta$ through vector bundle morphisms, with the same projected action on the common base $B$. This naturally induces an action of $G$ on the sections $C^\infty(\xi)$ and $C^\infty(\eta)$. In [1] we construct the reduced bundles $\xi^G$ and $\eta^G$, both over $B/G$, such that the set $C^\infty(\xi^G)$ of all sections of $\xi^G$ is in a bijective correspondence with the set $C^\infty(\xi)^G$ of all $G$-invariant sections of $\xi$ (and similarly for $\eta$).

Let $D : C^\infty(\xi) \to C^\infty(\eta)$ be an order-$k$ differential operator. The actions of $G$ on $C^\infty(\xi)$ and $C^\infty(\eta)$ induce naturally an action of $G$ on the set $\text{Diff}_k(\xi, \eta)$ of all order-$k$ differential operators from $\xi$ to $\eta$. Each $G$-invariant differential operator $D$ maps $C^\infty(\xi)^G$ to $C^\infty(\eta)^G$ and, hence, determines a reduced differential operator $D^G : C^\infty(\xi^G) \to C^\infty(\eta^G)$ between the sections of the reduced bundles.

Since the reduced bundles are constructed explicitly in [1] by restriction of the base $B$ to a submanifold, the construction of the reduced operator $D^G$ is naturally related to the process of restricting a differential operator to a submanifold of $B$. In general, this restriction is not a well-defined process in the sense that it requires additional information. This missing information can be supplied by considering an appropriately chosen auxiliary differential operator. For the restriction procedure to be self-consistent, the auxiliary operator must satisfy a condition called formal integrability.

If we consider the reduction of a $G$-invariant differential operator $D$, the auxiliary operator comes naturally from the Lie derivative of the action of $G$ (the infinitesimal symmetries). We give an explicit algorithm for reduction which involves only computing derivatives and solving a linear system of algebraic equations. We prove the formal integrability of the Lie derivative, which provides a theoretical justification of our algorithm.

The algorithm for computing the reduced differential operator $D^G$ elucidates the geometric structures arising naturally in the process of reduction. These issues are important, for example, in Kaluza-Klein theories and in model building with a desired symmetry in elementary particle physics. An example of this approach is our construction [2] of differential operators on Minkowski space that are invariant with respect to the (nonlinear) action
of the conformal group, starting from the (linear) action of the orthogonal
group on a bigger space.

Invariant linear differential operators have invariant total symbols. These
total symbols can be considered as invariant sections in appropriate vector
bundles. According to our general methodology, an invariant section corre-
sponds to a section in the reduced bundle. Therefore, the problem of descrip-
tion of all invariant linear differential operators can be restated as a problem
of description of certain reduced bundles. We utilize this approach to study
the relation between the set of all invariant linear order-1 differential oper-
ators from \( \xi \) to \( \eta \) and all linear order-1 differential operators between the
reduced bundles \( \xi^G \) and \( \eta^G \).

The plan of the paper is the following. In Sect. 2 we introduce the jet
bundle language for description of differential operators, paying special at-
tention to the concept of formal integrability (Sections 2.3 and 2.4). Sect. 3
is devoted to the problem of restriction of a differential operator to a sub-
manifold. In Sect. 4 we develop an algorithm for reduction of an invariant
differential operator, and resolve all related theoretical issues. In Sect. 5 we
discuss the relation between the set of invariant differential operators and the
differential operators in the reduced bundles.

Although this paper is a sequel to [1], the reader does not need to have
read all of [1]. In the present paper, we use mostly the material of Sections
2.1–2.3 of [1], and briefly recall the basic notions of [1] as needed.

2. Differential operators on vector bundles

Here we introduce the coordinate-free definition of differential operators, fol-
lowing [3, Chapter IV].

2.1. Jet bundles

Let \( \xi = (E, \pi, B) \) and \( \eta \) be finite-dimensional \( \mathbb{K} \)-vector bundles over the real
finite-dimensional manifold \( B \); the standard fiber \( \xi_b := \pi^{-1}(b) \) of \( \xi \) over \( b \in B \)
is a vector space over the field \( \mathbb{K} = \mathbb{R} \) or \( \mathbb{C} \). As usual, \( C^\infty(B) \) stands for the
ring of all smooth \( \mathbb{K} \)-valued functions on \( B \), and \( C^\infty(\xi) \) denotes the set of
all smooth sections of \( \xi \) (which is a module over \( C^\infty(B) \)). Throughout the
paper it is always assumed that the manifolds, vector bundles, functions and
sections of bundles are smooth (\( C^\infty \)).

Let \( I_b(B) \) be the ideal of the ring \( C^\infty(B) \) consisting of all functions
\( f \in C^\infty(B) \) that vanish at the point \( b \in B \):

\[
I_b(B) := \{ f \in C^\infty(B) : f(b) = 0 \} \subseteq C^\infty(B) .
\]

Let \( I^k_b(B) \) stand for the ideal of the ring \( C^\infty(B) \) consisting of all functions on
\( B \) that can be represented as a product of \( k \) functions from \( I_b(B) \).

Let \( Z^k_b(B) \) stand for the subspace of \( C^\infty(\xi) \) consisting of all sections
of \( \xi \) that can be represented as products of a function from \( I^{k+1}_b(B) \) and a
section of \( \xi \):

\[
Z_b^k(\xi) := \{ \psi \in C^\infty(\xi) : \psi = f \cdot \kappa, \text{ where } f \in I_b^{k+1}(B), \kappa \in C^\infty(\xi) \} .
\]

Clearly, all partial derivatives of \( \psi \in Z_b^k(\xi) \) (with respect to the coordinates in the base \( B \)) up to order \( k \) vanish at \( b \).

For each \( b \in B \) define \( J^k(\xi)_b := C^\infty(\xi)/Z_b^k(\xi) \), and let \( J^k(\psi)_b \) be the image of \( \psi \) under the canonical projection \( C^\infty(\xi) \to J^k(\xi)_b : \psi \mapsto J^k(\psi)_b \). The \( k \)-jet of the section \( \psi \in C^\infty(\xi) \) is defined by \( J^k(\psi)(b) := J^k(\psi)_b \) for any \( b \in B \). The \( k \)-jet \( J^k(\psi)_b \) is the coordinate-free concept for the section (“the field”) \( \psi \) and its derivatives up to order \( k \) at \( b \).

Let \( J^k(\xi) \) stand for the union \( \bigcup_{b \in B} J^k(\xi)_b \) endowed with the natural vector bundle structure. In more detail, this means the following. Let \((x^\mu, z^a)\) be local coordinates in \( \xi \), where \( \mu = 1, \ldots, n \) (with \( n = \dim B \)), \( a = 1, \ldots, \dim \xi \) (by definition, \( \dim \xi \) is the dimension of the standard fiber of \( \xi \)). They generate local coordinates in the jet bundle \( J^k(\xi)_b \),

\[
(x^\mu, z^a, z^{a_1}_\mu, z^{a_2}_{\mu_1\mu_2}, \ldots, z^{a_k}_{\mu_1\cdots\mu_k}) , \quad \left\{ \begin{array}{l}
1 \leq \mu_1 \leq \cdots \leq \mu_i \leq n , \ i = 1, \ldots, k \\
a = 1, \ldots, \dim \xi 
\end{array} \right.
\]

so that the coordinates of \( J^k(\psi)(b) \) are

\[
\left( J^k(\psi)(b) \right)^a_{\mu_1\cdots\mu_i} := \partial_{\mu_1\cdots\mu_i} \psi^a(b) ,
\]

where

\[
\partial_{\mu_1\cdots\mu_i} \psi^a(b) := \frac{\partial^i \psi^a}{\partial x^{\mu_1} \cdots \partial x^{\mu_i}} (b) .
\]

The transition functions gluing \( J^k(\xi) \) come from the standard formulae for transformation of partial derivatives under a change of variables. From the definition, \( J^0(\xi) = \xi \). The number of partial derivatives of order \( i \) of each component \( \psi^a \) is equal to \( \binom{n+i-1}{i} \), which is the number of unordered selections of \( i \) objects, with repetition allowed, out of \( n \) distinct objects. The dimension of the fibers of the vector bundle \( J^k(\xi) \) is, therefore,

\[
\dim J^k(\xi) = \sum_{i=0}^{k} \binom{n+i-1}{i} \dim \xi = \binom{n+k}{k} \dim \xi .
\]

For integers \( k \) and \( l \) satisfying \( k \geq l \geq 0 \), let

\[
\pi^{k,l} : J^k(\xi) \to J^l(\xi)
\]

stand for the natural projections (“cutting off” all derivatives of order \( l + 1, \ldots, k \)).

Let \( S^k(\tau^*)_b(B) \) be the \( k \)th symmetrized tensor product of the cotangent bundle of \( B \). There exists a natural morphism over the identity in \( B \)

\[
i : S^k(\tau^*)_b(B) \otimes \xi \to J^k(\xi) .
\]

The morphism \( i \) is defined as follows: if \( \omega \in T^*_b(B) \), and \( e \in \xi_b \), then

\[
i(S^k \omega \otimes e) = J^k \left( \frac{1}{k!} (f - f(b))^k \psi \right) (b) ,
\]
where \( f \in C^\infty(B) \) and \( \psi \in C^\infty(\xi) \) are such that \( \omega = df_b \) and \( e = \psi(b) \). In local coordinates \((x^\mu, z^a)\) of \( \xi \), the coordinates (2.2) of \( i(S^k \omega \otimes e) \) are

\[
(i(S^k \omega \otimes e))^a\mu_1 \cdots \mu_j = \begin{cases} 0 & \text{if } j < k , \\ \omega_{\mu_1} \cdots \omega_{\mu_k} e^a & \text{if } j = k . \end{cases}
\]

A basic fact in theory of jet bundles [3, Sect. IV.2] is the exactness of the following sequence:

\[
0 \longrightarrow S^k T^* (B) \otimes \xi \overset{i}{\longrightarrow} J^k(\xi) \overset{\pi_{k,k-1}}{\longrightarrow} J^{k-1}(\xi) \longrightarrow 0 .
\]

(2.8)

This sequence is related to the fact that all coordinate-free (i.e., geometrically natural) relations between derivatives come from the sequence

\[
J^k(\xi) \overset{\pi_{k,k-1}}{\longrightarrow} J^{k-1}(\xi) \overset{\pi_{k-1,k-2}}{\longrightarrow} \cdots \overset{\pi_{2,1}}{\longrightarrow} J^1(\xi) \overset{\pi_{1,0}}{\longrightarrow} J^0(\xi) .
\]

(2.9)

In particular, (2.8) tells us that the highest-order derivatives of a section \( \psi \in C^\infty(\xi) \) are distinguished in a geometrically natural way only if \( J^{k-1}(\psi)(b) = 0 \), and the subbundle \( \ker \pi_{k,k-1} \subset J^k(\xi) \) of the highest-order derivatives is isomorphic to \( S^k T^* (B) \otimes \xi \).

### 2.2 Differential operators and their symbols

#### 2.2.1. Notations and natural isomorphisms

We start by introducing some useful notations. Let \( \mathbb{F} \) be a subfield of the field \( \mathbb{K} \) (where \( \mathbb{K} = \mathbb{R} \) or \( \mathbb{C} \)), and \( B \) be a real finite dimensional manifold. Let \( \gamma \) be an \( \mathbb{F}\)-vector bundle over \( B \), and \( \zeta \) be a \( \mathbb{K}\)-vector bundle over \( B \).

A vector bundle morphism \( F \) from \( \gamma \) to \( \zeta \) over the identity in \( B \) is a mapping from \( \gamma \) to \( \zeta \) whose restriction \( F|_{\gamma_b} \) to a fiber \( \gamma_b \) is an \( \mathbb{F}\)-linear mapping from \( \gamma_b \) to \( \zeta_b \). Denote by \( \text{Hom}(\gamma, \zeta) \) the set of all such morphisms; \( \text{Hom}(\gamma, \zeta) \) is naturally endowed with a structure of a \( \mathbb{K}\)-vector space. In Sections 2–3 – before we consider the action of a Lie group on the vector bundles – all vector bundle morphisms (and, more generally, all fiber-preserving mappings) are over the identity in \( B \).

Let \( L(\gamma, \zeta) \) be the \( \mathbb{K}\)-vector bundle over \( B \) whose fiber \( L(\gamma, \zeta)_b \) over \( b \in B \) consists of all \( \mathbb{F}\)-linear mappings from \( \gamma_b \) to \( \zeta_b \). Let \( L^k(\gamma, \zeta) \) stand for the \( \mathbb{K}\)-vector bundle over \( B \) with \( L^k(\gamma, \zeta)_b \) equal to the set of all \( k\)-linear (with respect to \( \mathbb{F}\)) mappings from \( \gamma_b \) to \( \zeta_b \). Denote by \( L^k_s(\gamma, \zeta) \) the subbundle of \( L^k(\gamma, \zeta) \) with \( L^k_s(\gamma, \zeta)_b \) defined as the set of all symmetric \( k\)-linear (with respect to \( \mathbb{F}\)) mappings from \( \gamma_b \) to \( \zeta_b \).

Each fiber-preserving mapping \( F : \gamma \rightarrow \zeta \) over the identity in \( B \) (in particular, each vector bundle morphism \( F \in \text{Hom}(\gamma, \zeta) \)) induces a mapping \( F_* \) between the sections of \( \gamma \) and \( \zeta \) defined by

\[
F_* : C^\infty(\gamma) \rightarrow C^\infty(\zeta) : \psi \mapsto F_* \psi := F \circ \psi .
\]

(2.10)

Let \( \gamma^* \) stand for the vector bundle dual to \( \gamma \), i.e., each fiber \( (\gamma^*)_b \) is the dual to the \( \mathbb{F}\)-vector space \( \gamma_b \). Denote by \( C^\infty(\gamma^* \otimes \zeta) \) the set of all \( \mathbb{F}\)-linear mappings from \( \gamma^* \) to \( \zeta \) over the identity in \( B \); clearly, \( C^\infty(\gamma^* \otimes \zeta) \) is a \( \mathbb{K}\)-vector space. If \( F \in \text{Hom}(\gamma, \zeta) \), the mapping \( F_* \) from (2.10) can be considered as
an element of $C^\infty(\gamma^* \otimes \zeta)$, and we obtain that the three $\mathbb{K}$-vector bundles introduced above are naturally isomorphic:

$$\text{Hom}(\gamma, \zeta) \cong C^\infty(\gamma^* \otimes \zeta) \cong C^\infty(L(\gamma, \zeta)) \, .$$ (2.11)

### 2.2.2. A differential operator and its total symbol

Let $\xi$ and $\eta$ be $\mathbb{K}$-vector bundles over a common base $B$. A **differential operator** (DO) of order $k$ from $\xi$ to $\eta$ is a mapping $D : C^\infty(\xi) \to C^\infty(\eta)$ such that, for any $b \in B$, if $\psi_1 \in C^\infty(\xi)$ and $\psi_2 \in C^\infty(\xi)$, then $J^k(\psi_1)(b) = J^k(\psi_2)(b)$ implies $D(\psi_1)(b) = D(\psi_2)(b)$. In other words, $D\psi(b)$ depends only on the values of $\psi$ and its derivatives up to order $k$ at the point $b$. The set of all DOs of order $k$ from $\xi$ to $\eta$ will be denoted by $\text{Diff}_k(\xi, \eta)$. A DO $D \in \text{Diff}_k(\xi, \eta)$ can be identified with a fiber-preserving mapping $\tilde{D}$ from $J^k(\xi)$ to $\eta$ over the identity in $B$: in the notations of (2.10),

$$D = \tilde{D}_* J^k \, .$$ (2.12)

The mapping $\tilde{D}$ is called the **total symbol** of the DO $D$.

