Neville's Method

Recall the following definition which I gave in class today:

Definition: Let f be a function whose values at the n points x_0, x_1, …, x_n are known. Let $\{m_1, m_2, \ldots, m_k\}$ be a set of k distinct integers from the set $\{0, 1, 2, \ldots, n\}$. Let $P_{m_1,m_2,\ldots,m_k}(x)$ stand for the Lagrange polynomial that agrees with the function f at the k points x_{m_1}, x_{m_2}, …, x_{m_k}, i.e.,

$$P_{m_1,m_2,\ldots,m_k}(x_{m_1}) = f(x_{m_1}) \ , \quad P_{m_1,m_2,\ldots,m_k}(x_{m_2}) = f(x_{m_2}) \ , \ldots \ , \quad P_{m_1,m_2,\ldots,m_k}(x_{m_k}) = f(x_{m_k}) \ .$$

Naturally, $P_{m_1,m_2,\ldots,m_k}(x)$ is the only polynomial of degree $(k-1)$ that passes through the k points $(x_{m_1}, f(x_{m_1}))$, …, $(x_{m_k}, f(x_{m_k}))$.

The idea of the Neville's method is to use Lagrange polynomials of lower powers recursively in order to compute Lagrange polynomials of higher powers. This is useful, for example, if you have the Lagrange polynomial based on some set of data points $(x_i, f(x_k))$, $k = 0, 1, \ldots, n$, and you get a new data point, $(x_{n+1}, f(x_{n+1}))$. Neville's method is based on the following theorem:

Theorem. Let f be defined at the $(k+1)$ points x_0, x_1, …, x_k, and let x_i and x_j be two distinct points in this set. Let $P_{0,1,\ldots,i-1,i+1,\ldots,k}(x)$ be the Lagrange polynomial that agrees with f at x_0, x_1, …, x_{i-1}, x_{i+1}, …, x_k (only the point x_i is missing from this list). Similarly, let $P_{0,1,\ldots,j-1,j+1,\ldots,k}(x)$ be the Lagrange polynomial that agrees with f at x_0, x_1, …, x_{j-1}, x_{j+1}, …, x_k. Naturally, $P_{0,1,\ldots,i-1,i+1,\ldots,k}(x)$ and $P_{0,1,\ldots,j-1,j+1,\ldots,k}(x)$ are polynomials of degree $(k-1)$. Then the Lagrange polynomial $P_{0,1,\ldots,k}(x)$ through all the $(k+1)$ points x_0, x_1, …, x_k can be computed as follows:

$$P_{0,1,\ldots,k}(x) = \frac{(x - x_j)\, P_{0,1,\ldots,j-1,j+1,\ldots,k}(x) - (x - x_i)\, P_{0,1,\ldots,i-1,i+1,\ldots,k}(x)}{x_i - x_j} \ .$$

How does one use this theorem in practice? Suppose that you know the value of the unknown function f at the point $x_0 = 0$ and it is $f(x_0) = 0$. If you want to use Lagrange interpolation based on this point only, the only degree-0 polynomial whose graph passes through the point $(x_0, f(x_0)) = (0, 0)$ is the polynomial

$$P_0(x) = f(x_0) = 0 \ .$$

If you know that the point $(x_1, f(x_1)) = (\frac{1}{4}, \frac{1}{2})$ also belongs to the graph of f, and you want to construct the Lagrange polynomial based on this point *only*, this will be the polynomial

$$P_1(x) = f(x_1) = \frac{1}{2} \ .$$

Note that here we are using the new notations introduced in the definition above, so that the superscript 1 in the notation $P_1(x)$ does *not* mean that this is a polynomial of degree 1, but that this polynomial is based on the value of the function f at the point x_1.

Since we know two values of f, we would like to construct the degree-1 Lagrange polynomial $P_{0,1}(x)$ whose graph passes through the points $(x_0, f(x_0))$ and $(x_1, f(x_1))$. The above theorem tells us how to do this by using the expressions for $P_0(x)$ and $P_1(x)$:

$$P_{0,1}(x) = \frac{(x - x_1)\, P_0(x) - (x - x_0)\, P_1(x)}{x_0 - x_1} = \frac{(x - \frac{1}{4}) \cdot 0 - (x - 0) \cdot \frac{1}{2}}{0 - \frac{1}{4}} = 2\, x \ .$$

Now assume that you learn that the point $(x_2, f(x_2)) = (\frac{1}{16}, \frac{1}{4})$ also belongs to the graph of f. You can use this point to compute the degree-0 Lagrange polynomial $P_2(x)$. Then use your knowledge of $P_1(x)$ and

$P_2(x)$ to compute the degree-1 Lagrange polynomial $P_{12}(x)$ which is the only degree-1 polynomial whose graph passes through the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$, similarly to the way we found $P_{01}(x)$ above. Having found $P_{12}(x)$, you look back at $P_{01}(x)$ and realize that you can use the theorem above to find the degree-2 Lagrange polynomial $P_{0,1,2}(x)$. This is all you can do with the three points x_0, x_1, and x_2.