A DO $D \in \text{Diff}_k(\xi, \eta)$ is said to be **linear** if it is a $\mathbb{K}$-linear mapping. Let $\text{LDiff}_k(\xi, \eta)$ stand for the subset of $\text{Diff}_k(\xi, \eta)$ that consists of all linear DOs. It is clear from the definition that $J^k \in \text{LDiff}_k(\xi, J^k(\xi))$. A linear DO $D \in \text{LDiff}_k(\xi, \eta)$ can be identified with a vector bundle morphism $\tilde{D} \in \text{Hom}(J^k(\xi), \eta)$ through (2.12). Using (2.11), we can write the following natural isomorphisms:

$$\text{LDiff}_k(\xi, \eta) \cong \text{Hom}(J^k(\xi), \eta) \cong C^\infty(J^k(\xi)^* \otimes \eta) \cong C^\infty(L(J^k(\xi), \eta)) \, .$$ (2.13)

### 2.2.3. Universal properties of $J^k$ and $J^k(\xi)$

This short paragraph discusses some aspects of the algebraic interpretation of jet bundles, and may be skipped at first reading. Since each $D \in \text{Diff}_k(\xi, \eta)$ can be represented as a composition $\tilde{D}_* J^k = \tilde{D} \circ J^k$ as in (2.12), the DO

$$J^k : C^\infty(\xi) \to C^\infty(J^k(\xi)) : \psi \mapsto J^k(\psi)$$

is said to be **universal**. This terminology is consistent with the terminology from category theory (see, e.g., [4, Chapter IV]), which we discuss here in more detail for the case of linear DOs (the same construction works for non-linear DOs, but the terminology there is clumsier). Let $\mathcal{V}(B)$ be the category of all vector bundles $\eta, \zeta, \ldots$ over the manifold $B$; the morphisms in $\mathcal{V}(B)$ are the vector bundle morphisms, i.e., $F \in \text{Hom}(\eta, \zeta)$. Let $\xi \in \mathcal{V}(B)$ be some fixed vector bundle over $B$, and consider the category $\mathcal{LD}_k(\xi)$ of linear DOs of order $k$ from $\xi$ to some other bundle from $\mathcal{V}(B)$. Define the covariant functor $\mathcal{F} = (\mathcal{F}_{\text{Ob}}, \mathcal{F}_{\text{Mor}})$ from $\mathcal{V}(B)$ to $\mathcal{LD}_k(\xi)$ by

$$\mathcal{F}_{\text{Ob}}(\eta) = \text{LDiff}_k(\xi, \eta) \, ,$$

$$\mathcal{F}_{\text{Mor}}(F) = F_* : \text{LDiff}_k(\xi, \eta) \to \text{LDiff}_k(\xi, \zeta) : D \mapsto F_* D \equiv F \circ D \, ,$$
where \( \eta, \zeta \in \mathcal{V}(B) \), \( F \in \text{Hom}(\eta, \zeta) \), and we used the notation (2.10). Let \( J^k(\xi) \in \mathcal{V}(B) \) be the \( k \)-th jet bundle of \( \xi \), and \( J^k \in \text{Hom}(\xi, J^k(\xi)) \) be the operation of constructing the \( k \)-th jet of \( \xi \), defined in Sect. 2.1. The couple \((J^k, J^k(\xi))\) is a universal element for the functor \( \mathcal{F} \) in the following sense [4, Sect. IV.3]: for any \( \eta \in \mathcal{V}(B) \) and any \( D \in \mathcal{F}_{\text{Ob}}(\eta) = \text{LDiff}_k(\xi, \eta) \) there exists a unique morphism \( \tilde{D} \in \text{Hom}(J^k(\xi), \eta) \) such that \( \mathcal{F}_{\text{Mor}}(\tilde{D}) \circ J^k = D \) – this is nothing but the decomposition (2.10) of the linear DO \( D \).

2.2.4. The principal symbol of a linear DO. The definition of a principal symbol of a linear DO introduced below will be used only in Sect. 5. As in Sect. 2.2.1, let \( \mathbb{F} \) be a subfield of the field \( \mathbb{K} \), and \( \gamma \) and \( \zeta \) be respectively \( \mathbb{F} \)-vector bundle and a \( \mathbb{K} \)-vector bundle over the real finite dimensional manifold \( B \). Let \( P^k(\gamma, \zeta) \) stand for the vector bundle over \( B \) whose fiber \( P^k(\xi, \eta)_b \) is the \( \mathbb{K} \)-vector space \( P^k(\gamma_b, \zeta_b) \) of all homogeneous polynomials of degree \( k \) from \( \gamma_b \) to \( \zeta_b \). To be explicit, let \( \dim \gamma = m \), and \( h_1, \ldots, h_m \) be a basis in \( \gamma_b \); similarly, let \( \dim \zeta = d \), and \( e_1, \ldots, e_d \) be a basis in \( \zeta_b \). Then \( P^k(\gamma_b, \zeta_b) \) consists of all mappings of the form

\[
\gamma_b \ni \sum_{i=1}^m v^i h_i \mapsto \sum_j \sum_{a=1}^d P^a_j \xi^a_a \in \zeta_b ,
\]

where the first summation in the right-hand side is over all \( m \)-tuples \( j = (j_1, \ldots, j_m) \) of non-negative integers with \( j_1 + \cdots + j_m = k \), and \( \xi^j := (v^1)^{j_1} \cdots (v^m)^{j_m} \). We leave it to the reader to construct a natural isomorphism \( P^k(\gamma, \zeta) \cong L^k_s(\gamma, \zeta) \).

Let \( \xi \) and \( \eta \) be \( \mathbb{K} \)-vector bundles over \( B \), and \( D \in \text{LDiff}_k(\xi, \eta) \) be a linear DO between them. The principal symbol \( \sigma_k(D) \) of \( D \) is defined as the vector bundle morphism (over the identity in \( B \))

\[
\sigma_k(D) = \tilde{D} \circ i : S^k \tau^*(B) \otimes \xi \to \eta ,
\]

(2.14)

where \( i \) is the mapping from (2.6) and (2.7). If \( \mathbb{K} = \mathbb{R} \), both \( S^k \tau^*(B) \otimes \xi \) and \( \eta \) are \( \mathbb{R} \)-vector bundles, so that the mapping \( \sigma_k(D) \) is \( \mathbb{R} \)-linear. If \( \mathbb{K} = \mathbb{C} \), we assume that \( S^k \tau^*(B) \otimes \xi \) is complexified, so that \( S^k \tau^*(B) \otimes \xi \) is a \( \mathbb{C} \)-vector bundle, and in this case \( \sigma_k(D) \) is \( \mathbb{C} \)-linear. In more detail, if \( \omega \in T^*_b(B) \) and \( e \in \xi_b \), then \( (\omega \otimes \cdots \otimes \omega) \otimes e \in S^k T^*_b(B) \otimes \xi_b \) is the map

\[
T^*_b(B) \otimes \cdots \otimes T^*_b(B) \ni v_1 \otimes \cdots \otimes v_k \mapsto \omega(v_1) \cdots \omega(v_k) e \in \xi_b .
\]

To describe the principal symbols explicitly, one can use the natural isomorphisms (recall (2.11))

\[
\text{Hom}(S^k \tau^*(B) \otimes \xi, \eta) \cong C^\infty(L(S^k \tau^*(B) \otimes \xi, \eta)) \\
\cong C^\infty(L(S^k \tau^*(B), L(\xi, \eta))) \\
\cong C^\infty(L^k_s(\tau^*(B), L(\xi, \eta))) \\
\cong C^\infty(P^k(\tau^*(B), L(\xi, \eta))) .
\]

Thus, \( \sigma_k(D) \) at \( b \in B \) can be considered as a homogeneous polynomial of degree \( k \) defined on \( T^*_b(B) \) and taking values in \( L(\xi, \eta)_b \). To compute
\[ \sigma_k(D)(\omega)(e) \text{ for } \omega \in T^*_b(B), e \in \xi_b \text{ by using (2.7), take } f \in C^\infty(B) \text{ and } \psi \in C^\infty(\xi) \text{ satisfying } \omega = df_b \text{ and } e = \psi(b), \text{ then} \]

\[ \sigma_k(D)(\omega)(e) = D \left( \frac{1}{k!} (f - f(b))^k \psi \right)(b). \]  

(2.15)

Using (2.13) and the short exact sequence (2.8), one can show that

\[ 0 \longrightarrow \text{LDiff}_{k-1}(\xi,\eta) \xrightarrow{\text{emb}} \text{LDiff}_k(\xi,\eta) \xrightarrow{\sigma_k} \text{Hom}(S^k\tau^*(B) \otimes \xi,\eta) \longrightarrow 0 \]

is a short exact sequence [3, Sect. IV.3]. Here “emb” stands for the natural embedding, and \( \sigma_k \) is the mapping from an order-\( k \) linear DO \( D \) to its principal symbol \( \sigma_k(D) \).

### 2.3. Prolongation of a DO and formal integrability

One can differentiate simultaneously both sides of a differential equation \( D\psi = \phi \). In a coordinate-free language, the result of differentiating of a DO is called its prolongation. We will define this concept only for linear DOs (which is the only case that we will use in this paper).

Let \( D = \tilde{D}_*J^k \in \text{LDiff}_k(\xi,\eta) \) be a linear DO, and \( \tilde{D} \in \text{Hom}(J^k(\xi,\eta)) \) be its total symbol. The \( l \)-th prolongation of \( D \) is a linear DO \( P^l(D) \in \text{LDiff}_{k+l}(\xi,J^l(\eta)) \) is such that the following diagram commutes:

\[ C^\infty(J^{k+l}(\xi)) \xrightarrow{P^l(D)_*} C^\infty(J^l(\eta)) \]
\[ J^{k+l} \xrightarrow{D} J^l \]

The total symbol, \( \tilde{P^l(D)} \in \text{Hom}(J^{k+l}(\xi),J^l(\eta)) \), of \( P^l(D) \) is the only linear fiber-preserving mapping such that \( P^l(D) = \tilde{P^l(D)}_*J^{k+l} \) (in the notations of (2.10)).

For \( D \in \text{LDiff}_k(\xi,\eta) \), we set

\[ R^{k,l} := \ker \tilde{P^l(D)}, \]

(2.17)

which in general is a family of linear subspaces of the vector bundle \( J^{k+l}(\xi) \).

A linear DO \( D \in \text{LDiff}_k(\xi,\eta) \) is said to be formally integrable if for each \( l \geq 0 \) the following conditions are satisfied:

(a) (“regularity”, “constancy of rank”) \( R^{k,l} \) is a vector subbundle of \( J^{k+l}(\xi) \);  
(b) (“existence of formal solutions”) the natural projection

\[ \pi^{k+l+1,k+l} : R^{k,l+1} \rightarrow R^{k,l} \]

is an epimorphism (i.e., a surjective linear mapping).

For a formally integrable DO \( D \in \text{LDiff}_k(\xi,\eta) \), the subbundle \( R^{k,0} \subseteq J^k(\xi) \) is called its equation.

**Remark 2.1.** Clearly, if a DO is formally integrable, all its prolongations are also formally integrable DOs.
The formal integrability is crucially important for the dimensional reduction of invariant DOs considered in Sect. 4. Condition (b) from the definition of formal integrability deserves some discussion. Let $D \in \text{LDiff}_k(\xi, \eta)$ and $\psi \in C^\infty(\xi)$. By differentiating the equation $D\psi = 0$, we obtain the equation $P^l(D)\psi = 0$ of order $(k + l)$. In general, it may happen that the equation $P^l(D)\psi = 0$ contains some condition only on derivatives of order $\leq (k + l - 1)$ that was not contained in the prolongation $P^{l-1}(D)\psi = 0$. If this is the case, then one can never be sure that all the conditions on the partial derivatives of some order are obtained by any finite number of prolongations. Condition (b) of the definition of formal integrability guarantees that this cannot happen for a formally integrable DO. The example below clarifies this important point.

Example. Here is an example of a DO that is not formally integrable, suggested in the classic work of Janet [5, pp. 76–77] and discussed in [6, Introduction] and [7, pp. 97–98]. Let $\xi$ and $\eta$ be globally trivial vector bundles over the manifold $B = \{(x^1, x^2, x^3) : x^2 > 1\} \subset \mathbb{R}^3$ with standard fibers $\mathbb{R}$ and $\mathbb{R}^2$, respectively. (The condition $x^2 > 1$ is not essential for this example, but it will be convenient in the example in Sect. 3.3.3, where the results of this example are used.) Let the DO $M \in \text{LDiff}_2(\xi, \eta)$ be defined as $M := \left(\partial_{11} - x^2\partial_{33}\right)$. Then one can easily check that $\pi^{3,2} : R^{2,1} \rightarrow R^{2,0}$ is an epimorphism, i.e., that the equation $P^1(M)\psi = 0$ does not impose any additional conditions on the first and second derivatives that were not present in the equation $M\psi = 0$. The second prolongation, however, contains the condition $\partial_{233}\psi = 0$, which was not present in $R^{2,1}$. To see this, differentiate the first component of $M\psi = 0$ twice with respect to $x^2$, then note that the second component of $M\psi = 0$ (i.e., the equation $\partial_{22}\psi = 0$) implies that $\partial_{112}\psi = 0$ and $\partial_{2233}\psi = 0$. In formal language, this means that the projection $\pi^{3,3} : R^{2,2} \rightarrow R^{2,1}$ is not an epimorphism. The differential equation $M\psi = 0$ is not difficult to solve explicitly, and its solution can be shown to contain only 12 arbitrary constants, namely,

$$
\psi(x^1, x^2, x^3) = \alpha_1 x^1 \left[(x^1)^2 x^2 + (x^3)^2\right] x^3 + \alpha_2 \left[3(x^1)^2 x^2 + (x^3)^2\right] x^3
+ \alpha_3 x^1 x^2 x^3 + \alpha_4 x^1 \left[(x^1)^2 x^2 + 3(x^3)^2\right]
+ \alpha_5 \left[(x^1)^2 x^2 + (x^3)^2\right] + \alpha_6 x^1 x^2 + \alpha_7 x^1 x^3 + \alpha_8 x^2 x^3
+ \alpha_9 x^1 + \alpha_{10} x^2 + \alpha_{11} x^3 + \alpha_{12}.
$$

This happens because there are infinitely many conditions that the higher prolongations of $M\psi = 0$ imply on the lower-order derivatives (like the condition on $\partial_{112}\psi$ given above), which results in a general solution depending only on a finite number of parameters instead of depending on arbitrary functions.

2.4. Involutivity and formal integrability

The concept of involutivity of a collection of vector fields on a manifold is closely related to the formal integrability of the differential operator defined
as the set of all these vector fields. To clarify this, we start with two motivating examples. Let $\xi$ and $\eta$ be globally trivial vector bundles over the manifold $B = \{ (x^1, x^2, x^3) : x^1 > 0 \} \subset \mathbb{R}^3$ with standard fibers $\mathbb{R}$ and $\mathbb{R}^2$, respectively. Let $X$ and $Y$ be vector fields on $B$, and $M := \begin{pmatrix} X \\ Y \end{pmatrix} \in \text{LDiff}_1(\xi, \eta)$. Consider the following two cases:

- If $X = \partial_1$ and $Y = x^1 \partial_1 + \partial_2 + \partial_3$, then one can easily check that the general solution of the equation $M \psi = 0$ (i.e., of the system $\partial_1 \psi = 0$, $(x^1 \partial_1 + \partial_2 + \partial_3) \psi = 0$) is $\psi(x^1, x^2, x^3) = g(x^2 - x^3)$, where $g$ is an arbitrary differentiable function of one variable. Note that in this case $[X, Y] = X \in \text{span} \{X, Y\}$.
- If $X = \partial_1$, $Y = \partial_2 + x^1 \partial_3$, then the general solution of $M \psi = 0$ is $\psi(x^1, x^2, x^3) = \text{const}$. Note that in this case $[X, Y] = \partial_3 \notin \text{span} \{X, Y\}$.

The reason for the fact that in the latter case the general solution had smaller amount of “arbitrariness” (only one arbitrary constant instead of one arbitrary function of one variable) is due to the fact that the collection of vector fields $\{X, Y\}$ was not involutive. The involutivity of a set of vector fields is closely related to the formal integrability of the differential operator these vector fields define. We address this connection in the theorem below.

**Theorem 2.2.** Let $B$ be a real $n$-dimensional manifold, and $(x^\mu)$ be some local coordinates. Let $\xi = (B \times \mathbb{K}, \pi_1, B)$, $\eta = (B \times \mathbb{K}^d, \pi_1, B)$ be vector bundles over $B$, where $\mathbb{K}$ is some field. Let $X_a \ (a = 1, \ldots, d)$ be vector fields on $B$ that are linearly independent at each point, and the DO $D \in \text{LDiff}_1(\xi, \eta)$ be defined as

$$D(\psi)(b) = \begin{pmatrix} X_1(b)^\mu \partial_\mu \psi(b) \\ \vdots \\ X_d(b)^\mu \partial_\mu \psi(b) \end{pmatrix}, \quad \psi \in C^\infty(\xi). \quad (2.18)$$

Then $D$ is formally integrable if and only if the collection of vector fields $\{X_a\}_{a=1}^d$ is involutive.

*Proof.* The local coordinates $(x^\mu, z)$ in $\xi$ and $(x^\mu, w^a)$ in $\eta$ generate local jet bundle coordinates $(x^\mu, z, z_\mu)$ in $J^1(\xi)$, $(x^\mu, z, z_\mu, z_{\mu\nu})$ in $J^2(\xi)$, and $(x^\mu, w^a, w^a_\mu)$ in $J^1(\eta)$ (recall (2.1)). Let $b = (x^\mu) \in B$ be an arbitrary point. The total symbols of $D$ and its first prolongation are respectively

$$\tilde{D} : J^1(\xi) \rightarrow \eta : (x^\mu, z, z_\mu) \mapsto (x^\mu, w^a) \quad \text{with} \quad w^a = X_a(b)^\mu z_\mu,$$

and

$$\overline{P^1(D)} : J^2(\xi) \rightarrow J^1(\eta) : (x^\mu, z, z_\mu, z_{\mu\nu}) \mapsto (x^\mu, w^a, w^a_\mu)$$

with $w^a = X_a(b)^\mu z_\mu$, $w^a_\mu = \partial_\nu X_a(b)^\mu z_\mu + X_a(b)^\mu z_{\mu\nu}$. Let $R^{k,l}_b$ be the fiber over $b \in B$ of $R^{k,l} \subseteq J^{k+l}(\xi)$ (defined in (2.17)), and $\pi^{k+1,k} : J^{k+1}(\xi) \rightarrow J^k(\xi)$
be the canonical projections (2.5). In these notations,
\[ (x^\mu, z, z_\mu) \in R^{1,0}_b \iff X_a(b)^\mu z_\mu = 0 \]  
(2.19)
\[ (x^\mu, z, z_\mu, z_{\mu\nu}) \in R^{1,1}_b \iff \begin{cases} X_a(b)^\mu z_\mu = 0 \\ \partial_\nu X_a(b)^\mu z_{\mu\nu} + X_a(b)^\mu z_{\mu\nu} = 0 \end{cases} \]  
(2.20)
(it is understood that the conditions in the right-hand side hold for all \( a = 1, \ldots, d \) and \( \nu = 1, \ldots, n \)).

Multiplying \( \partial_c X_a(b)^\mu z_\mu + X_a(b)^\mu z_{\mu\nu} = 0 \) by \( X_c(b)^\nu \), and \( \partial_\nu X_c(b)^\mu z_\mu + X_b(b)^\mu X_c(b)^\mu z_{\mu\nu} = 0 \) by \( X_b(b)^\nu \) yields the system
\[ X_c(b)^\nu \partial_c X_a(b)^\mu z_\mu + X_c(b)^\mu X_a(b)^\mu z_{\mu\nu} = 0 \]
\[ X_a(b)^\nu \partial_c X_c(b)^\mu z_\mu + X_a(b)^\mu X_c(b)^\mu z_{\mu\nu} = 0 \]
from which, by subtracting the first equation from the second one, we obtain
\[ [X_a, X_c](b)^\mu z_\mu = 0 , \]
where \([X_a, X_c](b)^\mu\) is the \( \mu \)th component of the commutator of \( X_a \) and \( X_c \) at \( b \). This implies that
\[ (x^\mu, z, z_\mu) \in \pi^{2,1}(R^{1,1}_b) \iff \begin{cases} X_a(b)^\mu z_\mu = 0 \\ [X_a, X_c](b)^\mu z_\mu = 0 \quad \forall c = 1, \ldots, d. \end{cases} \]  
(2.21)

Assume that the DO \( D \) (2.18) is formally integrable. Then \( \pi^{2,1} : R^{1,1} \to R^{1,0} \) is an epimorphism, so from (2.19) and (2.21) we obtain that \( X_a(b)^\mu z_\mu = 0 \) implies \([X_a, X_c](b)^\mu z_\mu = 0\) for any \( c = 1, \ldots, d \). Since this holds for every point \( b \in B \), the formal integrability of \( D \) implies the involutivity of the collection of vector fields \( \{X_a\}_{a=1}^d \).

Conversely, assume that \( \{X_a\}_{a=1}^d \) is involutive. Then, by the Frobenius Theorem (see, e.g., [7, Sect. 2.4]), it is possible to choose local coordinates \((x^\mu)\) in \( B \) such that
\[ \text{span} \{X_1, \cdots, X_d\} = \text{span} \left\{ \frac{\partial}{\partial x^{n-d+1}}, \ldots, \frac{\partial}{\partial x^n} \right\} . \]

In these coordinates, the equation \( D\psi = 0 \) where \( D \) is defined in (2.18) is equivalent to the system
\[ \frac{\partial \psi}{\partial x^{n-d+1}} = 0 \]
\[ \vdots \]
\[ \frac{\partial \psi}{\partial x^n} = 0 , \]
hence the equations determining \( R^{1,l} \) become
\[ R^{1,0} = \{ z_\beta = 0 \} ; \]
\[ R^{1,1} = \{ z_\beta = 0 , z_\beta^\mu = 0 \} ; \]
\[ R^{1,2} = \{ z_\beta = 0 , z_\beta^\mu = 0 , z_\beta^\mu\nu = 0 \} ; \]
\[ R^{1,3} = \{ z_\beta = 0 , z_\beta^\mu = 0 , z_\beta^\mu\nu = 0 , z_\beta^\mu\nu\rho = 0 \} ; \]
etc.; here \(\mu, \nu, \rho, \ldots\) take values from 1 to \(d\), while \(\bar{\beta} = n - d + 1, \ldots, n\). This makes it obvious that all projections \(\pi^{l+2,l+1} : R^{1,l+1} \rightarrow R^{1,l}, l = 0, 1, 2, \ldots\) are epimorphisms, i.e., \(D\) is formally integrable. \(\square\)

The interested reader can find more about the relation between involutivity and formal integrability in [7, Ch. 4] and [8, Chapters IX and X].

### 3. Restriction of a DO to a submanifold

Let \(\xi\) and \(\eta\) be vector bundles over the manifold \(B\). Let \(C \subseteq B\) be a submanifold of \(B\) and \(i : C \hookrightarrow B\) be the natural embedding. We will denote by \(\xi_C\) or, equivalently, by \(\xi|_C\), the bundle \(i^*\xi\) induced by \(i\). In other words, \(\xi_C \equiv \xi|_C = (E', \pi', C)\) where \(E' := \pi^{-1}(C)\), \(\pi'\) is the restriction of \(\pi\) to \(E'\), and the fiber \((\xi|_C)_c = (\pi')^{-1}(c)\) of \(\xi|_C\) over a point \(c \in C\) is the same as \((\xi)_c = \pi^{-1}(c)\). If \(L_1, L_2\) are linear subspaces, \(L_1\) and \(L_2\) of the linear space \(L\) are transversal if:

- they are complementary in the sense of linear algebra, i.e., each \(v \in L\) can be decomposed as \(v = v_1 + v_2\) with \(v_1 \in L_1, v_2 \in L_2\);
- they intersect trivially: \(L_1 \cap L_2 = \{0\}\).

If \(L_1, L_2\) are transversal, we write \(L = L_1 \oplus L_2\).

We say that two submanifolds \(B_1\) and \(B_2\) of the manifold \(B\) intersect transversely at \(b\) if \(T_bB = T_bB_1 \oplus T_bB_2\). We say that two subbundles \(\xi_1\) and \(\xi_2\) of the vector bundle \(\xi\) are transversal and write \(\xi = \xi_1 \oplus \xi_2\) if \(\xi_b = (\xi_1)_b \oplus (\xi_2)_b\) for each \(b\) in the base of \(\xi\).

#### 3.1. Set-up and notations

In the construction of the reduced bundle from [1, Sect. 2.3], the base \(B/G\) of the reduced bundle \(\xi^G\) is “glued” by a family of submanifolds \(\{\tilde{U}_\alpha\}_{\alpha \in \mathcal{A}}\) (recall [1, Eqn. (2.17)]). Each \(\tilde{U}_\alpha\) is transversal to the orbits of action \(t\) of the Lie group \(G\) on the base \(B\), so that, for any \(\alpha \in \mathcal{A}\) and any \(b \in \tilde{U}_\alpha\), if \(O_b\) is the orbit of the point \(b\) under the action \(t\), then \(T_bB = T_b\tilde{U}_\alpha \oplus T_bO_b\). Let \(\xi\) and \(\eta\) be vector bundles over \(B\), and \(D \in \text{Diff}_k(\xi, \eta)\). Since the coordinate realizations of the reduced bundles \(\xi^G\) and \(\eta^G\) are glued from vector bundles over the submanifolds \(\tilde{U}_\alpha\) of \(B\), it is natural to ask how one can restrict a DO \(D \in \text{Diff}_k(\xi, \eta)\) to a submanifold of \(B\).

Let \(\tilde{U}\) be a submanifold of the common base \(B\) of \(\xi\) and \(\eta\), and \(\xi_{\tilde{U}} \equiv \xi|_{\tilde{U}}\) and \(\eta_{\tilde{U}}\) be the corresponding restrictions. If \(D \in \text{Diff}_k(\xi, \eta)\), then in general the natural embedding \(i : \tilde{U} \hookrightarrow B\) does not provide us with enough information in order to construct a restricted DO \(D_{\tilde{U}}\). Indeed, let \(\dim \tilde{U} = \tilde{n},\)
and assume that the local coordinates \((x^1, \ldots, x^n)\) in \(B\) are adapted to \(\tilde{U}\) in the sense that
\[
\tilde{U} = \{x^{\tilde{n}+1} = \cdots = x^n = 0\};
\]
in these coordinates, the natural embedding \(i : \tilde{U} \hookrightarrow B\) is given by
\[
i((x^1, \ldots, x^{\tilde{n}})) = (x^1, \ldots, x^{\tilde{n}}, 0, \ldots, 0).
\]
We will call \((x^1, \ldots, x^{\tilde{n}})\) internal for \(\tilde{U}\) and \((x^{\tilde{n}+1}, \ldots, x^n)\) external for \(\tilde{U}\) coordinates. Let \(\psi \in C^\infty(\xi)\) and
\[
\psi_{\tilde{U}} \equiv \psi_{\mid\tilde{U}} := \psi \circ i \in C^\infty(\xi_{\tilde{U}}) \quad (3.2)
\]
be the restriction of its domain to \(\tilde{U}\). After we restrict the domain of \(\psi\) to \(\tilde{U}\), we lose all information about the dependence of \(\psi\) on the external for \(\tilde{U}\) coordinates, hence \(D(\psi_{\tilde{U}})\) cannot be computed if \(D\) contains partial derivatives with respect to the external coordinates.

Because of the bijective correspondence (2.12) between the DOs from \(\text{Diff}_k(\xi, \eta)\) and their total symbols, we will concentrate to the problem of restricting the domain of a jet of a section \(\psi \in C^\infty(\xi)\). Taking the jet of a section does not “commute” with restricting the domain of the section. If we first find the \(k\)-jet \(J^k(\psi)\) of the section \(\psi \in C^\infty(\xi)\), and then restrict \(J^k(\psi)\) to \(\tilde{U}\), we obtain \(J^k(\psi)_{\tilde{U}} \equiv J^k(\psi)_{\mid\tilde{U}} = J^k(\psi) \circ i\), which is a section of \(J^k(\xi_{\tilde{U}})\). For any \(b \in \tilde{U}\), \(J^k(\psi)_{\tilde{U}}(b)\) (which is the same as \(J^k(\psi)(b)\)) contains derivatives with respect to all coordinates \(x^1, \ldots, x^n\), namely
\[
(J^k(\psi)_{\tilde{U}}(b))^a_{\mu_1 \cdots \mu_i} = \partial_{\mu_1 \cdots \mu_i} \psi^a(b), \quad \begin{cases} 1 \leq \mu_1 \leq \cdots \leq \mu_i \leq n, & i = 1, \ldots, k \\ a = 1, \ldots, \dim \xi \end{cases}
\]
(in the notations introduced in (2.2) and (2.3)).

On the other hand, if we first restrict \(\psi\) to \(\tilde{U}\) and after that compute the \(k\)th jet of the restriction \(\psi_{\tilde{U}}\) (3.2), the result, \(J^k_{\tilde{U}}(\psi_{\tilde{U}})\), is a section of \(J^k(\xi_{\tilde{U}})\). Here we introduced the notations \(J^k_{\tilde{U}}(\xi_{\tilde{U}})\) and \(J^k_{\tilde{U}} \in \text{LDiff}_k(\xi_{\tilde{U}}, J^k(\xi_{\tilde{U}}))\), which simply mean that we work with the sections of the restricted bundle \(\xi_{\tilde{U}}\) (as in (3.2)). For any \(b \in \tilde{U}\), \(J^k_{\tilde{U}}(\psi_{\tilde{U}})(b)\) contains only derivatives with respect to the internal for \(\tilde{U}\) coordinates \(x^1, \ldots, x^{\tilde{n}}:\)
\[
(J^k_{\tilde{U}}(\psi_{\tilde{U}})(b))^a_{\mu_1 \cdots \mu_i} = \partial_{\mu_1 \cdots \mu_i} \psi^a(b), \quad \begin{cases} 1 \leq \mu_1 \leq \cdots \leq \mu_i \leq \tilde{n}, & i = 1, \ldots, k \\ a = 1, \ldots, \dim \xi \end{cases}
\]
The dimensions of (the fibers of) \(J^k(\xi_{\tilde{U}})\) and \(J^k_{\tilde{U}}(\xi_{\tilde{U}})\) are
\[
\dim J^k(\xi_{\tilde{U}}) = \binom{n + k}{k} \dim \xi, \quad \dim J^k_{\tilde{U}}(\xi_{\tilde{U}}) = \binom{\tilde{n} + k}{k} \dim \xi. \quad (3.4)
\]
Denote by
\[
j^k : J^k(\xi_{\tilde{U}}) \to J^k_{\tilde{U}}(\xi_{\tilde{U}}) \quad (3.5)
\]
the natural projection given by “cutting off” all non-internal for \( \tilde{U} \) derivatives, i.e., derivatives containing at least one external for \( \tilde{U} \) partial derivative. In a coordinate-free language, for any \( b \in \tilde{U} \), the map \( j^k \) is given by

\[
j^k \left( J^k(\psi)_{\tilde{U}}(b) \right) = J^k(\psi_{\tilde{U}})(b) .
\]  

(3.6)

As usual, let

\[
j^k_* \equiv (j^k)_* : C^\infty(J^k(\xi)_{\tilde{U}}) \to C^\infty(J^k(\xi_{\tilde{U}})) : J^k(\psi)_{\tilde{U}} \mapsto J^k(\psi_{\tilde{U}})
\]  

(3.7)

be the map between the sections that is induced by \( j^k \) (as in (2.10)).

### 3.2. Internal for \( \tilde{U} \) DOs and their restriction to \( \tilde{U} \)

There is a situation in which the restriction of a DO \( D \in \text{Diff}_k(\xi, \eta) \) to \( \tilde{U} \) is naturally and uniquely determined by the embedding \( i : \tilde{U} \to B \). This happens when, for an arbitrary \( \psi \in C^\infty(\xi) \), the section \( D\psi \in C^\infty(\eta) \) evaluated at the points of \( \tilde{U} \) (which, in formal notation, is \( (D\psi)|_{\tilde{U}} \equiv (D\psi) \circ i \)) contains only differentiations with respect to internal for \( \tilde{U} \) coordinates. In this case the DO \( D \) is said to be \textit{internal} (for \( \tilde{U} \)).

Let \( D = \tilde{D}_*J^k \in \text{Diff}_k(\xi, \eta) \) be internal for \( \tilde{U} \), and \( b \in \tilde{U} \). Then \( (D\psi)(b) \) does not contain any non-internal for \( \tilde{U} \) derivatives, i.e., derivatives \( (J^k(\psi)_{\tilde{U}}(b))^a_{\mu_1 \cdots \mu_i} = \partial_{\mu_1 \cdots \mu_i} \psi^a(b) \) for which at least one of the indices \( \mu_1, \ldots, \mu_i \) exceeds \( n \). Clearly, in this case \( (D\psi)(b) \) can be expressed only in terms of the coordinates \( (J^k_{\tilde{U}}(\psi_{\tilde{U}})(b))^a_{\mu_1 \cdots \mu_i} \) of the \( k \)-jet \( J^k_{\tilde{U}}(\psi_{\tilde{U}})(b) \) of the restricted section \( \psi_{\tilde{U}} \) (see (3.3)). Therefore, for an internal for \( \tilde{U} \) DO \( D \), we can define the restricted to \( \tilde{U} \) DO \( D_{\tilde{U}} \in \text{Diff}_k(\xi_{\tilde{U}}, \eta_{\tilde{U}}) \) as follows: given a section \( \rho \in C^\infty(\xi_{\tilde{U}}) \), we can think of it as a restriction \( \psi_{\tilde{U}} = \psi \circ i \) of a section \( \psi \in C^\infty(\xi) \) to \( \tilde{U} \), and then set

\[
(D_{\tilde{U}}\rho)(b) := (D\psi)(b) , \quad b \in \tilde{U} .
\]

(3.8)

Since \( D \) is internal for \( \tilde{U} \), \((D\psi)(b)\) does not contain non-internal for \( \tilde{U} \) derivatives, so that the arbitrariness in the choice of \( \psi \in C^\infty(\xi) \) such that \( \psi_{\tilde{U}} = \rho \) is immaterial. Since differentiation is a local operation, the section \( \psi \) does not need to be defined on \( B \), but only on an open subset of \( B \) that contains \( \tilde{U} \). Clearly, if \( D \) is internal for \( \tilde{U} \), then the order of the restricted to \( \tilde{U} \) DO \( D_{\tilde{U}} \) is the same as the order of \( D \).