If you learn another point, $(x_3, f(x_3)) = (\frac{1}{9}, \frac{1}{3})$, that belongs to the graph of f, you can play the same game and find the degree-0 polynomial $P_3(x)$, the degree-1 polynomial $P_{23}(x)$, the degree-2 polynomial $P_{123}(x)$, and the degree-3 polynomial $P_{0123}(x)$.

If you are not interested in the interpolating polynomials, but only in the value of these interpolating polynomials at one point, say, $\bar{x} := \frac{1}{10}$, then you can do a calculation very similar to the above one, but with numbers (which is a bit easier than keeping track of the powers of x). The calculations below are actually the solution of part (a) of Problem 3 of the new Homework #6. The calculations look like this:

$$P_0(\tfrac{1}{10}) = f(x_0) = 0 \ ,$$

$$P_1(\tfrac{1}{10}) = f(x_1) = \tfrac{1}{2} \ ,$$

$$P_{0,1}(\tfrac{1}{10}) = \frac{(\frac{1}{10} - x_1)\, P_0(\frac{1}{10}) - (\frac{1}{10} - x_0)\, P_1(\frac{1}{10})}{x_0 - x_1} = \frac{(\frac{1}{10} - \frac{1}{4}) \cdot 0 - (\frac{1}{10} - 0) \cdot \frac{1}{2}}{0 - \frac{1}{4}} = \frac{1}{5} \ .$$

In the homework problem, you are given one more point, $(x_2, f(x_2)) = (\frac{1}{16}, \frac{1}{4})$, which you have to use (together with the previously found values) to find the values $P_2(\frac{1}{10})$, $P_{12}(\frac{1}{10})$, and $P_{0,1,2}(\frac{1}{10})$.

You can continue in this way if you know more points – for example, if in addition you know the point $(x_3, f(x_3)) = (\frac{1}{9}, \frac{1}{3})$, you can find $P_3(\frac{1}{10})$, $P_{23}(\frac{1}{10})$, $P_{1,2,3}(\frac{1}{10})$, and $P_{0,1,2,3}(\frac{1}{10})$; the last one has value $P_{0,1,2,3}(\frac{1}{10}) = \frac{2799}{8750}$.

It is convenient to arrange the above calculations in a table. The first column of the table below contains the values of x_i, the second one contains the degree-0 approximations $P_i(\frac{1}{10})$, which, as we know, are the same as the values of the corresponding $f(x_i)$. If we use only x_0 and x_1 and the corresponding values of f, we have the following table:

x_i	degree-0 approx.
x_0	$f(x_0) = P_0(\bar{x})$
x_1	$f(x_1) = P_1(\bar{x})$

Then we use the values $P_0(\bar{x})$ and $P_1(\bar{x})$ to find the degree-1 approximation to $f(\bar{x})$:

x_i	degree-0 approx.		degree-1 approx.
x_0	$f(x_0) = P_0(\bar{x})$	$\searrow$	
x_1	$f(x_1) = P_1(\bar{x})$	$\longrightarrow$	$P_{0,1}(\bar{x})$

The arrows in the table indicate that we used $P_0(\bar{x})$ and $P_1(\bar{x})$ to compute $P_{0,1}(x)$.

Then one computes all the entries in the next row, then the all the entries in the row after that, etc.:

x_i	degree-0 approx.		degree-1 approx.		degree-2 approx.		degree-3 approx.
x_0	$f(x_0) = P_0(\bar{x})$	$\searrow$					
x_1	$f(x_1) = P_1(\bar{x})$	$\longrightarrow$	$P_{0,1}(\bar{x})$	$\searrow$			
x_2	$f(x_2) = P_2(\bar{x})$	$\longrightarrow$	$P_{1,2}(\bar{x})$	$\longrightarrow$	$P_{0,1,2}(\bar{x})$	$\searrow$	
x_3	$f(x_3) = P_3(\bar{x})$	$\longrightarrow$	$P_{2,3}(\bar{x})$	$\longrightarrow$	$P_{1,2,3}(\bar{x})$	$\longrightarrow$	$P_{0,1,2,3}(\bar{x})$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Your task in Problem 3(b) of Homework #6 is to compute the numbers $P_2(\frac{1}{10})$, $P_{1,2}(\frac{1}{10})$, and $P_{0,1,2}(\frac{1}{10})$. You do *not* have to compute thepolynomials $P_2(x)$, $P_{1,2}(x)$, and $P_{0,1,2}(x)$! (Well, you can do it if you have nothing more interesting to do...)