In terms of the map \( j^k \) from (3.5) and (3.6), for an internal for \( \tilde{U} \) DO \( D \) we can naturally define the total symbol \( \tilde{D}_{\tilde{U}} \) of the restricted DO \( D_{\tilde{U}} \) by

\[
\tilde{D}_{\tilde{U}} = \tilde{D}_{\tilde{U}} \circ j^k ,
\]

(3.9)
where $\tilde{D}|_{\tilde{U}}: J^k(\xi) \rightarrow \eta_{\tilde{U}}$ is the restriction of the total symbol $\tilde{D} : J^k(\xi) \rightarrow \eta$ of $D$ to the submanifold $\tilde{U}$. In other words, $\tilde{D}|_{\tilde{U}}$ is defined so that the diagram

$$
\begin{array}{ccc}
C^\infty(J^k(\xi)_{\tilde{U}}) & \xrightarrow{j^k_*} & C^\infty(J^k_{\tilde{U}}(\xi)_{\tilde{U}}) \\
(\tilde{D}|_{\tilde{U}})_* & \downarrow & (\tilde{D}|_{\tilde{U}})_* \\
C^\infty(\eta_{\tilde{U}}) & \xrightarrow{(\tilde{D}|_{\tilde{U}})_*} & C^\infty(\eta_{\tilde{U}})
\end{array}
$$

(3.10)

be commutative.

To show that the definitions (3.8) and (3.9) are consistent for an internal for $\tilde{U}$ DO $D \in \text{Diff}_k(\xi, \eta)$, we derive (3.8) from the definition (3.6), (3.7) of the map $j^k$ and the definition (3.9) of $\tilde{D}|_{\tilde{U}}$: if $\rho \in C^\infty(\xi_{\tilde{U}})$ and $\psi \in C^\infty(\xi)$ is such that $\rho = \psi_{\tilde{U}}$, then

$$
D_{\tilde{U}}\rho = (\tilde{D}|_{\tilde{U}})_*J^k_{\tilde{U}}(\psi_{\tilde{U}}) = (\tilde{D}|_{\tilde{U}})_*j^k_*(J^k(\psi)_{\tilde{U}})
= (\tilde{D}|_{\tilde{U}} \circ j^k)_*(J^k(\psi)_{\tilde{U}}) = (\tilde{D}|_{\tilde{U}})_*(J^k(\psi)_{\tilde{U}})
= (\tilde{D}|_{\tilde{U}})_*(J^k(\psi))_{\tilde{U}} = (D\psi)_{\tilde{U}}.
$$

(3.11)

3.3. Restriction of a non-internal DO

3.3.1. Natural geometric objects in the problem. To restrict to $\tilde{U}$ a DO that is not internal for $\tilde{U}$, one needs information that does not come from the natural embedding $i : \tilde{U} \hookrightarrow B$. A natural geometric object that plays a crucial role is the subbundle $I^k_{\tilde{U}}$ of $J^k(\xi)_{\tilde{U}}$ consisting of the $k$-jets of all vanishing on $\tilde{U}$ sections of $\xi$, i.e., whose fiber over an arbitrary point $b \in \tilde{U}$ is

$$
(I^k_{\tilde{U}})_b := \{ J^k(\psi)(b) : \psi \in C^\infty(\xi) \text{ s.t. } \psi \circ i \equiv 0 \} \subseteq (J^k(\xi)_{\tilde{U}})_b.
$$

(3.12)

Let

$$
t^k : I^k_{\tilde{U}} \hookrightarrow J^k(\xi)_{\tilde{U}}
$$

(3.13)

stand for the natural embedding; to simplify the notations, we will write $I^k_{\tilde{U}}$ instead of $i^k(I^k_{\tilde{U}})$.

Let $(x^1, \ldots, x^{\tilde{n}})$ and $(x^{\tilde{n}+1}, \ldots, x^n)$ be respectively the internal and the external for $\tilde{U}$ local coordinates in $B$ (recall (3.1)). For the partial derivatives of a section of $\xi$, we recall the terminology used in Sect. 3.1 and 3.2: internal derivatives are those that contain only differentiations with respect to the internal coordinates; all other derivatives are non-internal; by definition, the zeroth derivative (i.e., the section itself) is internal. In jet bundle coordinates (2.1) in $J^k(\xi)_{\tilde{U}}$, the internal jet bundle coordinates in $J^k(\xi)_{\tilde{U}}$ are $z^a$ and those $z^a_{\mu_1 \cdots \mu_i}$ for which $\mu_1 \leq \tilde{n}, \ldots, \mu_i \leq \tilde{n}$, while the non-internal ones are $z^a_{\mu_1 \cdots \mu_i}$, for which at least one of the $\mu$’s is strictly greater than $\tilde{n}$. According to (3.4), the number of internal coordinates in $J^k(\xi)_{\tilde{U}}$ is $\left(\tilde{n}+k\right)\dim \xi$, while number of the non-internal ones is $\left[\left(\tilde{n}+k\right) - \left(\tilde{n}+k\right)\right]\dim \xi$.

To simplify the notations, we temporarily write “int” for the set of all internal coordinates in $J^k(\xi)_{\tilde{U}}$, and “non-int” for the set of all non-internal ones. Then $I^k_{\tilde{U}}$ consists of those elements of $J^k(\xi)_{\tilde{U}}$ all internal jet bundle
coordinates of which are zero, while the non-internal ones are arbitrary; symbolically this can be written as \( I^k_{\widetilde{U}} = \{(\text{int} = 0, \text{non-int})\} \). The natural projection \( j^k \) (3.5) maps the element \((\text{int}, \text{non-int}) \in J^k(\xi_{\widetilde{U}})\) to \((\text{int}) \in J^k_{\widetilde{U}}(\xi_{\widetilde{U}})\), preserving the values of all the internal jet bundle coordinates (i.e., \( j^k \) simply "cuts off" all non-internal coordinates). Obviously, \( j^k \circ i^k = 0 \), so that we obtain the following short exact sequence:

\[
0 \longrightarrow \{(\text{int} = 0, \text{non-int})\} \overset{i^k}{\longrightarrow} \{(\text{int}, \text{non-int})\} \overset{j^k}{\longrightarrow} \{(\text{int})\} \longrightarrow 0 . \tag{3.14}
\]

In coordinate-free notations, the short exact sequence (3.14) can be written as the horizontal short exact sequence in the diagram

\[
0 \longrightarrow I^k_{\widetilde{U}} \overset{i^k}{\longrightarrow} J^k(\xi_{\widetilde{U}}) \overset{j^k}{\longrightarrow} J^k_{\widetilde{U}}(\xi_{\widetilde{U}}) \longrightarrow 0 , \tag{3.15}
\]

in which we have also shown the maps from (3.9), as well as the maps \( \Pi^k \) and \( \Sigma^k \) which will be discussed below.

**3.3.2. Geometry of the restriction of a non-internal DO.** In the geometric language introduced above, the gist of the problem of restricting a DO \( D \in \text{Diff}_k(\xi, \eta) \) to a submanifold \( \widetilde{U} \) of the base \( B \) is that while \( J^k(\xi_{\widetilde{U}}) \) is isomorphic to the direct sum \( I^k_{\widetilde{U}} \oplus J^k_{\widetilde{U}}(\xi_{\widetilde{U}}) \), the bundle \( J^k_{\widetilde{U}}(\xi_{\widetilde{U}}) \) is not naturally embedded in \( J^k(\xi_{\widetilde{U}}) \). One way to define the total symbol \( \widetilde{D}_{\widetilde{U}} \) of the restricted DO \( D_{\widetilde{U}} \) is to choose a splitting of the short exact sequence in (3.15), i.e., a vector bundle morphism \( \Sigma^k \in \text{Hom}(J^k_{\widetilde{U}}(\xi_{\widetilde{U}}), J^k(\xi_{\widetilde{U}})) \) over the identity in \( \widetilde{U} \) that satisfies

\[
j^k \circ \Sigma^k = \text{Id}_{J^k_{\widetilde{U}}(\xi_{\widetilde{U}})} . \tag{3.16}
\]

Equivalently, we can choose a vector bundle morphism \( \Pi^k \in \text{Hom}(J^k(\xi_{\widetilde{U}}), I^k_{\widetilde{U}}) \) over the identity in \( \widetilde{U} \) such that

\[
\Pi^k \circ i^k = \text{Id}_{I^k_{\widetilde{U}}} , \quad \ker \Pi^k = \Sigma^k(J^k_{\widetilde{U}}(\xi_{\widetilde{U}})) . \tag{3.17}
\]

Then the total symbol of \( D_{\widetilde{U}} \) is given by

\[
\widetilde{D}_{\widetilde{U}} = D|_{\widetilde{U}} \circ \Sigma^k , \tag{3.18}
\]

as shown in the diagram (3.15).

Conditions (3.16) and (3.17) imposed on \( \Sigma^k \) and \( \Pi^k \) guarantee that if the DO \( D \) is internal for \( \widetilde{U} \), then \( D_{\widetilde{U}} \) defined by (3.18) is the same as its natural restriction to \( \widetilde{U} \) (discussed in Sect. 3.2).

We required that the maps \( \Sigma^k \) and \( \Pi^k \) be vector bundle morphisms for two reasons. Firstly, this is the case that occurs in dimensional reduction of invariant DOs considered in Sect. 4. Furthermore, if they are morphisms, the restriction \( D_{\widetilde{U}} \) of a linear DO \( D \) will be linear as well. In Remark 3.2 we
consider briefly the more general case when $\Sigma^k$ is a nonlinear fiber-preserving map.

Conditions (3.16) and (3.17) imply that the maps $\Sigma^k$ and $\Pi^k$ are completely defined by $\Sigma^k(J^k_k(\xi))$, which is a subbundle of $J^k(\xi)_{\bar{U}}$ transversal to $I^k_{\bar{U}}$ in $J^k(\xi)_{\bar{U}}$:

$$J^k(\xi)_{\bar{U}} = I^k_{\bar{U}} \oplus \Sigma^k(J^k_k(\xi))_{\bar{U}}. \quad (3.19)$$

Let us take an arbitrary point $b \in \tilde{U}$ and consider in practical terms the meaning of the splitting

$$J^k(\xi)_b = (I^k_{\bar{U}})_b \oplus \Sigma^k(J^k_k(\xi))_{\bar{U}})_b$$

of the fiber of $J^k(\xi)_{\bar{U}}$ over $b$. In the notations “int” and “non-int” introduced above, $(I^k_{\bar{U}})_b = \{(\text{int} = 0, \text{non-int})\}$ (recall the short exact sequence (3.14)). According to (3.4), the dimensions of the subspaces in the splitting are

$$\dim (I^k_{\bar{U}})_b = \left[ \binom{n+k}{k} - \binom{\tilde{n}+k}{k} \right] \dim \xi = \#\{\text{non-int}\},$$

$$\dim \Sigma^k(J^k_k(\xi))_{\bar{U}})_b = \binom{\tilde{n}+k}{k} \dim \xi = \#\{\text{int}\}. \quad (3.20)$$

To define a subspace $\Sigma^k(J^k_k(\xi))_{\bar{U}})_b$ of dimension $\#\{\text{int}\}$ in the linear space $\dim J^k(\xi)_b$ of dimension $\dim J^k(\xi)_b = \#\{\text{non-int}\} + \#\{\text{int}\}$, we can write a system of $\#\{\text{non-int}\}$ independent linear equations with $(\#\{\text{non-int}\} + \#\{\text{int}\})$ unknowns. To make this explicit, we denote by $Z_\mathcal{I}$, for $\mathcal{I} = 1, \ldots, \#\{\text{int}\}$, the set of all internal for $\tilde{U}$ coordinates in $J^k(\xi)_b$, and by $Z_{\mathcal{N}}$, for $\mathcal{N} = \#\{\text{int}\} + 1, \ldots, \#\{\text{int}\} + \#\{\text{non-int}\}$, the set of all non-internal coordinates in $J^k(\xi)_b$. In coordinates $(Z_\mathcal{I}, Z_{\mathcal{N}})$, the maps $\Sigma^k$ and $\Pi^k$ have the form

$$\Sigma^k : (Z_\mathcal{I}) \mapsto \begin{pmatrix} Z_\mathcal{I} \\ M(b)Z_\mathcal{I} \end{pmatrix}, \quad \Pi^k : \begin{pmatrix} Z_\mathcal{I} \\ Z_{\mathcal{N}} \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ -M(b)Z_\mathcal{I} + Z_{\mathcal{N}} \end{pmatrix}. \quad (3.20)$$

The matrix $M(b)$ is of size $\#\{\text{non-int}\} \times \#\{\text{int}\}$. This makes it clear that the set of all splittings of the horizontal short exact sequence in (3.15) is an affine space with a linear group $\text{Hom}(J^k_k(\xi)_{\bar{U}}, I^k_{\bar{U}})$. 
The process of restriction of a DO \( D \in \text{Diff}_k(\xi, \eta) \) to a submanifold \( \tilde{U} \) of the base \( B \) is illustrated well by the diagram (3.21).

The triangle in the upper left corner of (3.21) represents the relation (2.12) between the DO \( D \) and its total symbol \( \tilde{D} \). The dashed arrows with label \( \downarrow_{\tilde{U}} \) are the restrictions of the domains of the sections of \( \xi \), \( J^k(\xi) \), and \( \eta \) to \( \tilde{U} \), as in (3.2). The triangle involving \( C^\infty(\xi) \), \( C^\infty(J^k(\xi)) \), and \( C^\infty(\eta) \) represents the relation \( D_{\tilde{U}} = (\tilde{D}_{\tilde{U}})_* J^k_{\tilde{U}} \) between the desired restricted DO \( D_{\tilde{U}} \) and its total symbol \( \tilde{D}_{\tilde{U}} \). Note that the vertical arrows between the five rightmost objects in the diagram (3.21) come from the horizontal short exact sequence in (3.15). The triangle involving \( C^\infty(J^k(\xi)) \), \( C^\infty(J^k_{\tilde{U}}(\xi)) \), and \( C^\infty(\eta) \) clarifies the role of the splitting \( \Sigma^k \) in the construction of the restricted DO \( D_{\tilde{U}} \) – namely,

\[
D_{\tilde{U}} = (\tilde{D}_{\tilde{U}})_* J^k_{\tilde{U}} = (\tilde{D}_{\tilde{U}})_* \Sigma^k_{\tilde{U}} J^k_{\tilde{U}}.
\]

### 3.3.3. Defining the splitting of (3.15) through an auxiliary DO.

Thanks to (3.16), (3.17), the splitting of the horizontal short exact sequence in (3.15) is defined completely by specifying a subbundle of \( J^k(\xi)_{\tilde{U}} \) transversal to \( I^k_{\tilde{U}} \) (recall (3.19)). Such a subbundle can sometimes be defined through an auxiliary linear DO \( M \in L\text{Diff}_k(\xi, \eta) \). If \( \tilde{M}_{\tilde{U}} \in \text{Hom}(J^k(\xi), \eta) \) is the restriction to \( \tilde{U} \) of the total symbol of \( M \), then it may turn out that its kernel, \( \text{ker}(\tilde{M}_{\tilde{U}}) \subseteq J^k(\xi)_{\tilde{U}} \), is a subbundle transversal to \( I^k_{\tilde{U}} \) in \( J^k(\xi)_{\tilde{U}} \).
this case we can set
\[ \Sigma^k(J^k_U(\xi_U)) \equiv \ker \Pi^k := \ker(\widetilde{M}|_{\widetilde{U}}). \quad (3.23) \]

In the light of the discussion before (3.21), equation (3.23) means that we can express all non-internal jet bundle coordinates in terms of the internal jet bundle coordinates from the equation \((M\psi)(b) = 0\), where \(b\) is an arbitrary point in \(\widetilde{U}\). Having expressed all non-internal derivatives of \(\psi\) through the internal derivatives of \(\psi\), we substitute them in \((D\psi)|_{\widetilde{U}}\), and the result is \(D_U\psi_{\widetilde{U}}\) – an expression that contains only internal for \(\widetilde{U}\) derivatives of the restricted section \(\psi_{\widetilde{U}} \in C^\infty(\xi_{\widetilde{U}})\). From \(D_U\psi_{\widetilde{U}}\) we can read off the desired restricted DO \(D_{\widetilde{U}} \in \text{Diff}_k(\xi_{\widetilde{U}}, \eta_{\widetilde{U}})\).

To define the splitting (3.19), one can sometimes use a linear DO of order lower that \(k\). Let \(m < k\), and \(M \in \text{LDiff}_m(\xi, \eta)\). Then the \((k - m)\)th prolongation \(P^{k-m}(M)\) of \(M\) is a DO of order \(k\), and one can hope that the restriction to \(\widetilde{U}\) of its total symbol \(\widetilde{P}^{k-m}(M)\) can be used to define \(\Sigma^k(J^k_U(\xi_U))\), similarly to (3.23):
\[ \Sigma^k(J^k_U(\xi_U)) \equiv \ker \Pi^k := \ker\left([P^{k-m}(M)]|_{\widetilde{U}}\right). \quad (3.24) \]

While, in principle, it is possible to use an auxiliary linear DO \(M\) to define the splitting as in (3.23) or (3.24), finding such a DO may not be easy for several reasons.

- First of all, on what ground will one use certain DO \(M\) and not another one? In general, one can use some DO \(M\) if this would guarantee that certain properties are preserved. A fundamental example of this is the process of dimensional reduction of a DO invariant with respect to the action of a Lie group, in which case the needed auxiliary DO is the Lie derivative – see Sect. 4.

- A second problem is how to choose \(M\) so that the right-hand side of (3.23) or (3.24) indeed defines a subspace \(\Sigma^k(J^k_U(\xi_U))\) transversal to \(J^k_U(\xi_U)\) as in (3.19). Taking care of this transversality requirement is highly non-trivial. For an example of finding such an auxiliary DO see Sect. 6 of our paper [2].

- Last but not least, the auxiliary DO must be formally integrable – otherwise, unexpected complications may occur, as illustrated in the example below.

**Example.** This example shows the dangers of using an auxiliary DO \(M\) that is not formally integrable. We use the same notations as in the Example in Sect. 2.3 for \(B, \xi, \eta\), and the DO \(M := \left(\frac{\partial_{11} - x^2\partial_{33}}{\partial_{22}}\right) \in \text{LDiff}_2(\xi, \eta)\).

Let \(D := \left(\begin{array}{c} \partial_{233} \\ 0 \end{array}\right) \in \text{Diff}_3(\xi, \eta), \widetilde{U} = \{x^3 = 1\} \subset B\). Assume that we want to restrict \(D\) to \(\widetilde{U}\) by using a splitting of the horizontal short exact sequence in (3.15) that comes from \(M\) as an auxiliary DO.
The kernel of the first prolongation \( P^1(\xi_U) \) restricted to \( \tilde{U} \) — in other words, the solution of the equation \( P^1(\xi_U)(\psi)|_{U} = 0 \) — contains the equations 
\[
\partial_{11} \psi - x^2 \partial_{33} \psi = 0 \quad \text{and} \quad \partial_{112} \psi - x^2 \partial_{233} \psi - \partial_{33} \psi = 0,
\]
from which we can express the non-internal derivative in \( D \) as
\[
\partial_{233} \psi = \frac{1}{x^2} (\partial_{112} \psi - \partial_{33} \psi) = \frac{1}{x^2} \partial_{112} \psi - \frac{1}{(x^2)^2} \partial_{11} \psi.
\]
Therefore, if one uses \( \ker P^1(\xi_U) = \Sigma^3(J^3_U(\xi_U)) \) in (3.23) to define the splitting, then the restriction of the DO \( D \) to \( \tilde{U} \) is
\[
D_{\tilde{U}} = \begin{pmatrix} 1/2 \partial_{112} \psi(x^1, x^2) - 1/(x^2)^2 \partial_{11} \psi & 0 \\ 0 & 0 \end{pmatrix}.
\]
If \( \psi_{\tilde{U}}(x^1, x^2) := \psi(x^1, x^2, 1) \) is the restricted to \( \tilde{U} \) section, then it is easy to show that the general solution \( \psi_{\tilde{U}} \) of the restricted equation
\[
(D_{\tilde{U}} \psi_{\tilde{U}})(x^1, x^2) = \begin{pmatrix} 1/2 \partial_{112} \psi_{\tilde{U}}(x^1, x^2) - 1/(x^2)^2 \partial_{11} \psi_{\tilde{U}}(x^1, x^2) \\ 0 \\ 0 \end{pmatrix}
\]
is
\[
\psi_{\tilde{U}}(x^1, x^2) = h_1(x^1)x^2 + x^1 h_2(x^2) + h_3(x^2),
\]
where \( h_1, h_2, \) and \( h_3 \) are arbitrary smooth functions of one variable.

We can, however, treat \( D \) as a 4th-order DO and use \( \Sigma^4(J^4_{\tilde{U}}(\xi_U)) = \ker P^2(\xi_U) \subseteq J^4(\xi_{\tilde{U}}) \) to define the splitting, i.e., to express the non-internal derivatives in \( D \) from the equation \( P^2(\xi_U)(\psi)|_{U} = 0 \). As we showed in the example in Sect. 2.3, the second prolongation of the equation \( M \psi = 0 \) contains the condition \( \partial_{233} \psi = 0 \); clearly, this condition remains unchanged after restricting it to \( \tilde{U} \). Therefore the restricted DO for this choice of splitting becomes \( D_{\tilde{U}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \), so that the general solution of the reduced equation \( D_{\tilde{U}} \psi_{\tilde{U}}(x^1, x^2) = 0 \) consists of all smooth functions of two variables.

**Remark 3.1.** We note that, even if the auxiliary DO \( M \) provides a splitting of the horizontal short exact sequence in the diagram (3.15) and is formally integrable, the order of the restricted DO \( D_{\tilde{U}} \) may be greater than the order of the original DO \( D \). For a concrete example of this phenomenon we refer the reader to Sect. 6 of our paper [2].

**Remark 3.2.** One can define the total symbol \( \tilde{D}_{\tilde{U}} \) of the restricted DO \( D_{\tilde{U}} \) by choosing a nonlinear fiber-preserving mapping \( \Sigma^k : J^k_{\tilde{U}}(\xi_{\tilde{U}}) \to J^k(\xi_{\tilde{U}}) \) over the identity in \( \tilde{U} \) satisfying
\[
j^k \circ \Sigma^k = \text{Id}_{J^k(\xi_{\tilde{U}})}, \tag{3.25}
\]
and defining the total symbol of \( D_{\tilde{U}} \) by
\[
\tilde{D}_{\tilde{U}} = \tilde{D}|_{\tilde{U}} \circ \Sigma^k \tag{3.26}
\]
(cf. (3.22)). As before, the condition (3.25) imposed on \( \Sigma^k \) guarantees that for an internal for \( \tilde{U} \) DO its restriction defined by (3.26) is the same as its natural restriction to \( \tilde{U} \). Since our main goal is the reduction of invariant
DOs (Sect. 4), in which case the splitting $\Sigma^k$ is a linear map, we will not consider the case of nonlinear splittings in detail.

4. Dimensional reduction of invariant DOs

4.1. Group actions and Lie derivatives

We start this section by defining the concept of derivative of a Lie group action on a vector bundle, which plays a key role in the process of dimensional reduction of DOs invariant with respect to the action of a Lie group. For brevity, we will call it a Lie derivative of the action.

Recall the notations of Sect. 2.2 of [1]: let $G$ be a connected Lie group acting on the vector bundle $\xi = (E, \pi, B)$ by vector bundle morphisms $T = (t, T)$, where $t$ and $T$ are the actions of $G$ on the base $B$ and total space $E$, respectively. Assume that the action of $G$ on $\xi$ is such that $\xi$ is a reducible $G$-vector bundle (recall [1, Definition 2.8]). Let $x = (x^\mu)$ stand for some local coordinates in the base $B$; we will often identify a point $b \in B$ with its coordinates $x$. Assume that a basis $(e_i(x))$ has been chosen in each fiber $\xi_x = \pi^{-1}(x)$, and let $z = (z^a)$ be the coordinates in the fibers in this basis.

If the action $T$ preserves the fibers of $\xi$, then its general form is $T_g(x^\mu, z^a) = (t_g(x)^\mu, T_g(x, z)^a)$ for $g \in G$. If, in addition, the action is through vector bundle morphisms (i.e., if $T$ is linear in the fibers), then the general form of $T$ is

$$T_g(x^\mu, z^a) = (t_g(x)^\mu, T_g(x)^a_c z^c) , \quad g \in G . \quad (4.1)$$

By assumption, we consider only actions $T$ of the form (4.1).

The action $T$ of $G$ on $\xi$ defines an action of $G$ on $C^\infty(\xi)$: for $g \in G$ and $\psi \in C^\infty(\xi)$, the transformed section $g(\psi) \in C^\infty(\xi)$ is defined by

$$g(\psi) := T_g \circ \psi \circ t_g^{-1} . \quad (4.2)$$

In the local coordinates $(x^\mu, z^a)$,

$$g(\psi)^a(b) = T_g(t_g^{-1}(b))^a_c \psi^c(t_g^{-1}(b)) . \quad (4.3)$$

Let $\mathfrak{g}$ be the Lie algebra of $G$ with generators $\lambda_i$, and $e^{s\lambda}$ (with $s$ in an open interval in $\mathbb{R}$ containing 0) be the local 1-parameter subgroup of $G$ generated by $\lambda \in \mathfrak{g}$. Let $\mathfrak{g}^* \otimes \xi$ be a vector bundle whose sections are of the form $\Lambda \otimes \psi$, where $\Lambda \in \mathfrak{g}^*$ is an element of $\mathfrak{g}^*$ independent on the point in the base, and $\psi \in C^\infty(\xi)$. In other words, for any $b \in B$, $(\Lambda \otimes \psi)(b) = \Lambda \otimes \psi(b)$ takes an element $\lambda \in \mathfrak{g}$ and produces $\langle \Lambda, \lambda \rangle \psi(b) \in \xi_b$ (where $\langle \cdot, \cdot \rangle$ is the natural pairing between $\mathfrak{g}^*$ and $\mathfrak{g}$). To emphasize this peculiarity of $\mathfrak{g}^* \otimes \xi$, for the space of its sections we will use the notation $\mathfrak{g}^* \otimes C^\infty(\xi)$ instead of $C^\infty(\mathfrak{g}^* \otimes \xi)$.

The Lie derivative of the action $T$ of $G$ on $\xi$ is a linear first-order DO $L \in \text{LDiff}_1(\xi, \mathfrak{g}^* \otimes \xi)$ defined by

$$(L\psi)(\lambda) := \left. \frac{d}{ds} e^{s\lambda}(\psi) \right|_{s=0} \in C^\infty(\xi) , \quad \psi \in C^\infty(\xi) , \quad \lambda \in \mathfrak{g} \quad (4.4)$$
(where we used the notation (4.2)). In local coordinates, if \( b = (x^\mu) \), and \( \lambda_i \) is a generator of \( g \), then

\[
(L\psi)(\lambda_i)^a(b) = \frac{d}{ds} T_{\exp(s\lambda_i)} \left( t_{\exp(-s\lambda_i)}(b) \right) a_c \psi^c \left( t_{\exp(-s\lambda_i)}(b) \right) \bigg|_{s=0} \tag{4.5}
\]

\[=: -X_i(b)^\mu \partial_\mu \psi^a(b) + Z_i(b)^a \psi^c(b). \]

Here \( X_i = X_i^\mu \partial_\mu \in C^\infty(\tau(B)) \) with

\[
X_i(b)^\mu := \frac{d}{ds} T_{\exp(s\lambda_i)}(b)^\mu \bigg|_{s=0} \tag{4.6}
\]

are the fundamental vector fields of the action \( t \) of \( G \) on \( B \), and

\[
Z_i(b)^a := \frac{d}{ds} T_{\exp(s\lambda_i)}(b)^a \bigg|_{s=0}. \tag{4.7}
\]

Remark 4.1. The stationary subbundle \( \text{st} \xi \subseteq \xi \) (recall [1, Definition 2.7]) is invariant under the action of \( G \), so the Lie derivative can also be considered as a DO in \( \text{LDiff}_1(\text{st} \xi, \xi^* \otimes \text{st} \xi) \).

4.2. Invariant DOs and their reduction

4.2.1. Lie group action on a DO; reduced DO. Let \( \xi \) and \( \eta \) be reducible \( G \)-vector bundles over \( B \) with the same action \( t \) of \( G \) on the common base \( B \), and let \( T^\xi = (t, T^\xi) \) and \( T^\eta = (t, T^\eta) \) be the actions of \( G \) on the corresponding bundles. The actions of \( g \in G \) on \( C^\infty(\xi) \) and \( C^\infty(\eta) \),

\[
g^\xi : \psi \mapsto g^\xi(\psi) := T^\xi_g \circ \psi \circ t^{-1}_g, \tag{4.8}
\]

\[
g^\eta : \chi \mapsto g^\eta(\chi) := T^\eta_g \circ \chi \circ t^{-1}_g, \tag{4.9}
\]

define an action of \( g \) on \( \text{Diff}_k(\xi, \eta) \):

\[
g : \text{Diff}_k(\xi, \eta) \rightarrow \text{Diff}_k(\xi, \eta) : D \mapsto g(D) := g^\eta \circ D \circ (g^\xi)^{-1}. \tag{4.10}
\]

We say that a DO \( D \in \text{Diff}_k(\xi, \eta) \) is \( G \)-invariant if \( g(D) = D \), i.e.,

\[
g^\eta \circ D = D \circ g^\xi, \quad \text{for all } g \in G. \tag{4.11}
\]

Let \( \text{Diff}_k(\xi, \eta)^G \) stand for the set of all \( G \)-invariant order-\( k \) DOs from \( \xi \) to \( \eta \).

If \( \psi \in C^\infty(\xi)^G \) is a \( G \)-invariant section of \( \xi \) and \( D \in \text{Diff}_k(\xi, \eta)^G \), then \( D\psi \) is a \( G \)-invariant section of \( \eta \) — indeed, (4.10) yields \( g^\eta(D\psi) = D(g^\xi(\psi)) = D\psi \). Recall that in [1, Sect. 2.3] we gave an explicit construction of a reduced vector bundle \( \xi^G \). This bundle is such that the set \( C^\infty(\xi^G) \) of all its sections is in a bijective correspondence with the set \( C^\infty(\xi)^G \) of all \( G \)-invariant sections of \( \xi \), as in [1, Eqn. (2.16)]. Moreover, this bijection is a homomorphism from the \( C^\infty(B/G) \)-module \( C^\infty(\xi^G) \) to the \( C^\infty(B)^G \)-module \( C^\infty(\xi)^G \) (cf. [1, Remark 2.10]). Let

\[
\theta^\xi : C^\infty(\xi^G) \rightarrow C^\infty(\xi^G), \quad \theta^\eta : C^\infty(\eta^G) \rightarrow C^\infty(\eta^G) \tag{4.11}
\]

be the two bijections for \( \xi \) and \( \eta \), respectively. Then each \( G \)-invariant DO \( D \in \text{Diff}_k(\xi, \eta)^G \) maps \( C^\infty(\xi)^G \) to \( C^\infty(\eta)^G \) and, therefore, generates a reduced DO

\[
D^G := (\theta^\eta)^{-1} \circ D \circ \theta^\xi : C^\infty(\xi^G) \rightarrow C^\infty(\eta^G). \tag{4.12}
\]
between the sections of the reduced bundles $\xi^G$ and $\eta^G$.

**Remark 4.2.** If $\psi \in C^\infty(\xi)^G$, then $\psi \in C^\infty(st\xi)^G$ (recall [1, Definition 2.7]) – this was a central fact in the construction of $\xi^G$ in [1]. Thus, the above reasoning implies that $D \in \text{Diff}_k(\xi, \eta)^G$ can be naturally considered as a DO in $\text{Diff}_k(st\xi, st\eta)^G$.

### 4.2.2. Coordinate realizations of the reduced DO

To describe explicitly the reduced DO $D^G \in \text{Diff}_k(\xi^G, \eta^G)$, we recall the construction of the reduced bundles $\xi^G$ and $\eta^G$ from [1, Sect. 2.3]. Let $\{\tilde{U}_\alpha\}_{\alpha \in \mathcal{A}}$ be submanifolds of $B$ transversal to the orbits of the action $t$ of $G$ on $B$ for any $\alpha \in \mathcal{A}$ (and each orbit of $t$ intersects some $\tilde{U}_\alpha$). Let $\xi_\alpha = st\xi|_{\tilde{U}_\alpha}$ and $\eta_\alpha = st\eta|_{\tilde{U}_\alpha}$ be the restrictions of the stationary subbundles $st\xi$ and $st\eta$ to $\tilde{U}_\alpha$ [1, Eqn. (2.18)]; the stationary subbundles are sometimes called *kinematic bundles* in the literature [9, 10]. Let $\phi^\xi_{\alpha \beta} : \xi_{\beta, \alpha} \to \xi_{\alpha, \beta}$ and $\phi^\eta_{\alpha \beta} : \eta_{\beta, \alpha} \to \eta_{\alpha, \beta}$ [1, Eqn. (2.23)] be the transition isomorphisms that glue the reduced bundles $\xi^G$ and $\eta^G$ from their coordinate realizations $\{\xi_\alpha\}_{\alpha \in \mathcal{A}}$ and $\{\eta_\alpha\}_{\alpha \in \mathcal{A}}$. A section $\psi^G \in C^\infty(\xi^G)$ of the reduced vector bundle $\xi^G$ is realized as a collection $\{\psi_\alpha\}_{\alpha \in \mathcal{A}}$ of sections $\psi_\alpha \in C^\infty(\xi_\alpha)$ satisfying the compatibility conditions $\psi_\alpha = \phi^\xi_{\alpha \beta} \circ \psi_\beta$ whenever the pair of indices $\alpha$ and $\beta$ is such that there exists an orbit of the action $t$ of $G$ on $B$ that has nonempty intersections with both $\tilde{U}_\alpha$ and $\tilde{U}_\beta$ (see Eqns. (2.19)–(2.24) of [1]). The $G$-invariant section $\psi = \theta^\xi(\psi^G) \in C^\infty(\xi)^G$ (recall (4.11)) can be constructed by extending the values of the coordinate realizations $\psi_\alpha$ from $\tilde{U}_\alpha$ to $B$ by $G$-invariance; the compatibility conditions guarantee the consistency of this procedure. On the other hand, a $G$-invariant section $\psi \in C^\infty(\xi)^G$ naturally belongs to $C^\infty(st\xi)^G$ and, hence, determines uniquely sections $\psi_\alpha \in C^\infty(\xi_\alpha)$ by restriction of the domain of $\psi$ to $\tilde{U}_\alpha$:

$$\psi_\alpha := \psi|_{\tilde{U}_\alpha} \in C^\infty(st\xi|_{\tilde{U}_\alpha}) = C^\infty(\xi_\alpha), \quad \alpha \in \mathcal{A}.$$  \hspace{1cm} (4.13)

The family $\{\psi_\alpha\}_{\alpha \in \mathcal{A}}$ thus constructed satisfies the compatibility conditions automatically and, therefore, determines a section $\psi^G \in C^\infty(\xi^G)$.

Now we consider the construction of the reduced DO $D^G \in \text{Diff}_k(\xi^G, \eta^G)$ corresponding to a $G$-invariant DO $D \in \text{Diff}_k(\xi, \eta)^G$, in the light of (4.12) and the concrete realization of the reduced bundles $\xi^G$ and $\eta^G$. Without loss of generality, we assume that the same collection $\{\tilde{U}_\alpha\}_{\alpha \in \mathcal{A}}$ has been used in the construction of both $\xi^G$ and $\eta^G$. Since $\psi^G \in C^\infty(\xi^G)$ is realized as a family of sections of $\xi_\alpha$, the reduced DO $D^G$ can be defined as a family $\{D_\alpha\}_{\alpha \in \mathcal{A}}$ of DOs

$$D_\alpha \in \text{Diff}_k(\xi_\alpha, \eta_\alpha) = \text{Diff}_k(st\xi|_{\tilde{U}_\alpha}, st\eta|_{\tilde{U}_\alpha}).$$  \hspace{1cm} (4.14)

This family should be compatible with the transition isomorphisms $\phi^\xi_{\alpha \beta}$ and $\phi^\eta_{\alpha \beta}$ – namely, for a pair of indices $\alpha$ and $\beta$ such that some orbit of $t$ intersects both $\tilde{U}_\alpha$ and $\tilde{U}_\beta$, the following compatibility property must hold:

$$D_\alpha(\psi_\alpha) = \phi^\eta_{\alpha \beta} \circ D_\beta(\phi^\xi_{\beta \alpha} \circ \psi_\alpha) \quad \text{for each } \psi_\alpha \in C^\infty(\xi_\alpha).$$  \hspace{1cm} (4.15)
Taking into account that the coordinate realization $\psi_\alpha$ of a section $\psi^G \in C^\infty(\xi^G)$ is the restriction of the $G$-invariant section $\psi = \theta^G(\psi^G) \in C^\infty(\xi)^G$ to the submanifold $\tilde{U}_\alpha \subseteq B$, we set
\[
D_\alpha \psi_\alpha := (D\psi)|_{\tilde{U}_\alpha} \in C^\infty(\eta_\alpha) .
\] (4.16)

Since an invariant DO $D \in \text{Diff}_k(\xi, \eta)^G$ can naturally be considered as a DO in $\text{Diff}_k(\text{st} \xi, \text{st} \eta)^G$ (recall Remark 4.2), the construction of the coordinate realizations $D_\alpha$ (4.14) of the reduced DO $D^G$ is closely related to the restriction of $D$ to the submanifolds $\tilde{U}_\alpha$ of the base $B$ – a process discussed in Sect. 3. According to the discussion there, to define the restriction $D_\alpha$ of $D$ to $\tilde{U}_\alpha$, we have to choose a splitting $\Sigma^k_\alpha$ of short exact sequences of jet bundles for each $\alpha \in \mathcal{A}$. Moreover, because of (4.15), the family $\{\Sigma^k_\alpha\}_{\alpha \in \mathcal{A}}$ must satisfy appropriate compatibility conditions. The family $\{\Sigma^k_\alpha\}_{\alpha \in \mathcal{A}}$ of splittings comes naturally in the reduction of invariant DOs. The reason is that, if we know the value of a $G$-invariant section $\psi \in C^\infty(\xi)^G$ at $b \in \tilde{U}_\alpha$, we can reconstruct the values of $\psi$ at any point from the $t$-orbit of $b$. The knowledge of $\psi$ in a neighborhood of $\tilde{U}_\alpha$ allows us to find all derivatives of $\psi$, and then express the non-internal derivatives in terms on the internal ones, thus obtaining the desired splitting $\Sigma^k_\alpha$. It is not difficult to show that the family $\{\Sigma^k_\alpha\}_{\alpha \in \mathcal{A}}$ constructed in this way will satisfy automatically the compatibility conditions (4.15).

The implementation of the algorithm just described, however, is hampered by practical difficulties. To extend a section of $\xi_\alpha = \text{st} \xi|_{\tilde{U}_\alpha}$ to a neighborhood of $\tilde{U}_\alpha$ in $B$ is generally a difficult procedure that involves solving systems of nonlinear equations. To avoid the need of doing this, below we reformulate the problem of dimensional reduction of an invariant DO in the geometric language developed in Sect. 3, and in the following sections we develop the algorithm in detail and deal with issues related to its consistency.

We start by defining objects like the ones introduced in Sect. 3.3, but replacing $\tilde{U}$ by $\tilde{U}_\alpha$, and $\xi$ and $\eta$ by their stationary subbundles. Let $i_\alpha : \tilde{U}_\alpha \hookrightarrow B$ be the natural embedding, $I^k_\alpha$ be the subbundle of $J^k(\text{st} \xi)_{\tilde{U}_\alpha}$ with fiber over $b \in \tilde{U}_\alpha$ given by
\[
(I^k_\alpha)_b := \{ J^k(\psi)(b) : \psi \in C^\infty(\text{st} \xi) \text{ s.t. } \psi \circ i_\alpha \equiv 0 \} \subseteq (J^k(\text{st} \xi)_{\tilde{U}_\alpha})_b \quad (4.17)
\]
(cf. (3.12)). Let $i^k_\alpha : I^k_\alpha \hookrightarrow J^k(\text{st} \xi)_{\tilde{U}_\alpha}$ and
\[
J^k_\alpha : J^k(\text{st} \xi)_{\tilde{U}_\alpha} \rightarrow J^k_{\tilde{U}_\alpha}(\text{st} \xi|_{\tilde{U}_\alpha}) = J^k_{\tilde{U}_\alpha}(\xi_\alpha)
\]
be respectively the natural embedding and the natural projection as in (3.13), (3.5), (3.6); as before, we will write $I^k_\alpha$ instead of $i^k_\alpha(I^k_\alpha)$. Here we used the notation $J^k_{\tilde{U}_\alpha}$ introduced in Sect. 3.1 that reminds us that this is the $k$-jet of a vector bundle with base $\tilde{U}_\alpha$ (recall (3.3)). Consider the invariant DO $D$ as an element of $\text{Diff}_k(\text{st} \xi, \text{st} \eta)^G$, and let $\tilde{D} : J^k(\text{st} \xi) \rightarrow \text{st} \eta$ be its total symbol. Let $D_\alpha \in \text{Diff}_k(\xi_\alpha, \eta_\alpha)$ be the restriction of $D$ to $\tilde{U}_\alpha$, and
\(\widetilde{D}_\alpha : J^k(\xi_\alpha) \to \eta_\alpha\) be its total symbol, so that \(D_\alpha = (\widetilde{D}_\alpha)_* J^k_{\widetilde{U}_\alpha}\). Then the diagram (3.15) becomes

\[
\begin{array}{cccc}
0 & \longrightarrow & I^k_\alpha & \underset{\pi^k_\alpha}{\longrightarrow} & J^k(st \xi)_{\widetilde{U}_\alpha} & \overset{j^k_\alpha}{\longrightarrow} & J^k_{\widetilde{U}_\alpha}(\xi_\alpha) & \longrightarrow & 0.
\end{array}
\]

The total symbol \(\widetilde{D}_\alpha\) of the coordinate realization \(D_\alpha\) of the reduced DO \(D^G\) is then given as the composition \(\widetilde{D}_\alpha = \widetilde{D}|_{\widetilde{U}_\alpha} \circ \Sigma^k_\alpha\), as in (3.18).

The \(G\)-invariance of the DO \(D\) naturally determines a vector bundle morphism \(\Sigma^k_\alpha \in \text{Hom}(J^k_{\widetilde{U}_\alpha}(\xi_\alpha), J^k(st \xi)_{\widetilde{U}_\alpha})\) (over the identity in \(\widetilde{U}_\alpha\)) that splits the horizontal short exact sequence in (4.18), i.e., such that \(j^k_\alpha \circ \Sigma^k_\alpha = \text{Id}_{J^k_{\widetilde{U}_\alpha}(\xi_\alpha)}\) (cf. (3.16)). As explained in the beginning of Sect. 3.3.2, this condition guarantees that \(\Sigma^k_\alpha\) provides a representation of \(J^k(st \xi)_{\widetilde{U}_\alpha}\) as a direct sum,

\[
J^k(st \xi)_{\widetilde{U}_\alpha} = I^k_\alpha \oplus \Sigma^k_\alpha(J^k_{\widetilde{U}_\alpha}(\xi_\alpha)) ,
\]

and that \(\Sigma^k_\alpha\) is completely defined by its image, \(\Sigma^k_\alpha(J^k_{\widetilde{U}_\alpha}(\xi_\alpha))\). In Sect. 4.2.3, we will explain the explicit construction of \(\Sigma^k_\alpha(J^k_{\widetilde{U}_\alpha}(\xi_\alpha))\).

4.2.3. Lie derivatives and natural splittings \(\Sigma^k_\alpha\): the algorithm. To formalize the idea of obtaining the splittings \(\Sigma^k_\alpha\) – and, hence, the coordinate realizations \(D_\alpha\) of the reduced DO \(D^G\) – we employ the concept of Lie derivative \(L \in \text{LDiff}_1(\xi, g^* \otimes \xi)\) of the action \(\mathcal{T}\) of \(G\) on \(\xi\), introduced in Sect. 4.1. According to Remark 4.1, we can consider \(L\) as an element of \(\text{LDiff}_1(st \xi, g^* \otimes st \xi)\), as we will do here.

Let \(L = \widetilde{L}, J^1 \in \text{LDiff}_1(st \xi, g^* \otimes st \xi)\) be the Lie derivative, and \(\widetilde{L} \in \text{Hom}(J^1(st \xi), g^* \otimes st \xi)\) be its total symbol. The \((k - 1)st\) prolongation of \(L\) is a linear order-\(k\) DO

\[
P^{k-1}(L) = (P^{k-1}(L))_* J^k \in \text{LDiff}_k(st \xi, g^* \otimes J^{k-1}(st \xi))
\]

with total symbol \(P^{k-1}(L) \in \text{Hom}(J^k(st \xi), g^* \otimes J^{k-1}(st \xi))\). Here we use the notation \(g^* \otimes J^{k-1}(st \xi)\) instead of \(J^{k-1}(g^* \otimes st \xi)\) because the element of \(g^*\) does not depend on the point in the base – for the same reason we introduced the notation \(g^* \otimes C^\infty(\xi)\) instead of \(C^\infty(g^* \otimes \xi)\) in Sect. 4.1.

We introduce a special notation for the subbundle \(\ker P^{k-1}(L) \subseteq J^k(st \xi)\) with base restricted to \(\widetilde{U}_\alpha\):

\[
R^k_\alpha := (\ker P^{k-1}(L))|_{\widetilde{U}_\alpha} \subseteq J^k(st \xi)_{\widetilde{U}_\alpha}.
\]

We define the splitting \(\Sigma^k_\alpha\) by setting

\[
\Sigma^k_\alpha(j^k_{\widetilde{U}_\alpha}(\xi_\alpha)) := R^k_\alpha.
\]

Below we give the algorithm for dimensional reduction of an invariant DO that implements the choice (4.21), before proceeding with its theoretical
Algorithm for dimensional reduction of a $G$-invariant DO $D \in \text{Diff}_k(\xi, \eta)^G$:

**Step 1.** Let $\lambda_1, \ldots, \lambda_{\dim G}$ be a basis of $\mathfrak{g}$. Write down the system
\[
L\psi(\lambda_i) = 0, \quad i = 1, \ldots, \dim G.
\] (4.22)

**Step 2.** Find the $(k-1)$st prolongation of (4.22), i.e., compute all partial derivatives of each of the equations in (4.22) up to order $(k-1)$, thus obtaining the system
\[
L\psi(\lambda_i) = 0, \quad \partial_{\mu_1} L\psi(\lambda_i) = 0, \quad 1 \leq \mu_1 \leq n = \dim B,
\]
\[
\partial_{\mu_1 \mu_2} L\psi(\lambda_i) = 0, \quad 1 \leq \mu_1 \leq \mu_2 \leq n, \quad \vdots
\]
\[
\partial_{\mu_1 \cdots \mu_{k-1}} L\psi(\lambda_i) = 0, \quad 1 \leq \mu_1 \leq \mu_2 \leq \cdots \leq \mu_{k-1} \leq n;
\] (4.23)

here $i$ takes values $1, \ldots, \dim G$.

**Step 3.** Restrict all the equations from (4.23) to the submanifold $\tilde{U}_\alpha$. From the restricted equations express all non-internal for $\tilde{U}_\alpha$ partial derivatives of $\psi$ in terms of the internal for $\tilde{U}_\alpha$ partial derivatives.

**Step 4.** Substitute all non-internal partial derivatives in the DO $D$ with the expressions obtained for them in Step 3, thus obtaining the coordinate realization $D_\alpha \in \text{Diff}_k(\xi_\alpha, \eta_\alpha)$ (4.16).

**Step 5.** Repeat steps 3 and 4 for any $\alpha \in \mathcal{A}$ to obtain the coordinate realization $\{D_\alpha\}_{\alpha \in \mathcal{A}}$ of the reduced DO $D^G \in \text{Diff}_k(\xi^G, \eta^G)$ (4.12).

**Remark 4.3.** Note that the above algorithm for computing $D^G$ involves only elementary operations – computing derivatives and solving a system of linear algebraic equations. Moreover, the results in Sect. 4.2.4 guarantee that the linear system has constant rank and allows us to express all non-internal derivatives in a unique way.
so that \( x^\tilde{\mu} \) are coordinates in \( \tilde{U}_\alpha \), and
\[
\tilde{U}_\alpha = \{ x^\mu = 0 \} := \{ x^{\tilde{n}+1} = 0, \ldots, x^n = 0 \}.
\]
(4.24)

Similarly, split the indices \( i = 1, \ldots, \dim G \) into two groups:
\[
i = (\tilde{i}, \check{i}) \; , \; \quad \check{i} := (1, \ldots, \dim G_b) \; , \; \quad \tilde{i} := (\dim G_b + 1, \ldots, \dim G)
\]
in such a way that the elements \( \lambda_i \in \mathfrak{g} \) from the basis of \( \mathfrak{g} \) span the Lie algebra of \( G_b \):
\[
\text{span}_{\check{i} = 1, \ldots, \dim G_b} \{ \lambda_i \} = \text{Lie } (G_b).
\]
(4.25)

Clearly, \( \check{\mu} \) and \( \tilde{i} \) represent the same number of indices (namely, \( n - \tilde{n} \)).

Let \( X_i = X_i^\mu \partial_\mu \in C^\infty(\tau(B)) \) and \( Z_i \) be the objects defined in (4.6) and (4.7). In the new notations, the fields \( X_i \) can be split into two groups: \( X_i = (X_{\tilde{i}}, X_{\check{i}}) \), and their components,
\[
X_{\tilde{i}} = X_{\tilde{i}}^\tilde{\mu} \partial_\tilde{\mu} + X_{\tilde{i}}^\check{\mu} \partial_\check{\mu} \; , \quad X_{\check{i}} = X_{\check{i}}^\tilde{\mu} \partial_\tilde{\mu} + X_{\check{i}}^\check{\mu} \partial_\check{\mu},
\]
can be arranged in a block matrix form:
\[
(X_i^\mu) = \begin{pmatrix} X_{\tilde{i}}^\tilde{\mu} & X_{\tilde{i}}^\check{\mu} \\ X_{\check{i}}^\tilde{\mu} & X_{\check{i}}^\check{\mu} \end{pmatrix}.
\]
(4.26)

Because of the special choices of local coordinates (4.24) and basis of \( \mathfrak{g} \) (4.25), the following holds:
\[
(X_i(b)^\mu) = \begin{pmatrix} X_{\tilde{i}}(b)^\tilde{\mu} & X_{\tilde{i}}(b)^\check{\mu} \\ X_{\check{i}}(b)^\tilde{\mu} & X_{\check{i}}(b)^\check{\mu} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & X_{\check{i}}(b)^\check{\mu} \end{pmatrix}.
\]
(4.27)

The square matrix \( (X_{\check{i}}(b)^\check{\mu}) \) has full rank:
\[
\text{rank } (X_{\check{i}}(b)^\check{\mu}) = n - \tilde{n}
\]
(4.28)

(which, by continuity, implies that rank \( (X_{\check{i}}(b)^\check{\mu}) = n - \tilde{n} \) in an open neighborhood of \( b \)). Note that \( \tilde{U}_\alpha \) was not chosen in any special way (in particular, different points from \( \tilde{U}_\alpha \) may have different stationary subgroups), so that the vanishing of the components of \( X_i \) as in (4.27) occurs only at the point \( b \).

For brevity, do not write explicitly the values that the indices take; the notation for an index indicates the range of its values as follows: \( \mu = 1, \ldots, n; \tilde{\mu} = 1, \ldots, \tilde{n}; \mu = \tilde{n} + 1, \ldots, n; i = 1, \ldots, \dim G; \check{i} = 1, \ldots, \dim G_b; \tilde{i} = \dim G_b + 1, \ldots, \dim G \). We denote by \( (R^l_\alpha)_b \) the fiber over \( b \in \tilde{U}_\alpha \) of \( R^l_\alpha \subseteq J^1(\text{st } \xi)_{\tilde{U}_\alpha} \) (4.20), for \( l = 1, \ldots, k \). We omit the coordinates \( (x^\mu) \) of the base point from the notations of the jet bundle coordinates (2.1), and the index \( \alpha \) from \( R^l_\alpha \).

According to (4.5), an element \( (z^a, z^a_\mu) \) \( \in J^1(\xi)_b \) belongs to \( R^l_\alpha \) exactly when
\[
-X_{\check{i}}(b)^\check{\mu} z^a_\mu + Z_i(b)^a_c z^c = 0.
\]
(4.29)
Because of (4.27), these equations can be rewritten as
\[ Z_i(b)^a_c z^c = 0, \quad (4.30) \]
\[ -X_i(b)_\mu^a z^a_\mu + S_i(b)_c^a z^a_c = 0. \quad (4.31) \]

The equations (4.30) mean that if \((z^a, z^a_\mu) \in R^1_b\), then \((z^a)\) must belong to the stationary subbundle \(\text{st} \xi_b\); clearly,
\[ \dim \text{st} \xi = \dim \xi - \text{(the number of independent equations in (4.30))}. \]

Since by (4.28) the matrix \((X_i(b)_\mu^a)\) is invertible, from (4.31) we can express all non-internal first derivatives \(z^a_\mu\) as functions of \(z^a\). Therefore the dimension of (the fiber of) \(R^1_b\) is equal to the sum of \(\dim \text{st} \xi\) and the number of the internal derivatives \(z^a_\mu\) of the fields from \(\text{st} \xi\):
\[ \dim R^1_b = (1 + \tilde{n}) \dim \text{st} \xi. \]

Note that this is equal to \(\dim J^1_b(\xi_a) = (\tilde{n} + 1) \dim \text{st} \xi\).

An element \((z^a, z^a_\mu, z^a_{\mu\rho}) \in J^2(\xi)_b\) belongs to \(R^2_b\) if, in addition to (4.30) and (4.31), the following equations hold:
\[ -\partial_\rho X_i(b)_\mu^a z^a_\mu + \partial_\rho Z_i(b)_c^a z^a_c + Z_i(b)_c^a z^a_c = 0, \quad (4.32) \]
\[ -\partial_\rho X_i(b)_\mu^a z^a_\mu - X_i(b)_\mu^a z^a_{\mu\rho} + \partial_\rho Z_i(b)_c^a z^a_c + Z_i(b)_c^a z^a_c = 0 \quad (4.33) \]
(ordinary by differentiating (4.29) with respect to \(x^\rho\) and using (4.27)). Note that in (4.32) \(z^a_{\mu\rho}\) do not appear, while (4.33) contains all non-internal second derivatives \(z^a_{\mu\rho}\) (and no internal second derivatives). The invertibility of the matrix \((X_i(b)_\mu^a)\) (4.28) guarantees that all non-internal second derivatives \(z^a_{\mu\rho}\) can be expressed in terms of lower-order derivatives. From this we conclude that \((\dim R^2_b - \dim R^1_b)\) is no greater than the number of all internal second derivatives \(z^a_{\mu\rho}\), i.e.,
\[ \dim R^2_b - \dim R^1_b \leq \left(\tilde{n} + 2 - 1\right) \dim \text{st} \xi \quad (4.34) \]

(see the text preceding (2.4)).

An element \((z^a, z^a_\mu, z^a_{\mu\rho}, z^a_{\mu\rho\sigma}) \in J^3(\xi)_b\) belongs to \(R^3_b\) when it satisfies (4.30), (4.31), (4.32), (4.33), and
\[ -\partial_{\rho\sigma} X_i(b)_\mu^a z^a_\mu - \partial_{\rho\sigma} X_i(b)_\mu^a z^a_{\mu\sigma} - \partial_{\rho\sigma} X_i(b)_\mu^a z^a_{\mu\rho} \]
\[ + \partial_{\rho\sigma} Z_i(b)_c^a z^a_c + \partial_{\rho\sigma} Z_i(b)_c^a z^a_c + \partial_{\rho\sigma} Z_i(b)_c^a z^a_c + Z_i(b)_c^a z^a_c = 0, \quad (4.35) \]
\[ -\partial_{\rho\sigma} X_i(b)_\mu^a z^a_\mu - \partial_{\rho\sigma} X_i(b)_\mu^a z^a_{\mu\sigma} - \partial_{\rho\sigma} X_i(b)_\mu^a z^a_{\mu\rho} - X_i(b)_\mu^a z^a_{\mu\rho\sigma} \]
\[ + \partial_{\rho\sigma} Z_i(b)_c^a z^a_c + \partial_{\rho\sigma} Z_i(b)_c^a z^a_c + \partial_{\rho\sigma} Z_i(b)_c^a z^a_c + Z_i(b)_c^a z^a_c = 0 \quad (4.36) \]
Again, (4.35) does not contain any highest-order derivatives, while (4.36) contains all non-internal highest-order derivatives \(z^a_{\mu\rho\sigma}\) (and no other highest-order derivatives). Since all non-internal highest-order derivatives \(z^a_{\mu\rho\sigma}\) can
be expressed from \((4.36)\) due to \((4.28)\), we obtain, similarly to \((4.34)\),
\[
\dim R_b^3 - \dim R_b^2 \leq \left( \frac{n + 3 - 1}{3} \right) \dim \text{st} \xi .
\]
Continuing in this manner, we find
\[
\dim R_b^l - \dim R_b^{l-1} \leq \left( \frac{n + l - 1}{l} \right) \dim \text{st} \xi ,
\]
which, together with \((2.4)\), yields
\[
\dim R_b^l \leq \left( \frac{n + l}{l} \right) \dim \text{st} \xi = \dim J_b^l(\xi_\alpha) .
\]

On the other hand, if \(\psi \in C^\infty(\xi)^G\), then its \(l\)-jet at \(b\), \(J^l(\psi)_b\), belongs to \(R_b^l\) because \(L(\psi) = 0\) and, therefore, \(P^{l-1}(L)(\psi) = 0:\)
\[
\{ J^l(\psi)_b : \psi \in C^\infty(\xi)^G \} \subseteq R_b^l \quad \text{for any } b \in \tilde{U}_\alpha .
\]
The bijective correspondence \(\theta : C^\infty(\xi^G) \rightarrow C^\infty(\xi^G) [1, \text{Eqn. (2.16)}] \) implies a bijective correspondence
\[
\{ J^l(\rho)_b : \rho \in C^\infty(\xi_\alpha) \} \rightarrow \{ J^l(\psi)_b : \psi \in C^\infty(\xi)^G \} , \quad b \in \tilde{U}_\alpha
\]
– the function \(\psi \in C^\infty(\xi)^G\) corresponding to \(\rho \in C^\infty(\xi^G)\) is the extension of \(\rho\) to a neighborhood of \(\tilde{U}_\alpha\) by the action of \(G\). Therefore the dimension of \(R_b^l\) is no smaller than the dimension of the two linear spaces in \((4.39)\), i.e.,
\[
\dim R_b^l \geq \dim J_b^l(\xi_\alpha) = \left( \frac{n + l}{l} \right) \dim \text{st} \xi .
\]
From the opposite inequalities \((4.37)\) and \((4.40)\) it is clear that \(R_b^l\) and the two linear spaces in \((4.39)\) have the same dimension. This fact together with \((4.38)\) also imply that
\[
R_b^l = \{ J^l(\psi)_b : \psi \in C^\infty(\xi)^G \} \subseteq J^l(\text{st} \xi)_b , \quad b \in \tilde{U}_\alpha .
\]
This allows us to define the map \(\Sigma^k_\alpha : J^k(\text{st} \xi)_\tilde{U}_\alpha \rightarrow J^k(\text{st} \xi)\) in a way that satisfies all required properties, namely
\[
\Sigma^k_\alpha(J^k_\tilde{U}_\alpha(\rho)_b) := J^k(\psi)_b \in R_b^k \subseteq J^k(\text{st} \xi)_b ,
\]
where \(\psi\) is the extension of \(\rho\) by \(G\)-invariance.

We summarize the above reasoning in the following two theorems.

**Theorem 4.4.** The subbundle \(R^k_\alpha \subseteq J^k(\text{st} \xi)\) \((4.20)\) is transversal to \(I^k_\alpha\) in \(J^k(\text{st} \xi)\) \(\tilde{U}_\alpha\):
\[
J^k(\text{st} \xi)_{\tilde{U}_\alpha} = I^k_\alpha \oplus R^k_\alpha .
\]
Therefore \((4.21)\) defines a morphism \(\Sigma^k_\alpha \) \((4.41)\) splitting the horizontal short exact sequence in \((4.18)\).

**Theorem 4.5.** Under the assumptions for existence of reduced bundles, the Lie derivative \(L \in \text{LDiff}_1(\xi, g^* \otimes \xi)\) (and its restriction \(L \in \text{LDiff}_1(\text{st} \xi, g^* \otimes \text{st} \xi)\) is formally integrable.
The morphisms $\Sigma^k_\alpha$ determine the total symbols of the coordinate representations $D_\alpha$ (4.14) of the reduced DO $D^G$ (4.12) as in (3.18):

$$\widetilde{D}_\alpha = \widetilde{D}|_{\tilde{U}_\alpha} \circ \Sigma^k_\alpha : J^k_{U_\alpha}(\xi_\alpha) \to \eta_\alpha .$$

**Remark 4.6.** It is clear that the order of $D^G$ cannot exceed the order of the original DO $D$. This is in contrast with the process of restricting a DO to a submanifold by using an auxiliary DO, as explained in Remark 3.1.

### 4.3. Dimensional reduction of group action and invariant DOs

If there are additional geometric structures in the vector bundles that are compatible with the action of the group $G$, they induce analogous structures in the reduced bundles. An important example of this is the reduction of a connected component of the conformal group $\mathcal{C}^*$ to a complicated group action. We have employed this idea in [2] to construct DOs invariant with respect to the (nonlinear) action of the group on fields defined on $\mathbb{R}^{2,4}$. The action on $\mathbb{R}^{2,4}$ of the multiplicative group $\mathbb{R}^*$ of nonzero real numbers commutes with the action of $O(2,4)$. Starting with Maxwell’s equations on $\mathbb{R}^{2,4}$, which are naturally $O(2,4)$- and $\mathbb{R}^*$-invariant, we reduce them to $O_0(2,4)^*\mathbb{R}^*$-invariant equations on the projectization $\mathbb{R}^{5}$ of $\mathbb{R}^{2,4}$. Minkowski space $\mathbb{R}^{1,3}$ is realized as the projectivized light cone $\mathbb{P}^5$. 
in \( \mathbb{R}^{2,4} \). The \( \mathbb{R}^* \)-reduction is standard, while the restriction to \( Q\mathbb{R}^5 \) is related to restriction of DOs, discussed in Sect. 3. Using these techniques, we were able to reproduce in a systematic way many results derived previously in the literature. In [2] the reader can find many details (invariant subbundles, reduced gauge transformations, “universal” splitting relations for this construction, etc.) illustrating many of the issues discussed in the present paper.

5. Invariant DOs vs. DOs in the reduced bundle

Recall that the \( G \)-invariant sections \( C^\infty(\xi)^G \) of \( \xi \) are in a bijective correspondence with all sections \( C^\infty(\xi^G) \) of the reduced bundle \( \xi^G \) through the map \( \theta \) (4.11). However, for DOs the situation is different – many invariant DOs can lead to the same reduced DO. A simple example to keep in mind is the following.

Example. Let \( \xi = (\mathbb{R}^2 \times \mathbb{R}, \pi_1, \mathbb{R}^2) \), \( G \) be the additive group \((\mathbb{R}, +)\) acting on \( B \) through translations as follows:

\[
t_g((x^1, x^2)) = (x^1, x^2 + g), \quad g \in \mathbb{R}, \quad (x^1, x^2) \in \mathbb{R}^2,
\]

while the action of \( G \) on the fibers of \( \xi \) is trivial. Hence, the action on a section \( \psi \in C^\infty(\xi) \) is

\[
g(\psi)(x^1, x^2) = \psi(x^1, x^2 - g), \quad g \in \mathbb{R}.
\]

Let \((x^1, x^2, z)\) be coordinates in \( \xi \).

For the realization of the base \( B/G \) of the reduced bundle \( \xi^G \) we choose the submanifold \( \tilde{U}_\alpha = (\mathbb{R}, 0) \subset \mathbb{R}^2 \). The stationary group \( G_b \) of any \( b \in \mathbb{R}^2 \) is trivial, hence the coordinate realization \( \xi^G_\alpha \) of \( \xi^G \) is

\[
\xi^G_\alpha = (st \xi)|_{\tilde{U}_\alpha} = \xi(\mathbb{R}, 0),
\]

and the coordinates in \( \xi^G_\alpha \) are \((x^1, z)\).

Every \( \psi \in C^\infty(\xi^G) \) has the form \( \psi(x^1, x^2) = \hat{\psi}(x^1) \), where \( \hat{\psi} \) is an arbitrary function of one variable. The general form of a \( G \)-invariant linear 1st order DO \( D \) on \( \xi \) is \( D = \alpha(x^1)\partial_1 + \beta(x^1)\partial_2 + \gamma(x^1) \in \text{LDiff}_1(\xi, \xi)^G \). The reduced DO has the form \( D^G = \alpha(x^1)\partial_1 + \gamma(x^1) \in \text{LDiff}_1(\xi^G, \xi^G) \). Clearly, the set \( \text{LDiff}_1(\xi, \xi)^G \) is larger than the set \( \text{LDiff}_1(\xi^G, \xi^G) \).

In this section we first reformulate the problem of description of the set of all linear \( G \)-invariant DOs in terms of jet bundles (Sect. 5.1). Then in Sect. 5.2 we illustrate this approach on one important particular case of first-order linear DOs.

5.1. Description of all \( G \)-invariant DOs and reduction of jet bundles

Let \( \xi \) and \( \eta \) be \( G \)-reducible vector bundles over the real manifold \( B \), and \( D = \hat{D}, J^k \in \text{LDiff}_k(\xi, \eta) \). The actions of \( G \) on sections and DOs are defined in (4.8) and (4.9).
The action of $G$ on $\xi$ generates naturally an action $T^k_g : J^k(\xi) \to J^k(\xi)$ as follows: an element $J^k(\psi)_b \in J^k(\xi)_b$ is mapped to
\[ T^k_g J^k(\psi)_b := J^k (g^*(\psi)) (t_g(b)) \in J^k(\xi)_{t_g(b)}, \quad \psi \in C^\infty(\xi). \tag{5.4} \]
This action makes $J^k(\xi)$ a $G$-reducible vector bundle.

The actions (5.4) and $T^g$ induce an action of $G$ on $J(\xi)^* \otimes \eta$, and it is clear that
\[ \text{LDiff}_k(\xi, \eta)^G \cong C^\infty (J(\xi)^* \otimes \eta)^G \cong C^\infty ((J(\xi)^* \otimes \eta)^G), \tag{5.5} \]
i.e., the set of all $G$-invariant order-$k$ linear DOs is in bijective correspondence with the sections of the reduced bundle $(J(\xi)^* \otimes \eta)^G$. A local realization of $(J(\xi)^* \otimes \eta)^G$ over $\tilde{U}_\alpha$ is $(J(\xi)^* \otimes \eta)_\alpha = \text{st} (J(\xi)^* \otimes \eta)_{\tilde{U}_\alpha}$. A linear map $A_b : J^k(\xi)_b \to \eta_b$ belongs to $\text{st} (J^k(\xi)^* \otimes \eta)_b$ exactly when it is intertwining with respect to the representations of $G_b$ in $J^k(\xi)_b$ and $\eta_b$.

On the other hand,
\[ \text{LDiff}_k(\xi^G, \eta^G) \cong C^\infty (J^k(\xi^G)^* \otimes \eta^G), \tag{5.6} \]
so our goal is to clarify the relation between $C^\infty (J^k(\xi^G)^* \otimes \eta^G)$ and the rightmost term in (5.5). The crucial fact is the natural embedding
\[ \chi^k : J^k(\xi^G) \hookrightarrow J^k(\xi)^G. \tag{5.7} \]
The two bundles in (5.7) are bundles over $B/G$. Let $\tilde{U}_\alpha$ be a local realization of $B/G$. Then the coordinate realization of $J^k(\xi^G)_\alpha$ over $\tilde{U}_\alpha$ is
\[ J^k(\xi^G)_\alpha = J^k_{\tilde{U}_\alpha} ((\text{st} \xi)|_{\tilde{U}_\alpha}). \]
For a point $b \in \tilde{U}_\alpha$, each element of $J^k_{\tilde{U}_\alpha}(\xi^G)_b$ has the form $J^k_{\tilde{U}_\alpha}(\xi^G)_b$, where $\rho \in C^\infty((\text{st} \xi)|_{\tilde{U}_\alpha})$. The section $\rho$ can be uniquely extended by $G$-invariance to a neighborhood of $\tilde{U}_\alpha$ in $B$, and the $k$-jet of this extension is the image of $J^k(\rho)_b$ under the embedding (5.7). Recalling the notation $\theta^\xi$ from (4.11), we can write the embedding (5.7) as
\[ \chi^k (J^k_{\tilde{U}_\alpha}(\rho)_b) = J^k(\theta^\xi(\rho))_b. \]
While $J^k_{\tilde{U}_\alpha}(\rho)_b$ contains only derivatives internal for $\tilde{U}_\alpha$, $J^k(\theta^\xi(\rho))_b$ contains both internal and non-internal derivatives, and the non-internal ones are uniquely defined by the $G$-invariance of $\theta^\xi(\rho) \in C^\infty(\xi)^G$. Note that $\chi^k (J^k(\xi^G))$ is generally not the whole $J^k(\xi)^G$. Indeed, the fiber of $J^k(\xi)^G$ consists of $k$-jets of sections $\psi \in C^\infty(\xi)$ that are invariant with respect to the stationary group $G_b$, but in general not invariant with respect to the whole group $G$. The dual of (5.7) is the surjection
\[ (\chi^k)^* : (J^k(\xi)^G)^* \twoheadrightarrow J^k(\xi^G)^*. \tag{5.8} \]
According to the discussion in [1, Sect. 2.4], there is a natural embedding
\[ C : (J^k(\xi)^G)^G \otimes \eta^G \hookrightarrow (J^k(\xi)^* \otimes \eta)^G. \tag{5.9} \]
Let us assume that the stationary group $G_b$ has only completely reducible representations. Under this assumption, for any $G$-reducible vector bundle $\zeta$ over $B$ we have $(\zeta^G)^* = (\zeta^*)^G$ (see the reasoning preceding Theorem 4.4 of [1]). This implies that

$$
(J^k(\xi))^* = (J^k(\xi)^*)^G.
$$

Combining (5.8), (5.9), and (5.10), we obtain the following diagram.

$$
\begin{align*}
(J^k(\xi)^* \otimes \eta)^G & \quad \text{B} \quad \notag \\
(J^k(\xi)^* \otimes \eta^G) & = (J^k(\xi)^*)^G \otimes \eta^G \\
(\chi^k)^* \otimes \text{Id} & \quad \notag \\
J^k(\xi^G)^* \otimes \eta^G
\end{align*}
$$

The map $B$ in (5.11) is defined as follows. Let $b \in \tilde{U}_\alpha$. An element

$$
b \in \left( (J^k(\xi)^* \otimes \eta)^G \right)_b = \text{st} \left( J^k(\xi^b)^* \otimes \eta_b \right)
$$

can be considered as a linear operator $b : J^k(\xi)_b \to \eta_b$ that is intertwining with respect to the corresponding linear representations of $G_b$. Therefore, it defines a linear operator $b|_{\text{st} J^k(\xi)_b} : \text{st} J^k(\xi)_b \to \eta_b$, which can be thought of as an element of $(\text{st} J^k(\xi)_b)^* \otimes \text{st} \eta_b = ((J^k(\xi^G))^* \otimes \eta^G)_b$. The map $B$ is defined as $B(b) = b|_{\text{st} J^k(\xi)_b}$. Since $B \circ C$ is the identity\(^1\) and $C$ (5.9) is an embedding, it is clear that $B$ is surjective.

The composition

$$
A := [(\chi^k)^* \otimes \text{Id}] \circ B : (J^k(\xi)^* \otimes \eta)^G \to J^k(\xi^G)^* \otimes \eta^G
$$

corresponds to the reduction of an invariant linear DO $D \in \text{LDiff}_k(\xi, \eta)^G$. The induced map between the sections is (recall (2.10))

$$
A_* : C^\infty((J^k(\xi)^* \otimes \eta)^G) \to C^\infty((J^k(\xi^G)^* \otimes \eta^G))
$$

which, according to (5.5) and (5.6), is equivalent to

$$
A_* : \text{LDiff}_k(\xi, \eta)^G \to \text{LDiff}_k(\xi^G, \eta^G).
$$

The diagram (5.11) shows that each linear DO in the reduced bundles can be obtained by reduction of an invariant linear DO. The maps $B$ and $[(\chi^k)^* \otimes \text{Id}]$ in the diagram (5.11) describe the structure of all invariant linear DOs that give rise to the same DO in the reduced bundles.

\(^1\)Let us discuss this in more detail. An element $c \in ((J^k(\xi)^G)^* \otimes \eta^G)_b = \text{st} J^k(\xi^b)^* \otimes \text{st} \eta_b$ can be considered as a linear operator $c : \text{st} J^k(\xi)_b \to \text{st} \eta_b$. Let $J^k(\xi)_b = \text{st} J^k(\xi)_b \oplus M_1$ and $\eta_b = \text{st} \eta_b \oplus M_2$, where the subspaces $M_1$ and $M_2$ do not contain stationary elements with respect to the corresponding representations of $G_b$. Then $C(c) : \text{st} J^k(\xi)_b \oplus M_1 \to \text{st} \eta_b \oplus M_2$ is given by $C(c) = c \oplus 0$, and, by the definition of $B$, $B \circ C(c) = C(c)|_{\text{st} J^k(\xi)_b} = c$, hence $B \circ C$ is the identity.
Remark 5.1. Here we limited ourselves to considering only the description of all \(G\)-invariant linear DOs. The set \(L\text{Diff}_k(\xi, \eta)^G\) of all such operators is in a bijective correspondence with all sections of \((J^k(\xi)^* \otimes \eta)^G\) whose fiber, \((J^k(\xi)^* \otimes \eta)^G)_b = \text{st}(J^k(\xi)^* \otimes \eta)_b,\) over \(b \in \tilde{U}_\alpha\) consists of all linear operators \(A_b : J^k(\xi)_b \rightarrow \eta_b\) intertwining with respect to the corresponding representations of \(G_b\) (see the text after (5.5)). Similarly, the set \(\text{Diff}_k(\xi, \eta)^G\) of all \(G\)-invariant DOs (without imposing the linearity requirement) is in a bijective correspondence with all sections of a reduced bundle whose fiber over \(b \in \tilde{U}_\alpha\) is the set of all mappings from \(J^k(\xi)_b\) to \(\eta_b\) that are intertwining with respect to the corresponding (linear) representations of \(G_b\). Clearly, this is a more complicated problem that should be attacked case by case.

An example of a solved problem of this kind can be found in [11], where the authors explicitly described the set of all conformally invariant DOs acting on scalar functions on \(\mathbb{R}^n\). They also consider the description of all DOs on holomorphic functions on \(\mathbb{C}\) invariant with respect to the Möbius transformations \(z \mapsto \frac{az+b}{cz+d}, \quad \left(\begin{array}{cc} a & b \\ c & d \end{array}\right) \in \text{SL}(2, \mathbb{C})\); it turns out that a general DO of this kind can be constructed from a set \(D_3, D_4, \ldots\) of nonlinear DOs (where \(D_k\) is of order \(k\)) that are computed explicitly in the paper.

5.2. Structures occurring in the reduction of \(L\text{Diff}_1(\xi, \tau^*(B) \otimes \xi)^G\)

Here we will consider the description of one important class of linear DOs in which there are some additional structures that do not occur in the general case considered in Sect. 5.1. Namely, we study the reduction of \(G\)-invariant linear first-order DOs from \(C^\infty(\xi)\) to \(C^\infty(\tau^*(B) \otimes \xi)\) because the reduction of the cotangent bundle \(\tau^*(B)\) yields additional geometrically natural objects. Moreover, when the total symbol \(\tilde{D} : J^1(\xi) \rightarrow \tau^*(B) \otimes \xi\) of \(D \in L\text{Diff}_1(\xi, \tau^*(B) \otimes \xi)\) is a left inverse of the canonical embedding \(i : \tau^*(B) \otimes \xi \rightarrow J^1(\xi)\) from (2.8), i.e., the principal symbol is \(\sigma_1(D) = \tilde{D} \circ i = \text{Id}_{\tau^*(B) \otimes \xi},\) \(D\) is called a linear connection on \(\xi\) – an operator of central importance in gauge field theory.

Because of the natural isomorphisms (2.13), we identify an element of \(L\text{Diff}_1(\xi, \tau^*(B) \otimes \xi)\) with a section of \(J^1(\xi)^* \otimes \tau^*(B) \otimes \xi\). The actions of \(G\) on these two objects are equivalent, so that each \(D \in L\text{Diff}_1(\xi, \tau^*(B) \otimes \xi)^G\) corresponds to a unique \(G\)-invariant section \(\tilde{D} \in C^\infty(J^1(\xi)^* \otimes \tau^*(B) \otimes \xi)^G\). Each \(G\)-invariant section \(\tilde{D} \in C^\infty(J^1(\xi)^* \otimes \tau^*(B) \otimes \xi)^G\) can, in turn, be considered as an element of \(C^\infty((J^1(\xi)^* \otimes \tau^*(B) \otimes \xi)^G,\) where \((J^1(\xi)^* \otimes \tau^*(B) \otimes \xi)^G\) is the reduced bundle of \(J^1(\xi)^* \otimes \tau^*(B) \otimes \xi\).

For the reduced bundle \(\xi^G\) (with base \(B/G\)) we are interested in the DOs in \(L\text{Diff}_1(\xi^G, \tau^*(B/G) \otimes \xi^G)\) and their relation with \(L\text{Diff}_1(\xi, \tau^*(B) \otimes \xi)^G\). In other words, we want to compare the sections of \((J^1(\xi)^* \otimes \tau^*(B) \otimes \xi)^G\) with the sections of \((J^1(\xi^G)^* \otimes \tau^*(B/G) \otimes \xi^G)\). We restrict our analysis to the case when the stationary group \(G_b\) has only completely reducible representations.
Firstly, from the diagram (5.11), we obtain

\[
\begin{align*}
(J^1(\xi)^* \otimes \tau^*(B) \otimes \xi)^G \\
\text{B} \\
(J^1(\xi^G)^* \otimes (\tau^*(B) \otimes \xi)^G \\
(\chi^1)^* \otimes \text{Id}_{(\tau^*(B) \otimes \xi)^G} \\
J^1(\xi^G)^* \otimes (\tau^*(B) \otimes \xi)^G
\end{align*}
\]

(5.12)

where the arrows correspond to the surjective mappings constructed in Sect. 5.1.

The next ingredient in our analysis is the short exact sequence

\[
0 \longrightarrow (\tau^*(B)^*)^G \longrightarrow \tau^*(B)^G \longrightarrow n^* \tau^*(B/G) \longrightarrow 0
\]

(5.13)

(see [1, Lemma 4.3] or [12, Sect. 2]), which is the result of the dimensional reduction of the cotangent bundle \(\tau^*(B)\). The fibers of the vertical subbundle \(\tau^v(B)\) of \(\tau(B)\) are the subspaces tangent to the orbits of \(G\) in \(B\). In general, \((\tau^v(B)^*)^G\) is different from \((\tau^v(B)^G)^*\) (see [1, Sect. 4.1.5] and [2, Appendix]), but under the assumption for complete reducibility of the representations of \(G_b\), \((\tau^v(B)^*)^G = (\tau^v(B)^G)^*\). The diagrams (5.12) and (5.13) can be combined in the diagram below (where we kept the notations \(m^*\) and \(n^*\) from (5.13)).

\[
\begin{align*}
&J^1(\xi^G)^* \otimes (\tau^*(B) \otimes \xi)^G \\
\text{G-inv lin order-1 DOs } \xi \leadsto \tau^*(B) \otimes \xi \\
\text{B} \\
&J^1(\xi^G)^* \otimes (\tau^v(B)^G \otimes \xi^G \\
(\chi^1)^* \otimes \text{Id} \\
&J^1(\xi^G)^* \otimes (\tau^*(B/G) \otimes \xi^G \\
\text{Lin order-1 DOs } \xi^G \leadsto \tau^*(B/G) \otimes \xi^G \\
\text{E} \\
&J^1(\xi^G)^* \otimes (\tau^*(B) \otimes \xi)^G \\
\text{n}^* \\
&0 \\
\end{align*}
\]

(5.14)

The map \(E\) is the natural embedding occurring in the reduction of tensor product, discussed in [1, Sect. 2.4].

The diagram (5.14) reveals that every DO in \(\text{LDiff}^1(\xi^G, \tau^*(B/G) \otimes \xi^G)\) is obtained as a result of reduction of some – in general, not unique – DO from \(\text{LDiff}^1(\xi, \tau^*(B) \otimes \xi)^G\). If \([b] \in B/G\), then each element \(V_{[b]} \in (J^1(\xi^G)^* \otimes (\tau^*(B/G) \otimes \xi^G)_{[b]}\) gets mapped naturally by \(E \circ n^*\) to an element

\[
E(n^*(V_{[b]})) \in \left(J^1(\xi^G)^* \otimes (\tau^*(B) \otimes \xi)^G\right)_{[b]}.
\]
Since the maps $\mathcal{B}$ and $(\chi^1)^* \otimes \text{Id}$ are surjective, $E(n^*(V_{[b]}))$ always has a preimage in $(J^1(\xi)^* \otimes \tau^*(B) \otimes \xi)^G_{[b]}$, and it may happen that it has more than one preimage.

**Remark 5.2.** In the illustrative example in the introduction to Sect. 5, the stationary group $G_b$ of each $b \in \mathbb{R}^2$ is trivial, hence the maps $\mathcal{B}$ and $\mathcal{E}$ in the diagram (5.14) are isomorphisms. We leave it to the reader to write explicitly the maps $(\chi^1)^*$ and $n^*$ in this case.

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